A Appendix: Efficient Entry in Competing Auctions (James Albrecht, Pieter A. Gautier, Susan Vroman)

A.1 *Ex Post* Case

In this appendix, we verify equation (2) and derive expressions for $\Lambda(s, \theta(s); s)$, $\Pi(s, \theta(s); s)$, and $V(s, \theta(s))$. As noted in the text, in Albrecht et al. (2012), we analyzed the problem

$$\max_{r, \theta(r)} \Pi(r, \theta(r); 0) \text{ subject to } V(r, \theta(r)) = \overline{V}$$

using expressions for $\Pi(r, \theta(r); 0)$ and $V(r, \theta(r))$ that were derived in Peters and Severinov (1997). This is the problem of a seller of type $s = 0$ choosing a reserve price for a second-price auction who takes into account that the reserve price determines the expected arrival rate of buyers to the auction. In the process of showing that $r = 0$, that is, that the optimal reserve price equals the seller reservation value, we derived

$$\frac{\partial \Pi(0, \theta(0); 0)}{\partial \theta} + \theta(0) \frac{\partial V(0, \theta(0))}{\partial \theta} = 0$$

as one of the first-order conditions for the seller's constrained maximization problem. The same approach can be used to confirm equation (2),

$$\frac{\partial \Pi(s, \theta(s); s)}{\partial \theta} + \theta(s) \frac{\partial V(s, \theta(s))}{\partial \theta} = 0$$

for arbitrary s.

Here we present an alternative approach to verifying equation (2). We use the result that competition leads sellers to post efficient auctions; that is, sellers post reserve prices equal to reservation values. This gives equation (1) in the text, namely,

$$\Lambda(s,\theta(s);s) = \Pi(s,\theta(s);s) + \theta(s)V(s,\theta(s)).$$

Differentiating gives

$$\frac{\partial\Lambda(s,\theta(s);s)}{\partial\theta} = \frac{\partial\Pi(s,\theta(s);s)}{\partial\theta} + \theta(s)\frac{\partial V(s,\theta(s))}{\partial\theta} + V(s,\theta(s)),$$

and our proof of equation (2) consists of showing

$$\frac{\partial\Lambda(s,\theta(s);s)}{\partial\theta} = V(s,\theta(s)). \tag{7}$$

To do this, we develop expressions for $\Lambda(s,\theta(s);s)$ and $V(s,\theta(s))$. We also derive $\Pi(s,\theta(s);s)$ and verify that our expressions for seller and buyer expected payoffs are equivalent to the ones given in Peters and Severinov (1997).

We begin with an expression for $\Lambda(s,\theta(s);s)$, the expected surplus associated with the auction posted by seller s, which equals the expected maximum of s and the highest valuation greater than s drawn by one of the buyers. Suppose n buyers visit this seller and draw valuations of s or more. If $n=0$, then $\Lambda(s,\theta(s);s)=s$. If $n\geq 1$, then, conditional on n, the expected surplus is

$$\int_s^1 xd\left(\frac{F(x)-F(s)}{1-F(s)}\right)^n = 1 - \int_s^1 \left(\frac{F(x)-F(s)}{1-F(s)}\right)^n dx,$$

that is, the expected maximum of the n draws that are greater than or equal to s.

The number of buyers visiting seller s who draw valuations of s or more is Poisson with parameter $\theta(s)(1-F(s))$, so the unconditional expression for expected surplus is

$$\begin{aligned}
\Lambda(s,\theta(s);s) &= se^{-\theta(s)(1-F(s))} \\
&\quad + e^{-\theta(s)(1-F(s))}\sum_{n=1}^{\infty}\frac{(\theta(s)(1-F(s)))^n}{n!}\left(1-\int_s^1\left(\frac{F(x)-F(s)}{1-F(s)}\right)^n dx\right) \\
&= se^{-\theta(s)(1-F(s))} + e^{-\theta(s)(1-F(s))}\left(e^{\theta(s)(1-F(s))}-1\right) \\
&\quad - e^{-\theta(s)(1-F(s))}\int_s^1\sum_{n=1}^{\infty}\frac{(\theta(s)(F(x)-F(s)))^n}{n!}dx; \text{ i.e.,} \\
\Lambda(s,\theta(s);s) &= 1-\int_s^1 e^{-\theta(s)(1-F(x))}dx. \qquad (8)
\end{aligned}$$

Next, consider $V(s,\theta(s))$, the expected payoff for a buyer who chooses a seller of type s. Suppose this buyer draws a valuation $x \geq s$. This is the winning draw with probability $e^{-\theta(s)(1-F(x))}$. Conditional on winning, the buyer's payoff is the difference between x and the highest valuation drawn by any other buyers who visited this seller or s if no other buyer drew a valuation $y \geq s$. Now suppose there were n other buyers who drew $y \in [s,x)$. Conditional on n, the winning buyer's expected payoff is then

$$x - \int_s^x y d\left(\frac{F(y)-F(s)}{F(x)-F(s)}\right)^n = \int_s^x\left(\frac{F(y)-F(s)}{F(x)-F(s)}\right)^n dy$$

Summing against the probability mass function for n, the buyer's expected payoff, conditional on winning with valuation $x \geq s$, is

$$\sum_{n=0}^{\infty}\frac{e^{-\theta(s)(F(x)-F(s))}\left(\theta(s)(F(x)-F(s))\right)^n}{n!}\int_s^x\left(\frac{F(y)-F(s)}{F(x)-F(s)}\right)^n dy = \int_s^x e^{-\theta(s)(F(x)-F(y))}dy.$$

Multiplying by the probability of winning with valuation x at seller s gives

$$V(s,\theta(s);x) = \int_s^x e^{-\theta(s)(1-F(y))}dy. \qquad (9)$$

This is the expected payoff for a buyer who visits seller s conditional on drawing valuation x, where $x \geq s$. Finally, the unconditional expected payoff for a buyer who visits seller s is

$$\begin{aligned} V(s,\theta(s)) &= (1-F(s))\int_s^1 V(s,\theta(s);x)\frac{f(x)}{1-F(s)}dx \\ &= \int_s^1\int_s^x e^{-\theta(s)(1-F(y))}dyf(x)dx = \int_s^1 (1-F(x))e^{-\theta(s)(1-F(x))}dx \quad (10) \end{aligned}$$

where the last equality follows by integration by parts ($u = \int_s^x e^{-\theta(s)(1-F(y))}dy$, $v = -(1-F(x))$). With $s = 0$, equation (10) is the expression for expected buyer payoff derived in Peters and Severinov (1997).

Differentiating equation (8) and using equation (10) gives equation (7). This completes the derivation of equation (2).

For completeness, we also derive $\Pi(s,\theta(s);s)$. Consider a seller of type s who posts reserve price equal to s, and suppose this seller attracts n buyers who draw valuations of s or more. The seller gets a payoff of s if $n = 0$ or 1; if $n \geq 2$, the seller's expected payoff equals the expected value of the second highest valuation across the buyers, i.e., $E[Y_{n-1}]$. The distribution function of Y_{n-1} is

$$G_{n-1}(y) = \left(\frac{F(y)-F(s)}{1-F(s)}\right)^n + n\left(\frac{F(y)-F(s)}{1-F(s)}\right)^{n-1}\left(\frac{1-F(y)}{1-F(s)}\right) \text{ for } s \leq y \leq 1.$$

Thus, conditional on $n \geq 2$, the seller's expected payoff is

$$\int_s^1 ydG_{n-1}(y) = 1 - \int_s^1 \left(\frac{F(y)-F(s)}{1-F(s)}\right)^n dy - n\int_s^1 \left(\frac{(F(y)-F(s))^{n-1}(1-F(y))}{(1-F(s))^n}\right) dy.$$

Summing against the probability mass function for n,

$$\begin{aligned}
\Pi(s,\theta(s);s) &= se^{-\theta(s)(1-F(s))}\left(1+\theta(s)(1-F(s))\right)+e^{-\theta(s)(1-F(s))}\sum_{n=2}^{\infty}\frac{(\theta(s)(1-F(s))^n}{n!} \\
&\quad -e^{-\theta(s)(1-F(s))}\int_s^1\sum_{n=2}^{\infty}\frac{(\theta(s)(F(y)-F(s))^n}{n!}dy \\
&\quad -e^{-\theta(s)(1-F(s))}\int_s^1(1-F(y))\theta(s)\sum_{n=2}^{\infty}\frac{(\theta(s)(F(y)-F(s)))^{n-1}}{(n-1)!}; \text{ i.e.,} \\
\Pi(s,\theta(s);s) &= 1-\int_s^1 e^{-\theta(s)(1-F(x))}dx-\int_s^1\theta(s)(1-F(x))e^{-\theta(s)(1-F(x))}dx. \qquad (11)
\end{aligned}$$

Using equations (8), (10) and (11), it is straightforward to check equation (1). Finally, using

$$1-\int_s^1 e^{-\theta(s)(1-F(x))}dx=\theta(s)\int_s^1 xe^{-\theta(s)(1-F(x))}f(x)dx,$$

the seller's expected payoff can be rewritten as

$$\Pi(s,\theta(s);s)=\theta(s)\int_s^1\left(x-\frac{1-F(x)}{f(x)}\right)e^{-\theta(s)(1-F(x))}f(x)dx.$$

With $s=0$, this is the expression for expected seller payoff that is given in Peters and Severinov (1997).

A.2 *Ex Ante* Case

In this appendix, we give the details and prove the efficiency of free-entry equilibriuim for the *ex ante* case, i.e., the case in which buyers draw their valuations before deciding which seller to visit. As discussed in the text, competition leads sellers to post efficient mechanisms in the *ex ante* case just as it does in the *ex post case*; in particular, a seller with reservation value s posts a second-price auction with reserve price s. However, with a distribution of seller types in the market, the fact that buyers learn their valuations *ex ante* complicates the analysis. The extra complication arises because the distribution of buyers will vary across seller types. The surplus associated with an auction posted by a type s seller should therefore be written as $\Lambda(s, \theta(s), F(x; s); s)$; that is, the surplus depends on the posted reserve price, s, on the expected number of buyers attracted by that reserve price, $\theta(s)$, and on the distribution of valuations across the buyers visiting sellers of type s, $F(x; s)$, as well as on the seller's type, s. Similarly, the expected payoff for seller s is $\Pi(s, \theta(s), F(x; s); s)$, and the expected payoff for a buyer with valuation x who visits a seller posting reserve price s is $V(s, \theta(s), F(x; s); x)$. Applying the approach used in Appendix 1 gives

$$\Lambda(s, \theta(s), F(x; s); s) = 1 - \int_s^1 e^{-\theta(s)(1-F(x;s))} dx \tag{12}$$

$$\Pi(s, \theta(s), F(x; s); s) = \theta(s) \int_s^1 \left(x - \frac{1 - F(x; s)}{f(x; s)} \right) e^{-\theta(s)(1-F(x;s))} f(x; s) dx \tag{13}$$

$$V(s, \theta(s), F(x; s); x) = \int_s^x e^{-\theta(s)(1-F(y;s))} dy \tag{14}$$

For ease of notation, however, we use $\Lambda(s)$, $\Pi(s)$ and $V(s; x)$, respectively. We also simplify the notation by normalizing the measure of buyers, B, to one.

We use the following approach to characterize the equilibrium and the social planner solution in the *ex ante* case. First, we consider the case of two seller types with a mass m_1 of sellers of type s_1 and a mass m_2 of potential sellers of type s_2, where $0 \leq s_1 < s_2 < 1$. Second, we extend the analysis to the case of N seller types with masses $m_1, ..., m_N$ of seller types $0 \leq s_1 < ... < s_N < 1$. Finally, we move to a continuum of sellers by considering the appropriate limit.

A.2.1 Two Seller Types

We have argued in the text of the paper that equilibrium is characterized by a cutoff, x^*, and a measure, m_2^*, of type s_2 sellers such that

$$V(s_1; x^*) = V(s_2; x^*) \quad (15)$$

$$\Pi(s_2) = A + s_2. \quad (16)$$

Note that x^* is the *lowest* value of x satisfying equation (15); in particular, $V(s_1; x) > V(s_2; x)$ for all $x \in [s_1, x^*)$ while $V(s_1; x) = V(s_2; x)$ for all $x \in [x^*, 1]$.

The corresponding social planner problem is to choose a cutoff, $\widehat{x}$, and a measure, $\widehat{m}_2$, to maximize

$$m_1 \left(\Lambda(s_1) - (A + s_1)\right) + m_2 \left(\Lambda(s_2) - (A + s_1)\right).$$

Consider the partial derivative of the social planner maximand with respect to $\widehat{x}$; in particular, consider an increase in the cutoff from $\widehat{x}$ to $\widehat{x}+dx$. The key to understanding this derivative is to recognize that the only agents who change their behavior are buyers with valuations in the interval $[\widehat{x}, \widehat{x} + dx)$. Buyers with valuations $x \in [0, \widehat{x})$ randomized over sellers of type s_1 before the change; they continue to do so after the change. Similarly, buyers with valuations $x \in [\widehat{x} + dx, 1]$ randomized over all sellers before the change; they continue to do so afterwards.

Buyers with valuations $x \in [\widehat{x}, \widehat{x} + dx)$ randomized over all sellers before the increase in the cutoff; after the increase, these buyers randomize over sellers of type s_1. Thus, there are some buyers with valuations in $[\widehat{x}, \widehat{x} + dx)$ who would have participated in an auction run by a seller of type s_2 before the change but instead participate in an auction run by a seller of type s_1 after the change. To be more precise, approximately $\frac{m_2}{m_1+m_2}f(\widehat{x})dx$ buyers are expected to switch seller types. When a buyer switches from an auction with reserve price s_2 to one with reserve price s_1, there is an increase in surplus associated with the auction posted by the type s_1 seller but a decrease in surplus associated with the auction posted by the type s_2 seller. The social planner wants these two effects to offset each other. Were this not the case, e.g., if moving buyers with valuations close to $\widehat{x}$ from type s_2 sellers to type s_1 sellers increased the surplus associated with auctions posted by type s_1 sellers more than it decreased the surplus associated with auctions posted by type s_2 sellers, then the social planner should increase the cutoff value.

Consider the reallocation of a buyer with valuation $x \in [\widehat{x}, \widehat{x}+dx)$ from a type s_2 seller to a type s_1.seller. The increase in surplus at the auction posted by the type s_1 seller is the sum of three components: (i) the increase in the seller's expected payoff, (ii) the decrease in the expected payoffs of any "incumbent" buyers, and (iii) the expected payoff, $V(s_1;x)$, of the buyer who switched seller types. As we have argued in the text, the first two terms are exactly offsetting; thus, the increase in surplus associated with an auction posted by a type s_1 seller that gained a buyer of type x equals $V(s_1;x)$. By the same argument, the decrease in surplus associated with an auction posted by a type s_2 seller that lost a buyer of type x equals $V(s_2;x)$. Now let $dx \to 0$, so $x \approx \widehat{x}$. Satisfying the first-order condition of the social planner problem with respect to the cutoff value requires

$$V(s_1;\widehat{x}) = V(s_2;\widehat{x}).$$

The cutoff $\widehat{x}$ is the lowest value of x satisfying this equation. Equation (15) thus implies $\widehat{x} = x^*$; that is, the equilibrium and social planner cutoffs coincide.

Next consider the partial derivative of the social planner maximand with respect to $\widehat{m}_2$. We can write the first-order condition as

$$m_1 \frac{\partial \Lambda(s_1)}{\partial m_2} + \widehat{m}_2 \frac{\partial \Lambda(s_2)}{\partial m_2} + \Lambda(s_2) - (A + s_2) = 0. \tag{17}$$

Since the social planner wants sellers to post efficient mechanisms, we have

$$\Lambda(s_2) = \Pi(s_2) + \theta(s_2)V(s_2),$$

where

$$V(s_2) = \int_{x^*}^{1} V(s_2; x) \frac{f(x)}{1 - F(x^*)} dx \tag{18}$$

is a shorthand notation for the expected payoff per buyer visiting a type s_2 seller. Equation (17) implies equation (16) if

$$m_1 \frac{\partial \Lambda(s_1)}{\partial m_2} + \widehat{m}_2 \frac{\partial \Lambda(s_2)}{\partial m_2} = -\theta(s_2)V(s_2). \tag{19}$$

We now verify equation (19). Suppose a type s_2 seller enters the market. In expectation, this seller takes $\theta(s_2)$ buyers away from incumbent sellers; thus, $m_1 \frac{\partial \Lambda(s_1)}{\partial m_2} + \widehat{m}_2 \frac{\partial \Lambda(s_2)}{\partial m_2}$ is the business-stealing effect associated with the entrant. Any buyer who attempts to purchase the good from the new entrant has a valuation of x^* or more, and these buyers randomize their visits across both seller types. If a buyer with valuation x moves from an incumbent seller to the new entrant, the loss in surplus at the incumbent seller's auction is $V(s_2; x)$, and since $V(s_1; x) = V(s_2; x)$ for all $x \geq x^*$, i.e., high-valuation buyers are indifferent between the two seller types, this loss is the same irrespective of the type of the incumbent seller. The valuation x is a draw from the truncated density, $\frac{f(x)}{1 - F(x^*)}$; thus, the expected loss to the incumbent seller is $V(s_2)$, as given in equation (18). Multiplying this by the expected number of buyers who visit the new entrant gives equation (19).

A.2.2 N Seller Types

Suppose there are N seller types, $0 \leq s_1 < \cdots < s_N < 1$, with respective measures $m_1, \ldots, m_N$, where we consider the entry decision of the marginal seller type, s_N. In equilibrim, there will exist $N-1$ thresholds $x^*(s_2), \ldots, x^*(s_N)$ such that buyer types in $[x^*(s_k), x^*(s_{k+1})]$ randomize among sellers $s_1, \ldots, s_k$. Buyers of type $x < s_1$ do not participate in the market; that is, $x^*(s_1) = s_1$. Equilibrium is characterized by the cutoffs, $x^*(s_1), x^*(s_2), ...x^*(s_N)$, and a measure of sellers of type s_N such that

$$
\begin{aligned}
0 &= V(s_1; x^*(s_1)) \\
V(s_1; x^*(s_2)) &= V(s_2; x^*(s_2)) \qquad (20) \\
V(s_1; x^*(s_3)) &= V(s_2; x^*(s_3)) = V(s_3; x^*(s_3)) \\
&\ldots \\
V(s_1; x^*(s_k)) &= V(s_2; x^*(s_k)) = \ldots = V(s_k; x^*(s_k)) \qquad (21) \\
&\ldots. \\
V(s_1; x^*(s_N)) &= V(s_2; x^*(s_N)) = \ldots = V(s_N; x^*(s_N)) \\
\Pi(s_N) &= A + s_N. \qquad (22)
\end{aligned}
$$

The corresponding social planner problem is to choose cutoffs, $\widehat{x}(s_1), \widehat{x}(s_2), \ldots, \widehat{x}(s_N)$, and a measure, $\widehat{m}_N$, of type s_N sellers to maximize

$$\sum_{k=1}^{N} m_k \left(\Lambda(s_k) - (A + s_k) \right).$$

The proof that $\widehat{x}(s_k) = x^*(s_k)$ for $k = 1, ..., N$ is essentially the same as the one given for the case of two seller types.

First, it is obvious that $x^*(s_1) = \widehat{x}(s_1) = s_1$. Buyers with $x \leq s_1$ have no incentive to participate in the market nor does the social planner want them to do so. Second, given any collection of cutoffs for sellers of type s_3 and above and given any level of entry by type s_N sellers, the choice of $\widehat{x}(s_2)$ does not affect $\sum_{k=3}^{N} m_k \left(\Lambda(s_k) - (A + s_k)\right)$. The same argument that was used to characterize $\widehat{x}(s_2)$ in the two-seller case then implies that the social planner sets $\widehat{x}(s_2)$ so that

$$V(s_1; \widehat{x}(s_2)) = V(s_2; \widehat{x}(s_2)).$$

Comparing this with equation (20) gives $\widehat{x}(s_2) = x^*(s_2)$. The final step in the argument uses induction. Suppose $\widehat{x}(s_i) = x^*(s_i)$ for $i = 1, ..., k-1$, and take $\{\widehat{x}(s_{k+1}), \ldots, \widehat{x}(s_N), m_N\}$ as given. The choice of $\widehat{x}(s_k)$ has no effect on $\sum_{i=k+1}^{N} m_i \left(\Lambda(s_i) - (A + s_i)\right)$. Since $\{\widehat{x}(s_1), \widehat{x}(s_2), \ldots, \widehat{x}(s_{k-1})\}$ are assumed to have been set optimally, the social planner is indifferent between assigning the buyer with valuation $\widehat{x}(s_k)$ to seller s_k versus assigning that buyer to any seller with a lower reservation value. That is,

$$V(s_1; \widehat{x}(s_k)) = V(s_2; \widehat{x}(s_k)) = ... = V(s_k; \widehat{x}(s_k));$$

thus, by comparison with equation (21), $\widehat{x}(s_k) = x^*(s_k)$. By induction, the equilibrium and social planner cutoff values coincide for $i = 1, ..., N$.

Finally, in order that $\widehat{m}_N = m_N^*$, it must be that

$$\sum_{k=1}^{N} m_k \frac{\partial \Lambda(s_k)}{\partial m_M} = -\theta(s_N) V(s_N);$$

that is, the business-stealing effect associated with the entry of a type s_N seller has to equal the expected number of buyers drawn away by the entry of the marginal seller times the loss in surplus per buyer who leaves an incumbent's auction. The argument for why this equation holds is exactly the same as in the case with two seller types.

A.2.3 Continuum of Seller Types

In the model with N seller types, for each seller type s_k, there is a corresponding buyer type $x^*(s_k)$ who is indifferent between visiting a seller of type s_k versus any seller posting a lower reserve price. The function $x^*(s_k)$ is defined on a discrete set of points, $\{s_1, ..., s_N\}$. To move to a continuum of seller types, we let the distance between seller types s_{k+1} and s_k go to zero and derive a differential equation that gives a continuous function $x^*(s)$ as the limit of the N-seller case. The purpose of this subsection is to derive this equation. Since the continuum-of-seller-types solution is the limit of the discrete seller type case, our efficiency results carry over to the continuum.

As in the *ex post* case, we normalize the total measure of potential sellers to one, and we denote the distribution of reservation values across these seller by $G(s)$. We begin with a discrete distribution over seller types. Let $s_1 = 0$, $s_2 = \Delta s, ..., s_{k+1} = s_k + \Delta s$, and let $m_1 = g(s_1)\Delta s$, $m_2 = g(s_2)\Delta s$, etc. We denote the arrival rate of buyers to type s_k sellers by $\theta(s_k)$ and the distribution of valuations among buyers visiting type s_k sellers by $F(x; s_k)$.

Lemma 1

$$\theta(s_k) = \theta(s_{k+1}) + \frac{1}{\sum_{j=1}^{k} m_j} \int_{x^*(s_k)}^{x^*(s_k)} f(x)dx \text{ for } k = 1, ..., N-1 \quad (23)$$

$$\theta(s_k)F(x; s_k) = \frac{1}{\sum_{j=1}^{k} m_j} \int_{x^*(s_k)}^{x} f(z)dz \text{ for } x^*(s_k) \leq x \leq x^*(s_{k+1}). \quad (24)$$

Proof. Buyers with valuations $x \geq x^*(s_{k+1})$ randomize across all sellers of type s_{k+1} or below. Thus, a type s_k seller can expect as many buyers of this type as can a type s_{k+1} seller. In addition, a type s_k seller attracts some additional buyers, namely, those with valuations $x \in [x^*(s_k), x^*(s_{k+1}))$. Buyers with valuations in this range randomize over sellers of type s_k and below, and there is a mass $\sum_{j=1}^{k} m_j$ of such sellers. This gives equation (23). To understand equation (24), note that (i) the measure of the buyers with valuations between $x^*(s_k)$ and $x < x^*(s_{k+1})$ visiting type s_k sellers is $\left(\dfrac{m_k}{m_1 + ... + m_k}\right) \int_{x^*(s_k)}^{x} f(z)dz$ while (ii) the measure of type s_k sellers can be written as $m_k\theta(s_k)$. Since $F(x; s_k) = 0$ for $x \leq x^*(s_k)$, it follows that

$$F(x; s_k) = \frac{\left(\dfrac{m_k}{m_1 + ... + m_k}\right) \int_{x^*(s_k)}^{x} f(z)dz}{m_k\theta(s_k)} \text{ for } x^*(s_k) \leq x \leq x^*(s_{k+1}).$$

Multiplying both sides by $\theta(s_k)$ gives equation (24). ∎

The cutoff valuation $x^*(s_{k+1})$, i.e., the lowest buyer type who is indifferent between visitinga type s_k seller versus a type s_{k+1} seller, is defined by $V(s_k; x^*(s_{k+1})) = V(s_{k+1}; x^*(s_{k+1}))$. Using equation (14) gives

$$\int_{s_k}^{x^*(s_{k+1})} e^{-\theta(s_k)(1-F(x;s_k))}dx = \int_{s_{k+1}}^{x^*(s_{k+1})} e^{-\theta(s_{k+1})(1-F(x;s_{k+1}))}dx. \tag{25}$$

Note that $F(x; s_k) = 0$ for $x < x^*(s_k)$, and similarly $F(x; s_{k+1}) = 0$ for $x < x^*(s_{k+1})$. Equation (25) can thus be rewritten as

$$e^{-\theta(s_k)}\left(x^*(s_k) - s_k\right) + \int_{x^*(s_k)}^{x^*(s_{k+1})} e^{-\theta(s_k)(1-F(x;s_k))}dx = e^{-\theta(s_{k+1})}\left(x^*(s_{k+1}) - s_{k+1}\right) \text{ or}$$

$$x^*(s_k) - s_k + \int_{x^*(s_k)}^{x^*(s_{k+1})} e^{\theta(s_k)F(x;s_k)}dx = e^{\theta(s_k)-\theta(s_{k+1})}\left(x^*(s_{k+1}) - s_{k+1}\right)$$

Using

$$e^{\theta(s_k)F(x;s_k)} \simeq 1 + \theta(s_k)F(x; s_k) \text{ and } e^{\theta(s_k)-\theta(s_{k+1})} \simeq 1 + \theta(s_k) - \theta(s_{k+1})$$

equation (25) can be further rewritten as

$$x^*(s_{k+1}) - s_k + \int_{x^*(s_k)}^{x^*(s_{k+1})} \theta(s_k)F(x;s_k)dx = (1 + \theta(s_k) - \theta(s_{k+1}))\left(x^*(s_{k+1}) - s_{k+1}\right). \tag{26}$$

We use the notation $s_{k+1} = s$, $s_k = s - \Delta s$, $x^*(s_{k+1}) = x^*(s)$, and $x^*(s_k) = x^*(s) - \Delta x^*(s)$ and note that $G(s) = \sum_{j=1}^{k} m_j$. Then using equations (23) and (24), we can rewrite equation (26) as

$$x^*(s) - s + \Delta s + \int_{x^*(s)-\Delta x^*(s)}^{x} \int_{x^*(s)-\Delta x^*(s)}^{x} \frac{f(z)dz}{G(s)} dx = (x^*(s) - s)\left(1 + \frac{\int_{x^*(s)-\Delta x^*(s)}^{x^*(s)} f(x)dx}{G(s)}\right). \tag{27}$$

The left-hand side of equation (27) used equation (24); the right-hand side used equation (23). The term $\int_{x^*(s)-\Delta x^*(s)}^{x} \int_{x^*(s)-\Delta x^*(s)}^{x} \frac{f(z)dz}{G(s)} dx$ on the left-hand side of this equation is $o(\Delta x^*(s))$. On the right-hand side,

$$\int_{x^*(s)-\Delta x^*(s)}^{x^*(s)} f(x)dx = F(x^*(s)) - F(x^*(s) - \Delta x^*(s)) \simeq f(x^*(s))\Delta x^*(s).$$

Equation (27) therefore reduces to

$$\frac{\Delta x^*(s)}{\Delta s} = \frac{G(s)}{(x^*(s) - s)\, f(x)}. \tag{28}$$

Together with the initial condition, $x^*(0) = 0$, equation (28) determines the function $x^*(s)$.