Condorcet's Paradox (Part 1)

Player I A > B > C

Player II B > C > A

Player III C > A > B

1st round

A vs. B = 2 votes for A (I, III), 1 vote for B (II)

2nd round

B vs. C = 2 votes for B (I, II), 1 vote for C (III)

3rd round

C vs. A = 2 votes for C (II, III), 1 vote for A (I)

So, what is the preference ranking for the majority? It is this (a "Condorcet cycle"):

A > B > C > A > B > C > A > . . . . . > n

Is there a "will of the majority" then?

Condorcet's Paradox (Part 2)

Perhaps we can solve the paradox by having a run-off vote: We will pit A against B, then the winner of that contest against C.

Here are the preferences again:

Player I A > B > C

Player II B > C > A

Player III C > A > B

Here is a run-off race:

1st round:
A vs. B = 2 votes for A (I, III), 1 vote for B (II)
2nd round:
B vs. C = 2 votes for B (I, II), 1 vote for C (III)
Run-off:
A vs. B = 2 votes for A (I, III), 1 vote for B (II)

So, A is the "majority winner." Or is it?

Condorcet's Paradox (Part 3)

Is there a once-and-for-all "majority winner"? Note that the outcome changes depending on the order of voting.

Here are the preferences again:

Player I A > B > C

Player II B > C > A

Player III C > A > B

This time, let's change the order in which we vote on the preferences:

1st round:
A vs. C = 2 votes for C (II, III), 1 vote for A (I)
2nd round:
C vs. B = 2 votes for B (I, II), 1 vote for C (III)
Run-off:
B vs. C = 2 votes for B (I, II), 1 vote for C (III)

This time, B is the winner of the run-off, not A—but only because we changed the order of voting. Moral of the story: He who sets the agenda rules the world!

Arrow's Theorem

(Kenneth Arrow, Harvard University,
Nobel Prize in Economics, 1972)

There is no mechanism for translating the preferences of rational individuals into a coherent group preference that is not either itself irrational or dictatorial.

In other words, there is a trade-off between social rationality and the concentration of power.

[Formally, there is no mechanism for translating individual into group preferences that simultaneously satisfies four conditions:

1. universal admissibility (each member of the group may adopt any complete and transitive preference ordering)
2. Pareto optimality (if every member of G prefers j to k, the group preference must also reflect j over k)
3. independence from irrelevant alternatives (if j and k stand in a particular relationship in the preference functions of all members of G, then j and k must stand in the same relationship in the group function, regardless of changes in other irrelevant preferences), and
4. nondictatorship (there is no single member of G whose own preferences dictate group preferences independent of the preferences of other members).]

The All-American Scenario (Part 1)

I'll call this the "All-American" because it is the way we often like to believe that preferences are structured: A clear majority of people have the same preference ranking, a fact revealed by a simple plurality voting system.

I A>B>C

II A>B>C

III B>C>A

Plurality voting: A wins because it is preferred first by the majority (I, II).

Here is the outcome in paired voting:

1st round
  A vs. B = 2 votes for A (I, II), 1 vote for B (III)
2nd round
  B vs. C = 3 votes for B (I, II, III)
3rd round
  C vs. A = 2 votes for A (I, II), 1 vote for C (III)

Group preference ranking: A>B>C
No Condorcet cycle

The All-American Scenario (Part 2)

Let's try a different order of votes in paired voting with a run-off:

I A>B>C

II A>B>C

III B>C>A

1st round

B vs. C = 3 votes for B (I, III, III)

2nd round

C vs. A = 1 vote for C (III), 2 votes for A (I, II)

Run-off

A vs. B = 2 votes for A (I, II), 1 for B (III)

Again, A wins in a run-off. So the majority will is A. *Finding: In a 3 x 3 game, if a preference is ranked the same by more than one player, the majority will not cycle.*

But there should be two sources of uneasiness here:

1. Might player III be a "permanent minority"?
2. Might preference B be a minimally acceptable choice for all players and hence closer to the "majority will" than A?

The Simple World of 3 x 3 Condorcet Scenarios (Part 1)

Adapted from Shepsle and Bonchek, Analyzing Politics, chap. 4

In a world of only 3 preferences, there are 13 possible rankings available to any player:

"Strong rankings"
1. A>B>C
2. A>C>B
3. B>A>C
4. B>C>A
5. C>A>B
6. C>B>A

"Weak rankings"
7. (A=B)>C
8. (A=C)>B
9. (B=C)>A
10. A>(B=C)
11. B>(A=C)
12. C>(A=B)
13. A=B=C

Let's focus just on the strong rankings. With 3 players choosing any one of 6 available strong rankings, the number of possible "societies" is 216.
[Formally, this is $(m!)^n$ where m = alternative preferences and n = number of players.]

The Simple World of 3 x 3 Condorcet Scenarios (Part 2)

Adapted from Shepsle and Bonchek, Analyzing Politics, chap. 4

In these 216 strong-preference societies, there will be no Condorcet cycle in 204 votes. The probability of a Condorcet cycle is thus only .056.

Finding: The "majority will" is usually easy to ascertain in small groups with limited preferences. (Recall Olson here, too: Collective action problems rarely arise in very small groups.)

But as we increase the players and the preferences, the world gets more complicated.

- with 3 players and 4 alternatives, the probability of a Condorcet cycle rises to .111
- with 11 players and 4 alternatives, the probability of a Condorcet cycle is .160
- as we increase both the number of players and the available preferences, the probability approaches 1.0.

Finding: In a complex world of many players and many preferences in many combinations, Condorcet cycles are close to inevitable.

(Probability here is given by: Pr(m,n) = # of potential Condorcet cycles/$(m!)^n$, where m is the number of alternative preferences and n is the number of players.)

A Further Problem with the All-American Scenario (Part 1)

Let’s complicate the All-American Scenario by adding players:

I A>B>C
II A>B>C
III B>A>C
IV B>A>C
V C>A>B
VI C>A>B
VII C>A>B

In a plurality contest, C wins (3 votes for C, 2 for A, 2 for B)

But what happens in a pairwise vote?

A vs. B = 5 votes for A (I, II, V, VI, VII), 2 votes for B (III, IV)
B vs. C = 4 votes for B (I, II, III, IV), 3 votes for C (V, VI, VII)
C vs. A = 3 votes for C (V, VI, VII), 4 votes for A (I, II, III, IV)

A Further Problem with the All-American Scenario (Part 2)

The outcome is **not** a Condorcet cycle. In pairwise voting, the majority preference ranking is A>B>C.

But in any pairwise vote, players always prefer A or B to C.

Thus, while C wins the plurality vote, C is a "Condorcet loser."

So, is C really the "majority will"?

Answer: It depends on the system of voting that you use.