An "Anti-Folk Theorem" for a Class of Asynchronously Repeated Games

Roger Lagunoff* and Akihiko Matsui†

September 30, 1995‡

Abstract

It is a well known "Folk wisdom" that infinitely repeated games admit a multitude of equilibria, and that repetition always expands (weakly) the set of equilibrium payoffs. This paper demonstrates that in some types of games, the typical form of multiplicity is an artifact of the standard representation which assumes perfect synchronization in the timing of decisions between the players. We define here a more general family of repeated settings called *renewal games*. Specifically, a renewal game is a setting in which a stage game is repeated in continuous time, and at certain stochastic points in time determined by an arbitrary renewal process some set of players may be called upon to make a move. A stationary Markov process determines who moves at each decision node. We restrict attention in this paper to a subclass of renewal games called *asynchronously repeated games*, in which no two individuals can change their actions simultaneously. Special cases include the alternating move game and the Poisson revision game. In the latter, each player adjusts his action independently at Poisson distributed times.

Our main result concerns asynchronously repeated coordination games. First, it is shown that given $\epsilon > 0$, if players are sufficiently patient then every Perfect equilibrium payoff comes within ϵ of the Pareto dominant payoff if the stage game is pure coordination where the payoffs of all players in the stage game are identical up to an affine transformation. We also show that it is not generally true that repetition always expands (weakly) the set of equilibrium payoffs in asynchronously repeated games.

Journal of Economic Literature Classification Numbers: C72, C73.
Keywords: repeated games, asynchronously repeated games, renewal games, coordination games.

---

*Department of Economics, University of Pennsylvania, Philadelphia, PA 19104, USA, and Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA.

†Department of Economics, University of Pennsylvania, Philadelphia, PA 19104, USA, and Institute of Socio-Economic Planning, University of Tsukuba, Ibaraki 305, Japan.

‡We would like to thank Eddie Dekel, Hseuh-ling Huyn, Hideshi Itoh, Mamoru Kaneko, Peter Linhart, George Mailath, Stephen Morris, Roy Radner, Phil Reny, Andrew Postlewaite, Ariel Rubinstein, and numerous seminar participants for many useful discussions and suggestions.

1 Introduction

The theory of infinitely repeated games offers little predictive content. It is well known from the Folk Theorem that infinitely repeated games admit a multitude of equilibria.[1] To focus on a particularly robust multiplicity problem, it is known that each one shot Nash payoff is an equilibrium payoff of the infinite repetition of a stage game, and each payoff derived as a convex combination of one-shot equilibrium payoffs (both pure and mixed equilibria) is also an equilibrium payoff of the repeated game. In the simple, cannonical pure coordination game G_1 below, this means that any symmetric payoff pair (u, u) with u in the interval $[2/3, 2]$ is an equilibrium payoff of the infinite repetition of G_1. These payoffs are robust with respect to variations in both discount parameters and the equilibrium concept. That is, these payoffs remain equilibrium payoffs for all discount parameters, and they remain perfect equilibrium payoffs as well.

		2	
		s_2^*	$\bar{s}_2$
1	s_1^*	2, 2	0, 0
	$\bar{s}_1$	0, 0	1, 1

G_1

These observations are neither new nor surprising. What we emphasize here is the "Folk wisdom" that repetition always *expands* (weakly) the set of equilibrium outcomes simply because the repetition of a one-shot Nash equilibrium is always an equilibrium. Therefore, if the stage game has more than one equilibria, repetition will never overcome the multiplicity problem. Moreover, this is a particularly robust phenomena — *within the standard formulation of repeated play.* Our contribution here is to demonstrate why, in some types of games, this last addendum is necessary.

The standard model of repeated strategic play is a discretely repeated, simultaneous move game. This formulation assumes a perfect synchronization in the timing of actions between the players. Alternatively, this assumption may be interpreted as having each player move in ignorance of the other players' current move. The effect of this assumption in game G_1 above, for example, is that each player may take an action consistent with a Pareto inferior static equilibrium only because he expects that all others will do the same. Since all players move at once, no player can unilaterally signal his intent to do otherwise.

While the synchronized move is not an unreasonable model of repetition in certain settings, it is not clear why it should *necessarily* be the benchmark setting for repeated play. In

[1] See, for example, Abreu (1988) or Fudenberg and Maskin (1986). References to earlier results may be found in a survey by Aumann (1981).

the real world many repeated situations are asynchronous by nature. A dramatic example is the interchange between two branches of the same firm, one in New York and the other Tokyo, who must move asynchronously due to the time difference between the two cities. The "voting game" for a presidential candidate of a party in the U.S. has both synchronous and asynchronous moves since votes take place simultaneously for some group of states, but they are sequential across different groups of states. Simultaneous actions are, in fact, rare if players take actions randomly in time (consider the exchange of e-mail between co-authors!).

It seems natural in many contexts that players move asynchronously to initiate a unilateral move. While signalling can occur in the standard model, it remains part of the coordination built in to equilibrium beliefs rather than a physical attribute of the game. If no two players can move at once then even if a player expects that other players will choose the inferior equilibrium he must wait for them to actually do so since actions cannot be perfectly synchronized. The lack of synchronization may therefore eliminate a "coordinated mistake."

In this paper we describe a general class of games with repeated interaction, some of which break the perfect synchronization of the standard model. In so doing we are able to significantly pare down the set of equilibria in certain types of such games. We call games in this class *renewal games*. Specifically, a renewal game is defined as a setting in which a stage game is repeated in continuous time, and at certain stochastic points in time, determined by an arbitrary renewal process, some set of players may be called upon to make a move. A stationary Markov process operates on a set of "decision states," each decision state determining "who moves" at each jump in the renewal process. The process is assumed to have the property that every player will obtain a chance (almost surely) to make a move after any state. The standard repeated game model is shown to be a special case of a renewal game.

Though the characterization of equilibrium payoffs for general renewal games is our eventual goal, we will limit our attention in this paper to an interesting subclass of renewal games in which no more than one player may revise his action at any given time. We call the games that fit this description *asynchronously repeated games*.

Two special cases of asynchronously repeated games are the two player alternating move game (see, for example Maskin and Tirole (1988)), and the Poisson revision game studied in several evolutive models (see Lagunoff and Matsui (1995) for one formal description). In the former, the renewal process is deterministic, and each player moves at every other decision state without moving together with the other player. In the latter, each player can adjust his action at stochastic intervals determined by a Poisson process. Each player's adjustment process is independent of any other player's. Both examples share the general characteristic of asynchronously repeated games which is that no two individuals can change their actions simultaneously. In such games the players' adjustments may exhibit some inertia due to each player's (possibly stochastic) delay. We consider equilibria which are robust to taking the limit as the players' discount rates approach zero, or alternatively, the average delay between a player's revision opportunities goes to zero. In all such games, we do not bound the complexity of each player's behavior strategy except to assume that strategies are conditioned on the history of behavior at decision opportunities.

Our first result pertains to pure coordination stage games — games in which payoffs of all players are identical (up to an affine transformation). The importance of this class of games is emphasized by Marschak and Radner (1972) who try to understand the nature of team problems. We show that if players are sufficiently patient or, equivalently, if players can revise sufficiently quickly, then for any $\epsilon > 0$ every Perfect equilibrium payoff comes within ϵ of the Pareto dominant payoff. This result starkly contrasts with the Folk Theorem, as it essentially rules out all the inefficient payoffs in the above example, particularly the inefficient static Nash equilibrium (hence the term, "anti-Folk Theorem.") This result suggests that teams will solve their coordination problems in asynchronously repeated games.

Additionally, for a fixed discount rate close to zero we can find a neighborhood in payoff space of the pure coordination game in which the "anti-Folk Theorem" holds. Unfortunately, the uniqueness result seems to be limited to stage games close (in the sense just described) to the pure coordination game. We give an example in which the multiplicity of PE persists for patient players if costs of miscoordination are highly asymmetric. However, we show that there is a neighborhood of discount rates that support the Pareto dominant outcome as the unique PE payoff in a neighborhood of asynchronously repeated games. Again, this contrast with the standard model since inefficient Nash equilibrium payoffs of the stage game are not sustained in these neighborhoods of repeated games and discount rates.

Finally, we establish a "continuity result" for a one parameter family of renewal games. The standard repeated model is determined by one extremal value of the parameter, while an asynchronously repeated game is determined by the other. At issue is whether the "anti-Folk Theorem" represents a discontinuity when the standard model is perturbed. We show that the set of PE payoffs shrinks continuously from the set of Folk Theorem payoffs to the Pareto dominant singleton as the parameter varies between the standard and the asynchronous models, resp.

We emphasize that models of asynchronous interaction are not new. In differing contexts, Gale (1995), Rubinstein and Wolinsky (1995), Morris (1995), Perry and Reny (1993), Maskin and Tirole (1988), Farrell and Saloner (1985), and our previous work (some with Rafael Rob),[2], as well as others have studied consequences of asynchronous choice. In Section 6 we relate these to the present paper, and we look at other literature which contrasts with the Folk Theorem for repeated games.

To be clear, the present analysis is intended to be more suggestive than definitive. The results pertaining to the pure coordination games are essentially equivalent to unimprovability results in single agent dynamic programming problems (see, e.g., Howard (1960)).

[2]These include Matsui and Rob (1992), Lagunoff and Matsui (1995), and Lagunoff (1995).

The reason is that, unlike standard repeated games, asynchronously repeated games of pure coordination are identical to single person decision problems with perfect recall. It is the framework of the analysis rather than mathematics that distinguishes the present paper from preceding papers on single person decision problems. While a complete characterization of equilibrium payoffs in renewal games is preferable, our purpose here is to give some support for the study of a more general class of repeated interactions and to suggest that such an analysis will yield, at times strikingly, different results.

The paper proceeds as follows. Section 2 describes the model and defines the asynchronously repeated game and the equilibrium concept. Section 3 states and proves the uniqueness results for games of pure coordination. Section 4 discusses games of impure coordination. We show how the results may be sensitive to asymmetries in the costs of miscoordination. We provide a genericity result which describes the sense in which the results are robust to perturbations in the pure coordination structure. We also show that there are discount factors under which the "anti-Folk Theorem" holds for a more general class of coordination games. Section 5 gives the continuity result, and Section 6 discusses related literature.

2 A Model of Asynchronously Repeated Interaction

2.1 Stage Game

Let $G = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ denote a normal form stage game where I is the finite set of players, S_i $(i \in I)$ is the finite set of actions for player i, and $u_i : S \equiv \times_{i\in I} S_i \to \Re$ is the payoff function for player $i \in I$. Without loss of generality, assume that $S_i \cap S_j = \emptyset$ for all $i \neq j$. We will call an element of $s = (s_1, \ldots, s_{|I|}) \in S$ a *behavior profile* (or simply "profile"). Given some $\hat{s}_i \in S_i$, let $s\backslash\hat{s}_i = (s_1, \ldots, s_{i-1}, \hat{s}_i, s_{i+1}, \ldots, s_{|I|})$. The tuple of payoff functions is denoted by $u = (u_i)_{i\in I}$. A mixed strategy for i will be denoted by σ_i and has the standard properties: $\sigma_i : S_i \to [0, 1]$ and $\sum_{s_i} \sigma_i(s_i) = 1$. A *mixed profile* is given by $\sigma = (\sigma_i)_{i\in I}$. Finally, a game G is a *coordination game* if its Nash equilibria are Pareto ranked and there is some Nash equilibrium that strictly Pareto dominates every other profile of the game.

2.2 Renewal Games and Asynchronously Repeated Games

In this Section, we introduce a framework that encompasses a wide variety of repeated strategic environments. Consider a continuous repetition of a stage game G. After the first decision node, which occurs for all players at time zero, all players' decision points are determined by a semi-Markov process with finitely many states.[3] In the following, revision nodes refer to the decision nodes other than the first one at time zero.

A semi-Markov process is a stochastic process which makes transitions from state to state in accordance with a Markov chain, but in which the amount of time spent in each state before a transition occurs is random and follows a renewal process. For the sake of convenience, we separate the process into two parts, a renewal process and a Markov chain. Formally, let X_1, $X_2, \cdots$ be an infinite sequence of i.i.d. nonnegative random variables which follow a (marginal) probability measure ν with $E_\nu(X_1) < \infty$ and $\nu(X_1 > \epsilon) > \delta$ for some $\epsilon, \delta > 0$. It is also assumed that $\nu(X_1 = 0) = 0$ so that the orderliness condition for the renewals is guaranteed. Then let $T_0 = 0$ and $T_k = T_{k-1} + X_k = X_1 + \cdots + X_k$ $(k = 1, 2, \cdots)$. T_k is the time elapsed before the kth revision point. At each decision point a state ω is determined from a finite set Ω according to a Markov process $\{Y_k\}_{k=1}^{\infty} \in \Omega^\infty$ where $Y_k = \omega$ $(\omega \in \Omega)$ implies that state ω is reached at time T_k. We denote $p_{\omega\omega'} = Pr(Y_{k+1} = \omega' \mid Y_k = \omega)$ for $\omega, \omega' \in \Omega$. Let $\Omega_i \subseteq \Omega$ denote the nonempty set of states in which player i has a decision node. Let $\Omega_0 \subseteq \Omega$ be the set of "inertial" states in which no player has a decision node. By definition, $\Omega_0 = \Omega - (\cup_{i \in I} \Omega_i)$. We write $\vec{\Omega} = (\Omega_0, (\Omega_i)_{i \in I})$. We assume that the initial state, denoted by $\omega(0) \in \Omega$, is never reached again. By definition, $\omega(0) \in \Omega_i$ for all $i \in I$. Note that $\Omega_i \cap \Omega_j$“$\{\omega(0)\}$ $(i \neq j)$ may or may not be empty. To summarize, the renewal process, ν, determines when the decision nodes (the "jumps") occur, while the Markov transition, p, determines who moves at each node.

Using this semi-Markov process, a typical play of the game is described as follows. In the beginning, a strategy profile s^0 is chosen. Deterministically or stochastically, the first revision time is reached at time T_1. Suppose that ω is chosen by the Markov process, and let $I(\omega) = \{i \in I : \omega \in \Omega_i\}$ denoting the players who can move if nature chooses ω. If player i $(i \in I(\omega))$ takes s_i^1, then the strategy profile changes to $s^1 = s^0 \backslash (s_i^1)_{i \in I(\omega)}$. That is, each time there is a renewal and revision, only the corresponding coordinate(s) of the previous strategy profile is replaced by the revised one, while other coordinates remain unchanged. If we define $(s^k)_{k \geq 0}$ this way, i.e., s^k $(k = 1, 2, \cdots)$ is the strategy profile between T_k and T_{k+1}, then the flow payoff is realized and the discounted payoff for player $i \in I$ will be given by

$$r \sum_{k=0}^{\infty} \int_{T_k}^{T_{k+1}} e^{-r\tau} u_i(s^k) d\tau. \tag{1}$$

Figure 2 illustrates the process for a two-person game.

Definition 1 A *renewal game* is a tuple

$$\Gamma = \langle G,\ \nu, \vec{\Omega},\ (p_{\omega\omega'})_{\omega,\omega' \in \Omega},\ r \rangle,$$

[3] We can formulate the problem in such a way that the first action profile is chosen by nature as in models of evolution. It will be clear that the following description and results will not be altered by specification of choice at time zero.

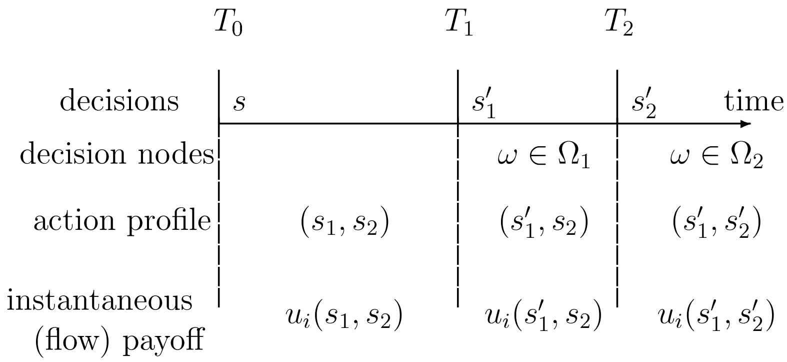

Figure 2: The play of the game

where $r > 0$ is a common discount rate, and for all $\omega \in \Omega$ and all $i \in I$, there exists a chain of states, $\omega^0, \omega^1, \omega^2, \ldots, \omega^M$ ($M < |\Omega|$), with $\omega = \omega^0$ and $\omega^M \in \Omega_i$ such that $p_{\omega^{m-1}\omega^m} > 0$ ($m = 1, \cdots, M$) (from any state, every player obtains a revision node).

Since the number of states is finite, renewal games have the property that from any state, every player obtains revision nodes infinitely many times, and the expected time interval between two revision nodes is finite. Standard discounted repeated games are renewal games which may be described in several ways. One straightforward way is: $\Omega = \Omega_i, \ \forall i \in I$, and $\nu(X_1 = 1) = 1$. However, we wish to specialize further to only those renewal games in which choice is asynchronous.

Definition 2 An *asynchronously repeated game* is a renewal game in which $\Omega_i \cap \Omega_j \setminus \{\omega(0)\} = \emptyset, \ \forall i \neq j$

In asynchronously repeated games, no two individuals have simultaneous revision opportunities. When $\omega \in \Omega_i$, we will write $i(\omega) = i$. Some examples are:

Example 1 (alternating move game). Let $I = \{1, 2\}$, $\Omega = I$, $X_1 \equiv 1$, and $p_{ij} = 1$ if $i \neq j$. Then the decision points are deterministic, and players' revision nodes alternate.

An example of an alternating move game of pure coordination is a situation in which two firms in the same product group desire a uniform accounting standard to simplify their consolidation work. However, they have different closing dates due to the nature of their business, which makes the decision points alternate. Another example is one in which two offices of a company are located in New York and Tokyo, respectively, so that their business hours do not overlap.

Example 2 (Poisson revision process). Let $\Omega = I$. And let X_1 follow an exponential distribution with parameter $\lambda > 0$, i.e., $\nu(X_1 < x) = 1 - e^{-\lambda x}$, and let $p_{ij} = p_j$ for all $i, j \in I$. Then players' revision points are independent of each other, and player i's decision points ($i \in I$) follow a Poisson process almost surely with parameter λp_i.

An example of this type is a replacement process, common in evolutive models, in which a player is defined as a lineage rather than a single individual entity. A son inherits his father's position only after the father's death, at which time the son can take his own action and commit to it through the rest of his life.

Example 3 (ϵ-approximation of the standard repeated game). Let $I = \{1, 2\}$, and

$$\Omega = \overbrace{\{\omega_0^1, \ldots, \omega_0^M\}}^{\Omega_0} \cup \overbrace{\{\omega_1^1, \omega_1^2\}}^{\Omega_1} \cup \overbrace{\{\omega_2^1, \omega_2^2\}}^{\Omega_2}$$

Then assume that Y_k's follow the process illlustrated in Figure 3 below. In the figure, the process proceeds through the inertial states until ω_0^M. At that time the process moves to either player 1's or player 2's decision node with probability 1/2 each. Let $X_1 \equiv 1/(M+2)$. Then if M is sufficiently large, the process approximates the standard repeated game in the sense that each player has a revision opportunity once in a unit of time, and that the two players' decision nodes are very close in timing.

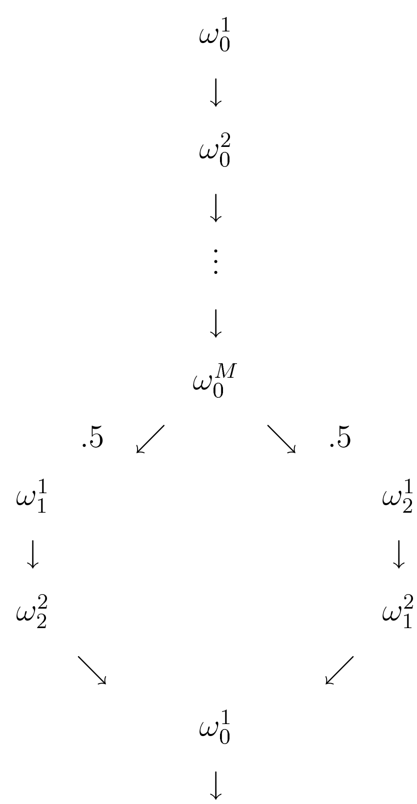

Figure 3: An ϵ-approximation of the standard repeated game.

2.3 Recursive Structure

One additional assumption that we will make will be to restrict the class of behavior strategies that individuals use. We assume that individuals only condition on the *sequences* of decision points and the actions taken at these points rather than on the time interval between them. This assumption does not restrict the strategy space at all if the renewal process is deterministic (the Markov process can be stochastic) as with Examples 1 and 3 in the previous section, and it significantly reduces the notational burden. It also turns out that it in no way alters main results of this paper.[4]

Restricting conditioning events to the "jumps" rather than on time intervals between jumps allows a straightforward recursive representation of individuals' continuation values in the asynchronous model. To formulate this, let $s(t) = (s_1(t), \ldots s_{|I|}(t))$ denote the behavior profile at time t, and let $N_t \equiv \inf\{k : T_k > t\}$ denote the number of renewals up to time t. Due to the assumption on X_k's, $N_t < \infty$ holds almost surely. Then define the space of histories H such that a history $h^t \in H$ is given by $h^t = (y^t, s^t)$ wherever N_t is finite, and

$$y^t = \{Y_k\}_{k \leq N_t},$$

and

$$s^t = \{s(T_k)\}_{k \leq N_t}.$$

The null history is denoted by h^0. Since at most one player switches his action at a time after the initial profile we write $s^t = (s(h^0), s_{Y_1}, \cdots, s_{Y_{N_t}})$ whenever convenient. We let $s(h)$ and $\omega(h)$ denote the current behavior and state at history h, resp., and let $i(h)$ denote the last player whose decision node was reached. We also define $h^{t-} = (y^t, \{s(T_k)\}_{k<N_t})$ so that h^{t-} includes the same information as history h^t except for the behavior profile at time t which may include a new decision by a player. We write H^- for the set of all such *conditioning histories* and denote an element by h^-.

A *strategy* for player i is a history contingent action given by the function $f_i : H^- \to \Delta(S_i)$. Although this formulation appears to require that i is unable to condition on the current behavior profile, recall that decisions are only made at the "jumps" in the renewal process.

Given a strategy profile f, the play of the game proceeds as follows. At time zero, all the players in I simultaneously take actions, $f_i(h^{0-})$ for $i \in I$. Suppose $T_1 = t_1 = x_1$ and $Y_1 = y_1 \in \Omega_i$. If $i = 0$, nothing changes except the Markov "clock." If $i \in I$, then $h^{t_1-} = (f(h^{0-}); y_1)$, player i's revision node is reached at time t_1, and he takes action $f_i(h^{t_1-})$. History at time t_1 becomes $h^{t_1} = (y_1; f(h^{0-}), f(h^{0-})\backslash f_i(h^{t_1-}))$. Given a strategy profile f and $h \in H^t$ (resp. $h^- \in H^{t-}$), $\{\tilde{s}(f|h)(\tau)\}_{\tau>t}$ (resp. $\{\tilde{s}(f|h^-)(\tau)\}_{\tau\geq t}$) denotes an induced (stochastic) path of action profile after h (resp. h^-).

[4]The interested reader can obtain a note by Lagunoff and Matsui (1995) which proves Theorems 1 and 2 for the case of full-fledged behavior strategies.

Given a history $h^t \in H$ and a strategy profile $f = (f_i)_{i \in I}$, we define the conditional discounted expected payoff to player i at time t by

$$V_i(f|h^t) = r \int_t^\infty e^{-r(\tau - t)} E\left[u_i\left(\tilde{s}(f|h^t)(\tau)\right)\right] d\tau,$$

where $E[\cdot]$ is the expectation operator. A strategy profile f^* is called a *perfect equilibrium (PE)* if for each $i \in I$, f_i^* is a best response to $(f_j^*)_{j \neq i}$ after every history h^t, i.e.,

$$V_i(f^*|h^t) \geq V_i(f^* \backslash f_i|h^t)$$

for any of player i's strategies f_i.

One immediate result in the asynchronously repeated game is that for almost all histories, mixed strategies will not be used at any revision opportunity. Hence, the mixed strategy minimax payoff which is always an equilibrium of the stage coordination game is not the benchmark here necessarily. Hereafter, we will often denote a pure strategy by $f_i(h^-)$ as well as a mixed strategy.

Given a history $h = h^t \in H$, let $h^t \circ (\omega; s_j)$ denote the concatenated history in which, after h^t, the next state $\omega \in \Omega_j$ is reached, at which player j takes s_j. Given $h^t \in H$, $h^t \circ \omega \in H^-$ is a path such that after h^t, state ω is reached (without specifying the revised action). Using this expression, the value after $h \circ \omega \in H^-$ is given by

$$V_i(f|h \circ \omega) \equiv V_i(f|h \circ (\omega; f_{i(\omega)}(h \circ \omega))).$$

The analysis will make extensive use of the following recursive formulation. The continuation value to i induced by f after history $h^t \in H$ with $\omega(h^t) = \omega$ may be expressed as

$$V_i(f|h^t) = (1 - \sum_{\omega' \in \Omega} \theta_{\omega\omega'}) u_i(s(h^t)) + \sum_{\omega' \in \Omega} \theta_{\omega\omega'} V_i(f|\ h^t \circ \omega'). \tag{2}$$

where

$$\begin{aligned} \theta_{\omega\omega'} &\equiv p_{\omega\omega'} \int_0^\infty e^{-rt} d\nu(t) \\ &= \text{expected discounted probability that } \omega' \\ &\quad\ \text{is the first state reached from } \omega. \end{aligned} \tag{3}$$

2.4 Existence

The following is a standard proof for the existence of perfect equilibria. What is proven here is actually the existence of so called Markov perfect equilibrium.

Theorem 0 *For any asynchronously repeated game $\Gamma = \langle G, \nu, \vec{\Omega}, p, r \rangle$ there exists at least one perfect equilibrium.*

Proof Partition H^- into $\wp = \{H^-_{\omega s}\}_{s\in S,\omega\in\Omega}$ such that

$$H^-_{\omega s} = \{h^- \in H^- | \omega(h^-) = \omega, s(h^-) = s\}, \forall \omega \in \Omega, \forall s \in S.$$

Observe that $\wp$ constitutes the "payoff relevent" set of states. Suppose that each player takes a $\sigma(\wp)$-measurable behavior strategy where $\sigma(\wp)$ is the σ-algebra generated by $\wp$. Then the play of the game follows a Markov process, and we can represent a strategy of player i by the "Markovian" function $\psi_i \in [\Delta(S_i)]^{\wp}$. The strategy represented by ψ_i is denoted by f_{ψ_i}.

For each $i \in I$ let $BR^i = \{BR^i_{\omega s}\}_{\omega\in\Omega,s\in S}$ satisfy

$$BR^i_{\omega(h)s}(\psi) = \arg\max_{f_{\psi_i}} V(f_\psi \backslash f_{\psi_i} |\, h) \tag{4}$$

Letting $BR = (BR_i)_{i\in I}$, equation (4) defines an upper hemicontinuous correspondence $BR : \times_i[\Delta(S_i)]^{\wp} \to \times_i[\Delta(S_i)]^{\wp}$ where $\times_i[\Delta(S_i)]^{\wp}$ is compact and convex. Therefore, by Kakutani's fixed point theorem, there exists ψ such that $\psi \in BR(\psi)$ holds. Standard arguments show that the corresponding strategy f_{ψ_i} is a best response to $(f_{\psi_j})_{j\neq i}$ within the class of *all* strategies after any history h (or, more precisely, after any h^-) since all j, $(j \neq i)$ only vary their behavior over "states" $\omega s \in H^-_{\omega s}$. Hence, f_ψ is a perfect equilibrium. □

3 Pure Coordination

A game G is a *pure coordination game* if $u_i = u_j = u$ for all i and j.[5] Let s^* denote the profile that gives each player his highest payoff u^*. The first part of our main result states that, independently of the discount rate r, the unique continuation value after any history in which the current profile is s^* must give the optimal payoff $u^*_i = u_i(s^*)$ for all $i \in I$.

Theorem 1 *Given any asynchronously repeated game $\Gamma = \langle G, \nu, \vec{\Omega}, (p_{\omega\omega'}), r\rangle$ in which G is a pure coordination game, for any perfect equilibrium f of Γ, for any history $h \in H$ with $s(h) = s^*$, and for all $i \in I$, $V_i(f|h) = u^*_i$ holds.*

proof of Theorem 1 Observe first that due to the identical payoffs, we can drop the subscript i on u_i and continuation values V_i. Fix a perfect equilibrium f. Recall that $s(h)$ is the current action profile given history $h \in H^t$. Define $\underline{V} = \inf_{\{h:\, s(h)=s^*\}} V(f|h)$. This is the infimum value of the game when the current behavior profile is s^*. Since the payoff space is bounded from below, this infimum exists. Fix $\epsilon > 0$. Then there exists $h = h_\epsilon \in H^t$ for some $t > 0$ such that $s(h) = s^*$ and

$$\underline{V} > V(f|h) - \epsilon. \tag{5}$$

[5]This can be weakened so that we require only that $u_i = \alpha u_j + \beta$ for $\alpha > 0$ and $\beta \in \Re$.

Recall from (2) the continuation value after h which is given by

$$V(f|h) = (1 - \sum_{\omega \in \Omega} \theta_{\omega(h)\omega})u^* \ + \sum_{\omega \in \Omega} \theta_{\omega(h)\omega} V(f|\ h \circ \omega). \tag{6}$$

If $\omega \in \Omega_0$, then $s(h \circ \omega)) = s^*$, and therefore, $V(f|h \circ \omega) \geq \underline{V}$. Observe also that since f is a perfect equilibrium strategy profile, it must be the case that for all $i \in I$ and all $\omega \in \Omega_i$,

$$V(f|\ h \circ\ \omega) \geq V(f|\ h \circ (\omega; s_i^*)\ ) \geq \underline{V}$$

where the second inequality holds due to the definition of $\underline{V}$. Substituting these inequalities into (6) and using (5), we obtain

$$\underline{V} > (1 - \sum_{\omega' \in \Omega} \theta_{\omega(h)\omega'})u^* + \sum_{\omega' \in \Omega} \theta_{\omega(h)\omega'} \underline{V} - \epsilon \tag{7}$$

Since $\sum_{\omega \in \Omega} \theta_{\omega(h)\omega} = \int_0^\infty e^{-r\tau} d\nu(\tau)$, (7) implies

$$\underline{V} > u^* - \frac{\epsilon}{1 - \int_0^\infty e^{-r\tau} d\nu(\tau)}.$$

Since ϵ is arbitrary and independent of r, $\underline{V} \geq u^*$ holds. □

The second part of the main result is that any continuation value in a perfect equilibrium is arbitrarily close to the Pareto efficient value for sufficiently patient players.

Theorem 2 *Given any asynchronously repeated game* $\Gamma = \langle G, \nu, \vec{\Omega}, (p_{\omega\omega'}), r \rangle$ *in which* G *is a pure coordination game, and given any* $\epsilon > 0$*, there exists* $\bar{r} > 0$ *such that if* $r \in (0, \bar{r})$*, then for all perfect equilibrium* f *of* Γ*, for all histories* $h \in H$*, and for all* $i \in I$*,* $V_i(f|\ h) > u_i^* - \epsilon$ *holds.*

proof of Theorem 2 The proof will use a backward induction argument to establish that after any history the equilibrium continuation must be within ϵ of the maximal payoff u^*. As in the first theorem, we drop the subscript i. Fix $\epsilon > 0$. For each $k = 0, 1, \cdots, |I|$, define

$$S^k = \{s \in S : \ \ |\{i \in I | s_i \neq s_i^*\}| = k\}.$$

S^k is the set of behavior profiles of the stage game that require k individuals to change their actions to get to profile s^*. By this definition, $S^0 = \{s^*\}$.

Fix a perfect equilibrium f. For each $s \in S$ and each $\omega \in \Omega$, we let $\underline{V}_\omega^s = \inf_{s(h)=s, \omega(h)=\omega} V(f|h)$, and $\underline{V}^s = \inf_{s(h)=s} V(f|h)$. Note that $\underline{V}^s = \min_{\omega \in \Omega} \underline{V}_\omega^s$.

The proof will proceed as follows: recall from Theorem 1 that after a history h with $s(h) \in S^0$, i.e., $s(h) = s^*$, the continuation value $V(f|h)$ is u^* for any perfect equilibrium f.

Given ϵ, the first step is to find a bound $\bar{r}$ such that if $r < \bar{r}$ then for all $k = 1, 2, \cdots, |I|$, there is some profile $s' \in S^{k-1}$ such that after any h with $s(h) \in S^k$, the continuation value satisfies

$$V(f|h) > \underline{V}^{s'} - \frac{\epsilon}{|I|}, \tag{8}$$

for any perfect equilibrium f. This turns out to be the crucial step. The choice of $\bar{r}$ must be determined independently of f. The next step is to verify from backward induction on S^k that (8) holds for all such $s' \in S^{k-1}$. Finally, we verify that $0 < r < \bar{r}$ implies that for all history $h \in H$, $V(f|h) > u^* - \epsilon$. Thus, the proof will be completed.

First, we determine the value of $\bar{r}$ independently of f. Recalling that the definition of $\theta_{\omega\omega'}$ is a function of r, define the bound $\bar{r}$ so as to satisfy

$$\min_{\substack{(\omega_0,\omega_1,\cdots,\omega_N)\in\Omega^{N+1}\\ 2\le N\le|\Omega|}} \left\{ \frac{\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0,\cdots,\omega_N)} : \ \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N} > 0 \right\} \ > \ 1 - \frac{\epsilon}{2|I|}\frac{1}{u^* - \min_s u(s)}, \tag{9}$$

where

$$\begin{aligned} \theta(\omega_0,\cdots,\omega_N) &= 1 - \sum_{\ell=1}^{N} \Pi_{n=1}^{\ell-1}\theta_{\omega_{n-1}\omega_n} \sum_{\omega'\neq\omega_\ell} \theta_{\omega_{\ell-1}\omega'} \\ &= 1 - \sum_{\omega'\neq\omega_1}\theta_{\omega_0\omega'} - \theta_{\omega_0\omega_1}\sum_{\omega'\neq\omega_2}\theta_{\omega_1\omega'} - \cdots - \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-2}\omega_{N-1}}\sum_{\omega'\neq\omega_N}\theta_{\omega_{N-1}\omega'}. \end{aligned}$$

Such an $\bar{r}$ can be found since both $\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}$ and $\theta(\cdot)$ converge to $p_{\omega_0\omega_1}\cdots p_{\omega_{N-1}\omega_N} > 0$ as r goes to zero. Also, let $\underline{\theta} = \min\{\theta(\omega_0,\cdots,\omega_N) : \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N} > 0\}$. This choice of $\bar{r}$ will be clarified in what follows.

Now, fix $k = 1, 2, \cdots, |I|$ and fix a profile $\hat{s} \in S^k$. By the definition of $\underline{V}^{\hat{s}}_{\omega}$ $(\omega \in \Omega)$, there exists $h_\omega \in H$ such that $s(h_\omega) = \hat{s}$, $\omega(h_\omega) = \omega$, and

$$\underline{V}^{\hat{s}}_{\omega} > V(f|h_\omega) - \frac{\epsilon\underline{\theta}}{2|I||\Omega|}. \tag{10}$$

We have

$$V(f|h_\omega) = (1 - \sum_{\omega'\in\Omega}\theta_{\omega\omega'})u(\hat{s}) \ + \ \sum_{\omega'\in\Omega}\theta_{\omega\omega'}V(f|\ h_\omega\circ\omega'). \tag{11}$$

Since f is a perfect equilibrium, for each $\omega' \in \Omega_j$ with $j \in I$,

$$V(f|\ h_\omega\circ\omega') \ge V(f|\ h_\omega\circ(\omega';\hat{s}_j)) \ge \underline{V}^{\hat{s}}_{\omega'}, \tag{12}$$

where the second inequality holds by the definition of $\underline{V}^{\hat{s}}_{\omega}$. Similarly, if $\hat{s}_j \neq s^*_j$, then

$$V(f|\ h_\omega\circ\omega') \ge V(f|\ h_\omega\circ(\omega';s^*_j)) \ge \underline{V}^{\hat{s}\backslash s^*_j}_{\omega'}, \tag{13}$$

Also, by the definition of $\underline{V}^{\hat{s}}_{\omega'}$ for $\omega \in \Omega_0$, we have

$$V(f|\ h_\omega \circ \omega') \geq \underline{V}^{\hat{s}}_{\omega'},\ \forall \omega' \in \Omega_0. \tag{14}$$

Substituting (12) and (14) into (11), we obtain

$$V(f|h_\omega) \geq (1 - \sum_{\omega' \in \Omega} \theta_{\omega\omega'})u(\hat{s})\ +\ \sum_{\omega' \in \Omega} \theta_{\omega\omega'}\underline{V}^{\hat{s}}_{\omega'}. \tag{15}$$

Inequalities (10) and (15) imply

$$\underline{V}^{\hat{s}}_{\omega} \geq (1 - \sum_{\omega' \in \Omega} \theta_{\omega\omega'})u(\hat{s})\ +\ \sum_{\omega' \in \Omega} \theta_{\omega\omega'}\underline{V}^{\hat{s}}_{\omega'} - \frac{\epsilon\underline{\theta}}{2|I||\Omega|}. \tag{16}$$

Also, if $s_{i(\omega^*)}(h) \neq s^*_{i(\omega^*)}$ holds for $\omega^* \in \Omega$, then

$$\underline{V}^{\hat{s}}_{\omega} \geq (1 - \sum_{\omega' \in \Omega} \theta_{\omega\omega'})u(\hat{s})\ +\ \sum_{\omega' \neq \omega^*} \theta_{\omega\omega'}\underline{V}^{\hat{s}}_{\omega'}\ +\ \theta_{\omega\omega^*}\underline{V}^{s\backslash s^*_{i(\omega^*)}}_{\omega^*} - \frac{\epsilon\underline{\theta}}{2|I||\Omega|}. \tag{17}$$

By definition, $\underline{V}^{\hat{s}}_{\omega_0} = \underline{V}^{\hat{s}}$ for some ω_0. Take such ω_0. There exists $i \in I$ such that $\hat{s}_i \neq s^*_i$. Then there exists a chain $\omega_0, \omega_1, \cdots, \omega_N$ in Ω ($N < |\Omega|$) with $\omega_N \in \Omega_i$ such that $p_{\omega_{n-1}\omega_n} > 0$ for all $n = 1, \cdots, N$. Sequentially substituting ω_n ($n = 0, 1, \cdots, N-1$ in place of ω in (16), and substituting ω_N in place of ω^* in (17) we obtain

$$\begin{aligned} \underline{V}^{\hat{s}} \quad > \quad & (1 - \textstyle\sum_{\omega' \in \Omega} \theta_{\omega_0\omega'})u(\hat{s}) + \sum_{\omega' \neq \omega_1} \theta_{\omega_0\omega'}\underline{V}^{\hat{s}}_{\omega'} \\ & +\theta_{\omega_0\omega_1}\left[(1 - \textstyle\sum_{\omega' \in \Omega} \theta_{\omega_1\omega'})u(\hat{s}) + \sum_{\omega' \neq \omega_2} \theta_{\omega_1\omega'}\underline{V}^{\hat{s}}_{\omega'}\right] \\ & + \cdots \\ & +\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-2}\omega_{N-1}}\left[(1 - \textstyle\sum_{\omega' \in \Omega} \theta_{\omega_N\omega'})u(\hat{s}) + \sum_{\omega' \neq \omega_N} \theta_{\omega_{N-1}\omega'}\underline{V}^{\hat{s}}_{\omega'}\right] \\ & +\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}\underline{V}^{s\backslash s^*_i}_{\omega_N} - \frac{\epsilon\underline{\theta}}{2|I|}. \end{aligned} \tag{18}$$

Using $\underline{V}^{\hat{s}}_{\omega} \geq \underline{V}^{\hat{s}}$ ($\forall \omega \in \Omega$), we have

$$\underline{V}^{\hat{s}} > \left[1 - \frac{\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0, \cdots, \omega_N)}\right]u(\hat{s})\ +\ \frac{\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0, \cdots, \omega_N)}\underline{V}^{s\backslash s^*_i} - \frac{\epsilon}{2|I|} \tag{19}$$

Since $\bar{r}$ was chosen to satisfy (9), if $r < \bar{r}$, then

$$\min_s u(s) > \left[1 - \frac{\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0, \cdots, \omega_N)}\right]u^*\ +\ \frac{\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0, \cdots, \omega_N)}\min_s u(s) - \frac{\epsilon}{2|I|} \tag{20}$$

Comparing (19) with (20), for any $r < \bar{r}$ then

$$\underline{V}^{\hat{s}} > \underline{V}^{s\backslash s^*_i} - \frac{\epsilon}{|I|}$$

holds where, by construction, $s \backslash s_i^* \in S^{k-1}$.

Since S^0 is the singleton set, $\{s^*\}$, we have $\underline{V}^s > u^* - \frac{\epsilon}{|I|}$ for all $s \in S^1$. It follows that $\underline{V}^s > \underline{V}^{s \backslash s_i^*} - \frac{\epsilon}{|I|}$ holds for all $s \in S^2$, and hence it holds for all $s \in S^k$. Therefore, by recursive substitution,

$$\underline{V}^{\hat{s}} > u^* - \epsilon, \ \forall \hat{s}$$

□

A Remark on the Logic of the Result. Before proceeding to the next section, we emphasize that since Theorems 1 and 2 holds for any asynchronously repeated game of pure coordination, what matters for such games is the fact that choice is asynchronous rather than the specific form of asynchronous choice. Given the pure coordination structure, the logic utilizes standard arguments from dynamic programming. Figure 4 below is a "false counterexample" to Theorem 1. For exposition, suppose that Γ denotes an alternating move, asynchronously repeated version of the pure coordination game G_1.

$$\begin{array}{ccccccc}
s^* & & s^* & & & s^* & \\
\downarrow & & \downarrow & & & \downarrow & \\
\vdots & & \vdots & & & \vdots & \\
\downarrow & & \downarrow & & & \downarrow & \\
s^* & & s^* & & & s^* & \\
\downarrow & & \downarrow & & & \downarrow & \\
(\bar{s}_1, s_2^*) & & (s_1^*, \bar{s}_2) & & & (\bar{s}_1, s_2^*) & \\
\downarrow & & \downarrow & & & \downarrow & \\
\bar{s}^1 & \overset{\text{deviation}}{\rightarrow} & \bar{s}^2 & \overset{\text{deviation}}{\rightarrow} & \cdots \rightarrow & \bar{s}^k & \rightarrow \\
\downarrow & & \downarrow & & & \downarrow & \\
\vdots & & \vdots & & & \vdots & \\
\downarrow & & \downarrow & & & \downarrow & \\
\bar{s}^{L_1} & & \bar{s}^{L_2} & & & \bar{s}^{L_k} & \\
\downarrow & & \downarrow & & & \downarrow & \\
(s_1^*, \bar{s}_2) & & (\bar{s}_2, s_1^*) & & & (s_1^*, \bar{s}_2) & \\
 & & & & & & \\
\textbf{Layer 1} & & \textbf{Layer 2} & & & \textbf{Layer k} &
\end{array}$$

Figure 4: "false" equilibrium paths in an alternating move game.

In Figure 4 the first layer describes the candidate "equilibrium" path. Subsequent layers are "equilibrium continuations" after deviations. In "equilibrium" the players remain in s^* for a fixed time before player 1 chooses $\bar{s}_1$, thus signalling a switch to the inferior equilibrium $\bar{s}$ which lasts for L_1 periods. Initially player 1, who is prescribed to be the first to switch to $\bar{s}_i$, may choose to deviate and remain in s^*. The second layer is a "punishment path" which prescribes a longer time length of L_2 periods spent in $\bar{s}$. Each successive layer lengthens the time in $\bar{s}$. The infimum $\underline{V}$ defined in the proof is approximated by the average of the payoffs in the kth layer for k large. We argue that there is some k large enough such that a deviation — remaining in s^* — is profitable. The reason is simple. The average payoff in layer k converges to 1 as $L_k \to \infty$ as $k \to \infty$. Therefore, the punishment for remaining in s^* in layer k is the additional $L_{k+1} - L_k$ periods which lowers the average payoff by $(n(L_{k+1} - L_k))/(n + L_{k+1})(n + L_k)$ where n is the number of periods spent in s^*. Clearly this cost approaches zero in k. By contrast the benefit to deviating is $2rE(X_1)$, which is independent of k. Once it is established that a player will deviate in the kth layer for k large enough, backward induction then destroys the equilibrium.

4 General Coordination Games

		2	
		s_2^*	$\bar{s}_2$
1	s_1^*	$u^*,\ u^*$	$\beta,\ \alpha$
	$\bar{s}_1$	$\alpha,\ \beta$	$1,\ 1$

G_2**:** $u^* \geq 2, \alpha, \beta < 1$

Figure 4

4.1 Example

Unfortunately, it turns out that uniqueness is not generally attainable in coordination games. Consider an alternating move game (given by Example 1 in the previous section) with its stage game given by G_2 above. G_2 is an imperfect coordination game since the costs of miscoordination are not identical. Using the full-dimensionality of this game, we construct a perfect equilibrium of which payoff is bounded away from the Pareto efficient payoff pair, (u^*, u^*). We construct such an equilibrium in the following way. First, we consider two phases, Phase I and Phase II. Phase I is divided into four subphases given by the following.

$$\overbrace{s^* \to \cdots \to s^*}^{k \text{ times}} \to \overbrace{(\bar{s}_1, s_2^*) \to \cdots \to (\bar{s}_1, s_2^*)}^{\ell \text{ times}} \to \quad \bar{s} \to \quad \overbrace{(s_1^*, \bar{s}_2) \to \cdots \to (s_1^*, \bar{s}_2)}^{m \text{ times}} \to$$

Phase I.A *Phase I.B* *Phase I.C* *Phase I.D*

This is the prescribed path in Phase I. After the last stage of Phase I.D, the system returns to the first stage of Phase I.A. In this construction, k, ℓ, and m satisfy the following:

$$\frac{u^*k + \alpha\ell + 1 + \beta m}{k + \ell + 1 + m} > \frac{u^*k + \beta\ell + 1 + \alpha m}{k + \ell + 1 + m} > 1, \tag{21}$$

and

$$\frac{u^* + \beta\ell + 1}{\ell + 2} < 1. \tag{22}$$

Such k, ℓ, and m exist. Indeed, we choose ℓ large enough to satisfy (22). Then choose m so that $\alpha\ell + \beta m$ is greater than $\beta\ell + \alpha m$, i.e., $\ell > m$ if $\alpha > \beta$, and vice versa. This will guarantee the first inequality of (21). Note that we cannot find such a m if the game is pure coordination, i.e., $\alpha = \beta$. Finally, take a sufficiently large k to satisfy the second inequality of (21).

Phase II is the same as Phase I except that $(\bar{s}_1, s_2^*)$ (resp. $(s_1^*, \bar{s}_2)$) is replaced by $(s_1^*, \bar{s}_2)$ (resp. $(\bar{s}_1, s_2^*)$). That is, Phase II is a mirror image of Phase I with respect to the players. The play of the game begins with the first stage of Phase I.A and stays in Phase I, following the above arrows, unless there is a deviation. If player 1 deviates, then the system moves to an appropriate subphase of Phase II. For example, if player 1 deviates in Phase I.B to take s_1^*, then the system goes to some state corresponding to player 2's move in Phase II.A.

If player 2 deviates, then we have the following transitions:

1. If player 2 deviates in Phase I.A, then player 2's prescribed action in the next move is to return to s_2^*, and player 1 will keep s_1^* until player 2 takes s_2^*. After player 2 returns, the system goes to the last stage of Phase I.A.

2. If player 2 deviates to take $\bar{s}_2$ earlier than prescribed in Phase I.B, then player 1 will keep $\bar{s}_1$ until player 2 takes s_2^*, and then the system moves to the first stage of Phase I.B. If player 2 deviates to keep taking s_2^* in the last stage of Phase I.B, then the system moves to the second last stage of Phase I.B.

3. If player 2 deviates to take s_2^* in Phase I.D, then the system moves to the last stage of Phase I.A. If player 2 deviates to keep taking $\bar{s}_2$ in the last stage of Phase I.D, then the system moves to the second last stage of Phase I.D.

Note that player 2 will have no revision point in Phase I.C. Prescribed actions and the transition in Phase II are the same as those in Phase I except that the roles of the players are reversed.

Now, we are in a position to check that incentive constraints are satisfied for a sufficiently small discount rate $r > 0$. If player 1 deviates in Phase I, his expected payoff converges to

$$\frac{u^* k + \beta \ell + 1 + \alpha m}{k + \ell + 1 + m} \tag{23}$$

as r goes to zero. On the other hand, his expected payoff in Phase I converges to

$$\frac{u^* k + \alpha \ell + 1 + \beta m}{k + \ell + 1 + m},$$

which exceeds (23). Thus, for a sufficiently small $r > 0$, player 1 has no incentive to deviate. To check player's incentive to deviate, we examine three cases indicated in the above construction.

1. Phase I.A: If player 2 deviates, she will get α for a while instead of u^*. Therefore, she has no incentive to deviate there. Even if she keeps $\bar{s}_2$, she will get only $\alpha < 1$.
2. Phase I.B: If player 2 deviates to take $\bar{s}_2$ earlier than prescribed, she will get 1 and then some extra β before the system reaches the stage where she deviated. Since the expected payoff along the equilibrium path exceeds one, and $\beta < 1$, player 2 has no incentive to deviate. In the last stage of Phase I.B, if she deviates, she will get $\beta < 1$ for two more periods, which does not increase her payoff.
3. Phase I.D: If player 2 deviates to take s_2^* earlier than prescribed, then she gets u^* for one period, β for ℓ periods, 1 for one period, and some α's before the system reaches the original stage where player 2 deviates. From (22), the expected average payoff before the system reaches the same stage is less than one. Thus, player 2 has no incentive to deviate. Finally, in the the last stage of Phase I.D, if she deviates, she will get $\alpha < 1$ for two more periods, which does not increase her payoff.

Hence, the strategy profile constructed above is a perfect equilibrium.

It should be noted that in a standard repeated game, we do not need this type of complicated construction since the strategy profile that prescribes $\bar{s}_i$ for player $i = 1, 2$ after any history is a subgame perfect equilibrium. On the other hand, it is shown that in an asynchronously repeated game, such a simple strategy does not constitute a subgame perfect equilibrium unless α is sufficiently larger than β.

4.2 Generic Payoffs and Optimality

Is the optimality result nongeneric in payoff space? It seems so if, for each game, we look at the set of equilibria as the discount rate approaches zero. If, on the other hand, we fix the discount sufficiently small, then for each pure coordination game, we always find an open neighborhood in payoff space such that in every game in the neighborhood, the Pareto efficient outcome is the unique PE outcome. The following theorem establishes it.

Theorem 3 *Given $\epsilon > 0$ and any asynchronously repeated game $\langle G, \nu, (p_{\omega\omega'}), r\rangle$ in which $G = (I, S, u)$ is a pure coordination game with u^* as the unique Pareto efficient outcome, there exists $\bar{r} > 0$ such that for any $r \in (0, \bar{r})$, there exists an open subset $\mathcal{U} \subset \Re^{\times_{i\in I} S_i}$ with $u \in \mathcal{U}$ such that in an asynchronously repeated game $\langle (I, S, u'), \nu, (p_{\omega\omega'}), r\rangle$ with $u' \in \mathcal{U}$, every continuation value in any perfect equilibrium is at least $u^* - \epsilon$.*

Theorem 3 and the example that uses stage game G_2 show that order of limits matters. The example with G_2 fixes the stage game and then varies r. By contrast, the hypothesis of Theorem 3 fixes r and then varies G. The former will surely be more familiar to those familiar with the Folk Theorem. The latter may be more useful if it is the discount rate r, rather than the stage payoffs, which is pinned down by exogenous data.

proof of Theorem 3 Take as given an asynchronously repeated game $\langle G, \nu, (p_{\omega\omega'}), r\rangle$ in which $G = (I, S, u)$ is a pure coordination game, and $s^* \in S$ is the unique Pareto efficient outcome. Also take $\epsilon > 0$ as given. Any affine transformation of u will give the same result in the following analysis. Analogously to the proof of Theorem 2 we set $\bar{r}$ so as to satisfy

$$\min_{\substack{(\omega_0,\omega_1,\cdots,\omega_N)\in\Omega^{N+1}\\ 2\le N\le|\Omega|-1}} \left\{ \frac{\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}}{\theta(\omega_0,\cdots,\omega_N)} : \ \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N} > 0 \right\} \ > \ 1 - \frac{\epsilon}{3|I|}\frac{1}{u^* - \min_s u(s)},$$

where, just as in (9),

$$\theta(\omega_0,\cdots,\omega_N) = 1 - \sum_{\omega'\neq\omega_1}\theta_{\omega_0\omega'} - \theta_{\omega_0\omega_1}\sum_{\omega'\neq\omega_2}\theta_{\omega_1\omega'} - \cdots - \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-2}\omega_{N-1}}\sum_{\omega'\neq\omega_N}\theta_{\omega_{N-1}\omega'}.$$

Such an $\bar{r}$ can be found since both $\theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}$ and θ converge to $p_{\omega_0\omega_1}\cdots p_{\omega_{N-1}\omega_N} > 0$ as $\bar{r}$ goes to zero. Fix $r \in (0, \bar{r})$. Again let $\underline{\theta} = \min \theta(\omega_0, \cdots, \omega_N)$.

Consider a neighborhood of u given by $\mathcal{U} = \{u' \in \Re^S | \ \forall s \in S, \ |u'_i(s) - u_i(s)| < \eta/4\}$ where η satisfies

$$0 < \eta < \frac{1}{2}\min\left\{u^* - u^{**}, \ \frac{\epsilon}{2|I||\Omega|}, \ \frac{\epsilon}{3(|I|+1)}[1 - \int_0^\infty e^{-r\tau} d\nu(\tau)]\right\},$$

where $u^{**} = \max_{s \neq s^*} u(s)$, i.e., the second highest payoff. In the following, we consider an asynchronously repeated game $\Gamma' = \langle (I, S, u'), \nu, \vec{\Omega}, (p_{\omega\omega'}), r \rangle$ with $u' \in \mathcal{U}$.

Now, fix a perfect equilibrium f of Γ'. For each $i \in I$, each $s \in S$, and each $\omega \in \Omega$, we let $\underline{V}^s_{i\omega} = \inf_{s(h)=s, \omega(h)=\omega} V_i(f|h)$, and $\underline{V}^s_i = \inf_{s(h)=s} V_i(f|h)$. Note that $\underline{V}^s_i = \min_{\omega \in \Omega} \underline{V}^s_{i\omega}$.

We first show the counterpart of Theorem 1. For any $\delta > 0$ there exists $h \in H$ such that $s(h) = s^*$ and

$$\underline{V}^{s^*}_i > V_i(f|h) - \delta, \ \forall i \in I. \tag{24}$$

The continuation value for player $i \in I$ after h is given by

$$V_i(f|h) = (1 - \sum_{\omega \in \Omega} \theta_{\omega(h)\omega}) u'_i(s^*) \ + \ \sum_{\omega \in \Omega} \theta_{\omega(h)\omega} V_i(f| \ h \circ \omega). \tag{25}$$

If $\omega(h) \in \Omega_0$, then $s(h \circ \omega) = s^*$ and therefore, $V_i(f|h \circ \omega) \geq \underline{V}^{s^*}_i$. Observe also that since f is a perfect equilibrium strategy profile, it must be the case that for all $\omega \in \Omega_j$ $(j \in I)$,

$$V_j(f| \ h \circ \omega) \geq V_j(f| \ h \circ (s^*_j; \ \omega)) \geq \underline{V}^{s^*}_j \tag{26}$$

where the second inequality holds due to the definition of $\underline{V}^{s^*}_j$. Since $|u(s) - u_i(s)| < \eta/4$, $|V_j(f|h) - V_i(f|h)| < \eta/2$ holds for all $i, j \in I$, all f and all $h \in H$, and therefore, (26) implies

$$V_i(f| \ h \circ \ \omega) \geq \underline{V}^{s^*}_i - \eta. \tag{27}$$

Substituting (27) into (25) and using (24), we obtain

$$\underline{V}^{s^*}_i > (1 - \sum_{\omega' \in \Omega} \theta_{\omega(h)\omega'}) u'_i(s^*) + \sum_{\omega' \in \Omega} \theta_{\omega(h)\omega'} \underline{V}^{s^*}_i - \delta - \eta \tag{28}$$

Inequality (28) implies

$$\underline{V}^{s^*}_i > u'_i(s^*) - \frac{\delta + \eta}{1 - \int_0^\infty e^{-r\tau} d\nu(\tau)}.$$

Since δ is arbitrary and independent of other variables, the definition of η implies $\underline{V}^{s^*}_i > u^* - \epsilon/(|I|+1)$.

This second part mirrors the proof of Theorem 2. We will show that for all $k = 1, 2, \cdots, |I|$, after h with $s(h) \in S^k$, its continuation value satisfies

$$V_i(f|h) > \underline{V}^{s'}_i - \frac{\epsilon}{|I|+1}, \tag{29}$$

for some $s' \in S^{k-1}$ if $r < \bar{r}$. Backward induction implies that (29) holds for all $s' \in S^{k-1}$ if $r < \bar{r}$. Once we show (29) for all s' and k's, we verify that $0 < r < \bar{r}$ implies that for all history $h \in H$, $V_i(f|h) > u'_i(s^*) - \epsilon$. Moreover, recall that the choice of $\bar{r}$ is independent of f. Thus, the proof will be completed.

Fix $k = 1, 2, \cdots, |I|$ and $\hat{s} \in S^k$. By the definition of $\underline{V}^{\hat{s}}_{i\omega}$ ($\omega \in \Omega$), there exists $h_\omega \in H$ such that $s(h_\omega) = \hat{s}$, $\omega(h_\omega) = \omega$, and

$$\underline{V}^{\hat{s}}_{i\omega} > V_i(f|h_\omega) - \frac{\epsilon\underline{\theta}}{3(|I|+1)|\Omega|}. \tag{30}$$

We have

$$V_i(f|h_\omega) = (1 - \sum_{\omega' \in \Omega} \theta_{\omega\omega'})u'_i(\hat{s}) \ + \ \sum_{\omega' \in \Omega} \theta_{\omega\omega'} V_i(f|\ h_\omega \circ \omega'). \tag{31}$$

Since f is a perfect equilibrium, for each $\omega' \in \Omega_j$ with $j \in I$,

$$V_j(f|\ h_\omega \circ \omega') \geq V_j(f|\ h_\omega \circ (\hat{s}_j; \omega) \geq \underline{V}^{\hat{s}}_{j\omega'}, \tag{32}$$

where the second inequality holds by the definition of $\underline{V}^{\hat{s}}_{j\omega}$. Similarly, if $\hat{s}_j \neq s^*_j$, then

$$V_j(f|\ h_\omega \circ \omega') \geq V_j(f|\ h_\omega \circ (s^*_j; \omega)) \geq \underline{V}^{\hat{s}``s^*_j}_{j\omega'}, \tag{33}$$

Also, by the definition of $\underline{V}^{\hat{s}}_{i\omega'}$ for each $\omega \in \Omega_0$, we have

$$V_i(f|\ h_\omega \circ \omega') \geq \underline{V}^{\hat{s}}_{i\omega'}, \ \forall \omega' \in \Omega_0. \tag{34}$$

Since $|u(s) - u_i(s)| < \eta/4$, then $|V_j(f|h) - V_i(f|h)| < \eta/2$ holds for all $i, j \in I$, all f, all s, and all $h \in H$, and therefore, (32) and (33) imply

$$V_i(f|\ h_\omega \circ \omega') \geq \underline{V}^{\hat{s}}_{i\omega'} - \eta, \tag{35}$$

and

$$V_i(f|\ h_\omega \circ \omega') \geq \underline{V}^{\hat{s}\backslash s^*_j}_{i\omega'} - \eta, \tag{36}$$

respectively. Substituting these inequalities into (31), we obtain

$$V_i(f|h_\omega) \geq (1 - \sum_{\omega' \in \Omega} \theta_{\omega\omega'})u'_i(\hat{s}) \ + \ \sum_{\omega' \in \Omega} \theta_{\omega\omega'} \underline{V}^{\hat{s}}_{i\omega'} - \eta. \tag{37}$$

Inequalities (30) and (37) imply

$$\underline{V}^{\hat{s}}_{i\omega} \geq (1 - \sum_{\omega' \in \Omega} \theta_{\omega\omega'})u'_i(\hat{s}) \ + \ \sum_{\omega' \in \Omega} \theta_{\omega\omega'} \underline{V}^{\hat{s}}_{i\omega'} - \frac{\epsilon\underline{\theta}}{3(|I|+1)|\Omega|} - \eta. \tag{38}$$

Since i was arbitrarily chosen, (38) holds for all $i \in I$. By definition, $\underline{V}^{\hat{s}}_{i\omega_0} = \underline{V}^{\hat{s}}_i$ for some ω_0. Take such ω_0. There exists $i \in I$ such that $\hat{s}_i \neq s^*_i$. Then there exists a chain $\omega_0, \omega_1, \ldots, \omega_N$ with $N < |\Omega|$ and $\omega_N \in \Omega_i$ such that $p_{\omega_{n-1}\omega_n} > 0$ for all $n = 1, \cdots, N$. Sequentially substituting ω_n $(n = 0, 1, \cdots, N$ in place of ω in (38) and applying (36), we obtain

$$\begin{aligned}
\underline{V}_i^{\hat{s}} \;\geq\; & (1 - \textstyle\sum_{\omega' \in \Omega} \theta_{\omega_0 \omega'}) u_i'(\hat{s}) + \displaystyle\sum_{\omega' \neq \omega_1} \theta_{\omega_0 \omega'} \underline{V}_{i\omega'}^{\hat{s}} \\
& + \theta_{\omega_0 \omega_1} \left[(1 - \sum_{\omega' \in \Omega} \theta_{\omega_1 \omega'}) u_i'(\hat{s}) + \sum_{\omega' \neq \omega_2} \theta_{\omega_1 \omega'} \underline{V}_{i\omega'}^{\hat{s}} \right] \\
& + \cdots \\
& + \theta_{\omega_0 \omega_1} \cdots \theta_{\omega_{N-2} \omega_{N-1}} \left[(1 - \textstyle\sum_{\omega' \in \Omega} \theta_{\omega_N \omega'}) u_i'(\hat{s}) + \displaystyle\sum_{\omega' \neq \omega_N} \theta_{\omega_{N-1} \omega'} \underline{V}_{i\omega'}^{\hat{s}} \right] \\
& + \theta_{\omega_0 \omega_1} \theta_{\omega_{N-1} \omega_N} \underline{V}_{i\omega_N}^{\hat{s} \backslash s_i^*} - \frac{\epsilon \underline{\theta}}{3(|I|+1)} - \eta |\Omega|.
\end{aligned} \tag{39}$$

Using $\underline{V}_{i\omega}^{\hat{s}} \geq \underline{V}_i^{\hat{s}}$ $(\forall \omega \in \Omega)$, we have

$$\underline{V}_i^{\hat{s}} \geq \left[1 - \frac{\theta_{\omega_0 \omega_1} \cdots \theta_{\omega_{N-1} \omega_N}}{\theta(\omega_0, \cdots, \omega_N)} \right] u_i'(\hat{s}) \; + \; \frac{\theta_{\omega_0 \omega_1} \cdots \theta_{\omega_{N-1} \omega_N}}{\theta(\omega_0, \cdots, \omega_N)} \underline{V}_i^{\hat{s} \backslash s_i^*} - \frac{2\epsilon}{3(|I|+1)}$$

Thus, for all $r < \bar{r}$,

$$\underline{V}_i^{\hat{s}} > \underline{V}_i^{\hat{s} \backslash s_i^*} - \frac{\epsilon}{|I|+1},$$

where, by construction, $\hat{s} \backslash s_i^* \in S^{k-1}$. □

4.3 More on Genericity

Though uniqueness of the Pareto optimal outcome is not generally possible, i.e., not possible for all discount rates, for general coordination games with asymmetric costs of miscoordination we do establish asymptotic uniqueness for an interval of discounts rates. This stands in contrast with standard repeated games. Pareto inferior Nash equilibria of the stage game are always equilibria of the standard repeated game, regardless of the value of the discount rate. We show that the inferior Nash equilibrium is not sustainable in a neighborhood of discount rates for the class of symmetric 2×2 coordination games given in G_3.

2

1		s_2^*	$\bar{s}_2$
	s_1^*	$u^*,\ u^*$	$\beta,\ 0$
	$\bar{s}_1$	$0,\ \beta$	$1,\ 1$

G_3

Theorem 4 *Consider any asynchronously repeated game* $\Gamma = \langle G_3, \nu, \vec{\Omega}, (p_{\omega\omega'}), r\rangle$ *with* G_3 *as above where* $u^* > 2$ *and* $\beta > 0$*. Then there exists a nonempty open set* $R \in (0, \infty)$ *such that if* $r \in R$ *then for any perfect equilibrium* f*,*

$$V_i(f|h) = u^*,\ \forall i,\ \forall h \text{ with } s(h) = s^*,$$

and with probability one, there is some history h *with* $s(h) = s^*$ *is reached.*

Note that from the proof it will be clear that the parametric part of G_3 is not essential to the argument.

proof of Theorem 4 Fix a perfect equilibrium f. First, let $\underline{V}_i = \inf_{s(h)=s^*} V_i(f|h)$ $(i = 1, 2)$. Assume the contrary, i.e., that $\underline{V}_1 < u^*$. Note that $\underline{V}_2 < u^*$ holds, too. By the recursive formula for V and the above contrapositive assumption, for all $\epsilon > 0$ there exists $h = h_\epsilon \in H$ such that player 2 switches to $\bar{s}_2$ at $h' = h \circ \omega'$ with positive probability where $p_{\omega(h)\omega'} > 0$, and

$$\underline{V}_1 > V_1(f|h) - \epsilon$$

and

$$\underline{V}_1 > V_1(f|h \circ (\omega'; \bar{s}_2)) \tag{40}$$

hold. By perfection of f,

$$V_2(f|h \circ (\omega'; \bar{s}_2)) \geq V_2(f|h \circ (\omega'; s_2^*)) \geq \underline{V}_2. \tag{41}$$

Let $\Theta(i|\omega')$ $(i = 1, 2)$ be given by

$$\begin{aligned}\Theta(i|\omega') &= \sum_{(\omega_0,\omega_1,\cdots,\omega_N)\in\vec{\Omega}(i|\omega')} \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N} \\ &= \sum_{(\omega_0,\omega_1,\cdots,\omega_N)\in\vec{\Omega}(i|\omega')} p_{\omega_0\omega_1}\cdots p_{\omega_{N-1}\omega_N}\left[\int_0^\infty e^{-r\tau}d\nu(\tau)\right]^N\end{aligned}$$

where

$$\vec{\Omega}(i|\omega') = \{(\omega_0, \cdots, \omega_N) \in \Omega |\ \omega_0 = \omega', \forall n < N : \omega_n \notin \Omega_i, \omega_N \in \Omega_i\}.$$

That is, $\vec{\Omega}(i|\omega')$ is the set of sequences of ω's which start with ω' and reach a state in Ω_i after N opportunities for the first time. Similarly, $\Theta(ij|\omega')$ $(i \neq j)$ is given by

$$\Theta(ij|\omega') = \sum_{(\omega_0,\omega_1,\cdots,\omega_N)\in\vec{\Omega}(ij|\omega')} \theta_{\omega_0\omega_1}\cdots\theta_{\omega_{N-1}\omega_N}$$

where

$$\vec{\Omega}(ij|\omega') = \{(\omega_0, \cdots, \omega_N) \in \Omega |\ \omega_0 = \omega', \exists n : \omega_n \in \Omega_i, \forall n < N : \omega_n \notin \Omega_j, \omega_N \in \Omega_j\}.$$

That is, $\vec{\Omega}(ij|\omega')$ is the set of sequences of ω's similar to $\vec{\Omega}(j|\omega')$ but player i gets at least one revision opportunity before j.

Observe that we can write any continuation $V_i(f|h)$ as a weighted sum

$$A_h u^* + B_h u_i(s_1^*, \bar{s}_2) + C_h \cdot 1 + D_h u_i(\bar{s}_1, s_2^*)$$

with $A_h + B_h + C_h + D_h = 1$. Let h' satisfy $s(h') = (s_1^*, \bar{s}_2)$ and such that $\bar{s}$ is assumed to be reached from h'. Observe that

$$V_1(f|h') \;-\; V_2(f|h') \;=\; \beta(B_{h'} - D_{h'})$$

Next, observe that $B_{h'} > 1 - \Theta(1|\omega)$ as the total fraction of time spent in $(s_1^*, \bar{s}_2)$ must be at least the discounted probability that player 1 does not have a revision opportunity since $\bar{s}$ is assumed to be reached from h'. Also, $D_{h'} < \Theta(12|\omega)$ since $\Theta(12|\omega)$ is the fastest time it takes for the players to reach $(\bar{s}_1, s_2^*)$ and stay there permanently. Since the players will not remain in $(\bar{s}_1, s_2^*)$ permanently under f, $D_{h'} < \Theta(12|\omega)$. Hence, the following inequality holds:

$$V_1(f|h \circ (\omega'; \bar{s}_2)) \;-\; V_2(f|h \circ (\omega'; \bar{s}_2)) \;\geq\; \beta(\; 1 - \Theta(1|\omega') - \Theta(12|\omega') \;). \tag{42}$$

Observe that the right hand side (RHS) of (42) tends to be positive as r goes to infinity, and tends to be negative as r approaches to zero. Let $\underline{r}^1 > 0$ be such that the RHS of (42) is zero. Such an $\underline{r}^1$ exists since the RHS of (42) is continuous. Then (40) through (42) imply that for $r > \underline{r}^1$,

$$\underline{V}_1 > \underline{V}_2.$$

Switching 1 with 2, we can make the same argument to obtain

$$\underline{V}_2 > \underline{V}_1$$

for $r > \underline{r}^2$ where $\underline{r}^2$ is similarly defined. These two inequalities contradict with each other if $r > \max\{\underline{r}^1, \underline{r}^2\}$. Therefore, for this r, $\underline{V}_i = u^*$ for $i = 1, 2$.

Next, from an action profile $(s_1^*, \bar{s}_2)$, if player 2's revision node is reached, he changes his action to s_2^*. Therefore, from this profile player 1 obtains at least

$$\beta(1 - \Theta(2|\omega)) + \Theta(2|\omega)u^* \tag{43}$$

provided that the current state is $\omega \in \Omega$. Likewise, player 2 obtains at least

$$\beta(1 - \Theta(1|\omega)) + \Theta(1|\omega)u^* \tag{44}$$

from $(\bar{s}_1, s_2^*)$ at state ω.

Now, we must show that for some $r > \max\{\underline{r}^1, \underline{r}^2\}$, at least one of the players, say player i, switches to s_i^* from profile $(\bar{s}_1, \bar{s}_2)$ and at *some* state $\omega \in \Omega_i$. Suppose $\underline{r}^1 \geq \underline{r}^2$. The other case is analyzed in the same way. Since $\Theta(1|\omega) > \Theta(12|\omega)$, (42) implies that there exists

$\hat{r} > \underline{r}^1$ under which $\Theta(1|\omega) > 1/2$. Let $R = (\max\{\underline{r}^1, \underline{r}^2\}, \hat{r}) \subset \Re$. Then for all r in R, (44) is greater than one since $u^* \geq 2$. It implies that staying away from s^* cannot be a perfect equilibrium. Thus, s^* is reached with probability one. But once it is reached, the action profile never leaves it if $r > \max\{\underline{r}^1, \underline{r}^2\}$. Therefore, s^* becomes the unique outcome in the long run. □

5 A Continuity Result

Finally, we establish an "approximation Theorem" stating that the set of equilibrium payoffs for a one parameter family of renewal games with pure coordination stage games varies "continuously" between the Folk Theorem payoffs and $\{u^*\}$ as the approximating parameter varies between an asynchronously repeated game and a standard repeated game. The result demonstrates that there is no discontinuity between the Folk Theorem and "anti-Folk Theorem" results.

Consider a renewal game $\langle G, \nu, p_{\omega\omega'}, r\rangle$ in which G is a pure coordination game and $\Omega = 2^I$. Now define a "δ-approximation" to a standard repeated game as a tuple

$$\Gamma^\delta = \langle G, \nu^\delta, \vec{\Omega}, p^\delta, r\rangle$$

where

$$\nu^\delta(X_k \in [1-\delta, 1+\delta]) \geq 1-\delta, \ \forall k. \tag{45}$$

and

$$\sum_{j\in I} p^\delta_{C\{j\}} = \delta \text{ and } \quad p^\delta_{CI} = 1-\delta, \ \forall C \subseteq I. \tag{46}$$

Here, there is probability $1-\delta$ that all players move at once at revision opportunities at the jumps of close to unit length. If $\delta = 1$ then p places full support on the singleton sets and so (a.s.) only one player moves at a time.

Given G let

$$\begin{aligned} \bar{u}^* &= u(s^*) \\ \underline{u}^* &= \min\{u(\sigma) : \sigma \ \text{ is a mixed Nash eqm profile}\} \end{aligned}$$

and let

$$\begin{aligned} &E(\Gamma) = E(\langle G, \nu, \vec{\Omega}, p, r\rangle) = \\ &\left\{\bar{V} \in \Re^I \middle| \ \exists \text{ Perfect equilibrium } f \text{ and } \exists h \in H^0 \text{ s.t. } \bar{V}_i = V_i(f|h), \forall i\right\}. \end{aligned}$$

The set $E(\Gamma)$ denotes the Perfect equilibrium payoffs of Γ. Now let

$$\bar{E}(\Gamma) = \lim_{r\to 0} E(\Gamma) \text{ in Housdorff metric}$$

The set $\bar{E}(\Gamma)$ is the limit of the set of Perfect equilibrium payoffs of Γ taken as r approaches zero.

Theorem 5 *Given a renewal game Γ^δ with pure coordination, there is a $\eta(\delta) > 0$ such that*

(i) $\bar{E}(\Gamma^\delta) = [\bar{u}^* - \eta(\delta), \bar{u}^*]$.

(ii) $\eta(\cdot)$ *is continuous, and* $\eta(\delta) \to 0$ *as* $\delta \to 1$ *while* $\eta(\delta) \to \bar{u}^* - \underline{u}^*$ *as* $\delta \to 0$.

Before proceeding with the proof we should emphasize that the result does *not* prove that the standard repeated model is robust since the δ approximation is just one way of defining the standard model as a limit of sequences of renewal games. Recall Example 3 in Section 2 which provides an alternative definition. In that formulation, the standard model is not robust since it is the limit of sequences of asynchronously repeated games defined by Example 3 (as opposed to being a limit of sequences of games in which *some* decisions are made asynchronously — which is the case here).

proof of Theorem 5 Fix $\epsilon > 0$. Fix $\delta > 0$, and $r > 0$, and an equilibrium f of Γ^δ given r. The argument mirrors that of Theorems 1 and 2. We first give a lower bound to PE payoffs, then show that the payoffs that exceed this bound less ϵ are achieved if r is sufficiently small. This lower bound will be shown to be $\bar{u}^* - \eta(\delta)$ with $\eta(\delta)$ satisfying the requisite properties.

Given the δ approximation, Γ^δ, define (as in Theorem 1) $\underline{V} = \inf\{V(f|h) : s(h) = s^*\}$. Then for any $\varepsilon > 0$ there is some history h with $s(h) = s^*$ such that

$$\underline{V} > [1 - \sum_{\omega' \in \Omega} \theta_{\omega(h)\omega'}]\bar{u}^* + \sum_{\omega' \in \Omega} \theta_{\omega(h)\omega'} V(f|\ h \circ \omega') - \varepsilon \tag{47}$$

Given the construction of the δ approximation we can rewrite the right hand side of (47) as

$$[1 - \sum_{\omega' \in \Omega} \theta_{\omega(h)\omega'}]\bar{u}^* + \int e^{-rt} d\nu \left(\sum_{i \in I} p^\delta_{\omega(h)\{i\}} V(f|\ h \circ \{i\}) + (1-\delta) V(f|\ h \circ I) \right) - \varepsilon$$

With Perfection it follows that $V(f|\ h \circ \{i\}) \geq V(f_{-i}, \hat{f}_i|\ h \circ \{i\})$ for all $\hat{f}_i$. In particular this inequality holds for any $\hat{f}_i$ with $\hat{f}_i(h \circ \{i\}) = s_i^*$. Therefore, we have

$$\underline{V} > \frac{1 - \int e^{-rt} d\nu}{1 - \delta \int e^{-rt} d\nu} \bar{u}^* + \frac{(1-\delta) \int e^{-rt} d\nu V(f|\ h \circ I) - \varepsilon}{1 - \delta \int e^{-rt} d\nu} \tag{48}$$

If individuals use their minimax strategies in state $h \circ I$ then (48) becomes

$$\underline{V} > \frac{1 - \int e^{-rt} d\nu}{1 - \delta \int e^{-rt} d\nu}\ \bar{u}^* + \frac{(1-\delta) \int e^{-rt} d\nu\ \underline{u}^* - \varepsilon}{1 - \delta \int e^{-rt} d\nu} \tag{49}$$

Since (49) holds for all $\varepsilon > 0$, there is a number $\eta(\delta) > 0$ which satisfies the limit properties of (ii). It is given by

$$\eta(\delta) = \lim_{\varepsilon \to 0} \left[\frac{(1-\delta) \int e^{-rt} d\nu}{1 - \delta \int e^{-rt} d\nu} \bar{u}^* - \frac{(1-\delta) \int e^{-rt} d\nu \underline{u}^* - \varepsilon}{1 - \delta \int e^{-rt} d\nu} \right] \tag{50}$$

Observe that $\eta(\cdot)$ is continuous. Now the same backward induction argument as in Theorem 2 shows that there is some $\bar{r} > 0$ such that if $0 < r \leq \bar{r}$ then $V(f|h)$ is greater than $\bar{u}^* - \eta(\delta) - \epsilon$ given any initial history h.

It remains to show that any payoff which exceeds this lower bound can be achieved as a limit payoff of a sequence of Perfect equilibrium payoffs as $r \to 0$. Let $(\hat{u}_1, \hat{u}_2)$ denote a payoff that exceeds the lower bound in (49). We construct the equilibrum f to approximate $(\hat{u}_1, \hat{u}_2)$ if r is small. For any history h and at any state $\omega = \{i\},\ i \in I$, let $f_i(h \circ \{i\}) = s_i^*$. That is, at all asynchronous decision states, play remains in or returns to s^* as soon as possible. If h is on the equilibrium path, we define at decision state $\omega = I,\ f_i(h^t \circ I) = \tilde{\sigma}_i$ for each i where $\tilde{u}_i = \sum_{s \in S} \tilde{\sigma} u_i(s)$ is defined implicitly by

$$\hat{u}_i = \frac{1 - \int e^{-rt} d\nu}{1 - \delta \int e^{-rt} d\nu}\ \bar{u}^* + \frac{(1-\delta) \int e^{-rt} d\nu\ \tilde{u}_i}{1 - \delta \int e^{-rt} d\nu} \tag{51}$$

Since there is a mixed strategy that can achieve any symmetric payoff in $[\underline{u}^*, \bar{u}^*]$ such a $\tilde{\sigma}$ and $\tilde{u}$ can be found. If there is ever a deviation from this strategy we let $f_i(h^t \circ I) = \underline{\sigma}_i$ after the deviation history h^t where $\underline{\sigma}$ achieves the minimax payoff $\underline{u}^*$. Since $\hat{u}_i > \bar{u}_i - \eta(\delta)$, the minimax punishment deters deviations from the prescribed f. Also, since $\bar{u}_i - \eta(\delta)$ is itself a Perfect equilibrium continuation payoff, f is a Perfect equilibrium which achieves payoff profile $\hat{u}$. □

6 Related Literature

The mathematics used to prove the result for pure coordination games follows the one used in the proof of the unimprovability result due to Howard (1960), who shows that piecewise optimization, i.e., local optimization at every decision node, induces global optimization in a single person decision problem with perfect recall. Local optimization at every decision node is equivalent to a Nash equilibrium in a game where any two decision nodes belong to different agents. Thus, these two proofs are essentially the same since an asynchronously repeated game with a pure coordination game as its stage game is identical with a single person decision problem with many agents. Thus, though the proofs of other results obtained in the present paper cannot be reduced to corollaries of the unimprovability result, it is the framework of the analysis rather than mathematics that distinguishes the present paper from preceding papers on single person decision problems. Incidentally, Abreu (1988) uses the same logic as Howard to prove the folk theorem in standard repeated games. It should be noted, therefore, that the mathematical structure of standard repeated games is very different from that of asynchronously repeated games. Even if the stage game is pure coordination, the Pareto inferior outcome is sustained as an equilibrium. This is because in a repeated game with simultaneous choice, the assumption of perfect recall must be sacrificed in order to view it as a single person decision problem.

The breakdown of simultaneity assures optimality in another framework given in Rubinstein and Wolinsky (1995). They consider a standard infinite repetition of an extensive form game of pure coordination. In their model the optimal outcome is also obtained. Strictly speaking, theirs is not an asynchronously repeated game since payoffs only occur at the end of each period. Since the perfect equilibrium of their extensive form stage game is unique, the set of equilibrium payoffs does not shrink with repetition. Nevertheless, the extensive form stage game breaks the "synchronized mistake" in much the same way as in our model. We view our results as one natural way of achieving their "extensive form" intuition in repeated games, although we do not yet know if our genericity results extend to the standard model of repetition of extensive form games.

Other asynchronous choice models have been studied to some extent in various settings. As a points of comparison with related models, we emphasize the work on bargaining by Perry and Reny (1993) , the dynamic oligopoly problems studied by Maskin and Tirole (1988), Farrell and Saloner (1985), Gale (1995), and Morris (1995), and our previous work (some with Rafael Rob), on asynchronously repeated games with evolutive dynamics (Matsui and Rob (1992), Lagunoff and Matsui (1995), and Lagunoff (1995)).

In the Perry/Reny model, individuals can move at any time, with a small delay after each player moves. They show that if players can react to offers instantaneously then the set of equilibria are bounded by the first and second mover payoffs, resp., of the Rubinstein bargaining model. The Perry/Reny model offers the advantage over our framework that the choice of timing of moves is endogenous subject to a fixed waiting time that must be endured after a player makes an offer. However, since players have the option of moving simultaneously, it appears that in the adaptation of their model to the repeated game, each of the static Nash equilibria of a coordination game will persist as Perfect equilibria.

Maskin and Tirole rule out simultaneous moves by considering an alternating move game in which, in one variant of the model, each firm sets capacity given the (temporarily) fixed capacity of rivals. The timing structure of the model is subsumed in our general definition of asynchronously repeated games, however, their interest is with strategic behavior that depends only on payoff relevent information, i.e., natural state variables.

Works by Farrell and Saloner (1985), Gale (1995), and Morris (1995) are also related to the present paper. Farrell/Saloner and Gale show that in an n-person coordination game with asynchronous choice and finitely many moves, a Pareto efficient outcome is attained if players are sufficiently patient. Backward induction is used to establish that when, say, first $k < n$ players cooperate, the $(k + 1)$th player also cooperates. Though its logic looks similar to ours at a first glance, it is very different. The difference is manifested in Game G_2. In this game, s^* is uniquely attained in the framework of Farrell/Saloner and Gale, while $\bar{s}$ is also reached infinitely often in our framework. The difference is that there is no fear of "retaliation" for the last person in theirs, while there is always a chance of "retaliation" in our framework. When the game played is a pure coordination game, this "retaliation" does not work since one who "retaliates" also harms himself to the same degree. Morris considers a strategic environment similar to Farrell/Saloner and Gale except that the entry times into the game of each of the n individuals are not common knowledge. Each player observes his own "clock" but knows only the support of the distribution over others' clocks. In contrast to the first two, Morris shows that optimal coordination is, in fact, never achieved. As with the other two, the nonrepetitive strategic environment prevents "retaliation." Without this threat, synchronized coordination unravels when the timing of moves is not common knowledge.

More closely related to the present paper are the Matsui/Rob, Lagunoff/Matsui, and Lagunoff evolutive models which assume the Poisson revision process of Example 2 in Section 2. New individuals were assumed to enter the game stochastically and asynchronously. In the first two models heterogeneous forecasts across generations provided the impetus for change. In the last model, perfect foresight was assumed and, for this reason, is closest to the present equilibrium model.[6]

There is also a sizable literature which characterizes equilibria of repeated games that differ in other respects from the standard model. Two distinct (though related) approaches may be found. In one, restrictions are placed on repeated game equilibria by considering those that are limit points of sequences of equilibria of perturbed games with perturbations in the information structure. Often these perturbations allow the players to precommit to particular outcomes through reputation or informational leakage. Examples include Fudenberg and Levine (1987), Matsui (1989), and Aumann and Sorin (1989). All of these have unique "limit equilibria" for a certain class of stage games.

Fudenberg and Levine (FL) (1987), for instance, characterize outcomes of a repeated game in which a long run player faces a sequence of short run or "myopic" players. They show that only payoffs that dominate a certain "stackelberg" payoff arise when the short run players have a small amount of uncertainty regarding the long run player's "type."[7] However, since FL are primarily interested in modeling reputation, they consider only "limit equilibria" as the uncertainty becomes negligible. This, together with the presence of myopic players on one side of the game eliminates some of multiplicity created by infinite repetition.

[6]Related work includes Blume (1993), Matsui (1994), and Matsui and Matsuyama (1995). Individuals' decisions in these papers are asynchronous, however, the players are either assumed to be myopic, or are randomly matched in a large population so that no player accounts for the intertemporal effects of his own behavior.

[7]The "stackelberg" payoffs are those that arise if the long run player could commit to a behavior strategy taking as given the myopic best response of the short run players. It coincides with the Pareto dominant payoff in pure coordination games.

A second approach uses bounds on strategic complexity to restrict equilibria in repeated games. Examples are Rubinstein (1986), Abreu and Rubinstein (1988), Kalai, Samet, and Stanford (1988), and Cho and Li (1995). In some of these, e.g., Rubinstein (1986), Nash equilibria of a standard repeated game may be eliminated as solutions even if complexity costs enter into preferences lexicographically.[8]

Both of the above approaches tend to look at a more general class of stage games than is considered in the present work. As with standard repeated games, we differ from these by considering fully rational players (subject to the simplifying recursivity assumption), and we consider all the Perfect equilibria of the repeated game.

We have found the general characterization of equilibrium payoffs of all asynchronously repeated games to be difficult and elusive. There is no obvious analogue to the minimax payoff of standard repeated games. Moreover, the success of punishment threats depend on the relative frequencies of decision opportunities of the various players. We hope that others will agree that a full characterization of the rich variety of repeated strategic interaction warrants further exploration.

References

[1] Abreu, D. (1988), "On the Theory of Infinitely Repeated Games with Discounting," *Econometrica*, 56: 383-398.

[2] Abreu, D. and A. Rubinstein, "The Structure of Nash Equilibrium in Repeated Games with Finite Automata," *Econometrica*, 56: 1259-1281.

[3] Aumann, R. (1981), "Survey of Repeated Games," in *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern*, Manheim: Bibliographisches Institut.

[4] Aumann, R. and S. Sorin (1989), "Cooperation and Bounded Recall," *Games and Economic Behavior*, 1: 5-39.

[5] Blume, L. (1993), "The Statistical Mechanics of Strategic Interaction," *Games and Economic Behavior*, 5: 387-424.

[6] Cho, I.-K. and H. Li (1995), "Complexity and Neural Networks," mimeo, University of Chicago.

[7] Farrell, J. and G. Saloner (1985), "Standardization, Compatibility, and Innovation," *Rand Journal of Economics*, 16: 70-83.

[8] Without these costs it is not true necessarily that the structure of equilibrium payoffs differ fundamentally from the Folk Theorem. See Kalai and Stanford (1988).

[8] Fudenberg, D. and D. K. Levine (1989), "Reputation and Equilibrium Selection in Games with a Patient Player," *Econometrica*, 57: 759-78.

[9] Fudenberg, D. and E. Maskin (1986), "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54: 533-56.

[10] Gale, D. (1995), "Dynamic Coordination Games," *Economic Theory*, 5: 1-18.

[11] Howard, R. (1960) *Dynamic Programming and Markov Processes*, John Wiley and Sons: New York.

[12] Kalai, E., Samet , and W. Stanford (1988), "A Note on Reactive Equilibria in the Repeated Prisoner's Dilemma," *International Journal of Game Theory*, 176-86.

[13] Kalai, E., and W. Stanford (1988), "Finite Rationality and Interpersonal Complexity in Repeated Games," *Econometrica*, 56: 397-410.

[14] Lagunoff, R. (1995), "On the Evolution of Pareto Optimal Behavior in Repeated Coordination Problems," mimeo, University of Pennsylvania.

[15] Lagunoff, R. and A. Matsui (1995), "Evolution in Mechanisms for Public Projects," *Economic Theory*, 6: 195-223.

[16] Marschak, J. and R. Radner (1972), *The Economic Theory of Teams*, Yale University Press, New Haven.

[17] Maskin, E. and J. Tirole (1988), "A Theory of Dynamic Oligopoly, I: Overview and Quantity Competition with Large Fixed Costs," *Econometrica*, 56: 549-70.

[18] Matsui, A. (1989), "Information Leakage Forces Cooperation," *Games and Economic Behavior*, 1: 94-115.

[19] Matsui, A. (1993), "Evolution and Rationalizabilty," mimeo, University of Pennsylvania, March.

[20] Matsui, A. and K. Matsuyama (1995), "An Approach to Equilibrium Selection," *Journal of Economic Theory*, 65: 415-34.

[21] Matsui, A. and R. Rob (1992), "Evolution with Forward Looking Players," mimeo, University of Pennsylvania.

[22] Morris, S. (1995), "Cooperation and Timing," mimeo, University of Pennsylvania.

[23] Perry, M. and P. Reny (1993), "A Non-cooperative Bargaining Model with Strategically Timed Offers," *Journal of Economic Theory*, 59: 50-77.

[24] Rubinstein, A. (1986), "Finite Automata Play the Repeated Prisoner's Dilemma," *Journal of Economic Theory*, 39: 83-96.

[25] Rubinstein, A. and A. Wolinsky (1995), "Remarks on Infinitely Repeated Extensive-Form Games," *Games and Economic Behavior*, 9: 110-115.