An octonion model for physics

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Abstract

The no-zero-divisor division algebra of highest possible dimension over the reals is taken as a model for various physical and mathematical phenomena mostly related to the Four Color Conjecture. A geometric form of associativity is the common thread.
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1 Introduction

We explore some consequences of octonion arithmetic for a hidden variables model of quantum theory and ideas regarding the propagation of gravity. This approach may also have utility for number theory and combinatorial topology.

Octonion models are currently the focus of much work in the physics community. See, e.g., Okubo [29], Gursey [14], Ward [36] and Dixon [11]. Yaglom [37, pp. 94, 107] referred to an "octonion boom" and it seems to be accelerating. Historically speaking, the inventors/discoverers of the quaternions, octonions and related algebras (Hamilton, Cayley, Graves, Grassmann, Jordan, Clifford and others) were working from a physical point-of-view and wanted

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their abstractions to be helpful in solving natural problems [37]. Thus, a connection between physics and octonions is a reasonable though not yet fully justified suspicion.

It is easy to see the allure of octaves since there are many phenomena in the elementary particle realm which have 8-fold symmetries. Another reason is simple extremality: one can't go any further than the octonions without sacrificing the basic operations of arithmetic.

Of course, multiplication in the octaval arithmetic fails to be either commutative or associative, but that could be a blessing in disguise. If multiplication depends on the *order* of the elements being multiplied together and even on how they are grouped, then at one fell swoop, geometry enters the calculation in an organic way. The Principle of Indeterminacy could then arise in a natural fashion from relativistic considerations, making quantum theory a consequence of an underlying 8-dimensional hidden-variable process, very much in the flavor of the theories of de Broglie and Bohm. Uncertainty of measurement would be a corollary of our inability to absolutely order events or to absolutely control the way in which they are grouped.

In this paper, we will consider an application of higher dimensional arithmetic to the Four Color Problem of combinatorial topology (see Appel and Haken [4], Saaty and Kainen [30]). This problem has a strong physical flavor (Kauffman and Saleur [23], Bar Natan [5]). Indeed, the essence of the problem is now seen to involve a weak form of associativity [22], and so a connection with the octonions is plausible. Since Cayley was the first to write about the Four Color Problem [30], for his octaval arithmetic to play a role in the solution would be a nice irony.

The organization is as follows: In section 2, we discuss emergence of higher dimensions in lower, in physical and mathematical contexts. Sections 3 and 4 review the Four Color Conjecture and its formulation in terms of special trees and the properties of vector cross-product. Sections 5 and 6 deal with an algebraic formulation suggested by quantum computing and quantum algebra. In section 7, we return to some relevant features of the octonions and the paper concludes with a section of remarks and an appendix on the definition of octonions.

2 Higher dimensional emergence

There are various places in mathematics and physics where phenomena occur which manifest something intrinsically higher-dimensional in a space of lower dimension. In physics, although we live in a world that appears to be three-dimensional, current theories (e.g., of strings or "branes") use much higher dimensions (see, e.g., Giveon and Kutasov [13]) and this idea, beginning with Kaluza and later (Oscar) Klein, has an interesting history (see, e.g., Y. Vladimirov [35] and A. Lichnerowicz [27]).

Naively, measurements are real numbers but complex numbers are now ubiquitous in physics (e.g., in optics and electronics). As complex numbers extend the real line, Hamilton's quaternions provide a useful four-dimensional tool with direct applications to optics and mechanics. Dirac noted that quaternions can explain such three-dimensional phenomena as why two full rotations are needed in order to untangle knots. Since physics is intimately connected with symmetries, it is not too surprising to find such applications (see, e.g., Altman [2]).

There are only four division algebras over the reals. Except for the octonions, the others (real, complex and quaternion) have been found useful in physics, as have the Clifford algebras which generalize them. This makes the octonions a natural target for physicists and mathematicians [11]. So far there does not seem to be a clear situation where octaves have found an essential application to physics [36, vii]. However, Freudenthal [12] and Tits [34] have found that the five singular simple Lie groups (which are not in the four infinite families) are related to the isometries of octaval planes [37, p. 107].

In mathematics too, one finds higher-dimensional objects casting shadows in the lower dimensions. For example, the Penrose nonperiodic tiling of the plane is a projection of something in dimension at least 4 (Katz, [24]) and the Hardy-Ramanujan formula (Andrews [3]) shows that the number of partitions of a positive integer n may be expressed in terms of the $24k$-th roots of unity. While integers are 0-dimensional, the partition formula suggests that a 24-fold symmetry is involved - as would be the case if it involves objects in 4-dimensions.

The quaternions and octonions can be used to prove theorems expressing an integer as a sum of four or eight squares; see Hardy and Wright [16], Coxeter [7]. While these results can be derived in other ways, they fit a higher-dimensional model. Analagously, Felix Klein [25] showed how to display otherwise obscured aspects of the dynamics of a top with four complex variables to represent space and time, satisfying $x^2 + y^2 + z^2 - t^2 = 0$. Though Klein was quite emphatic on not intending "complex time" to be taken literally, the term is now referenced in a number of papers in the literature. See, e.g., [6] regarding Sonia Kovalevskaya's contribution in this regard.

While Klein's analysis does not specify the algebraic properties of these variables (i.e., whether they correspond to complexified quaternions or to octonions - see, e.g., [36]), it reminds us of the advantages of using higher dimensional formulations to simplify the descriptions of phenomena which live in three-dimensions. These advantages include not only algebraic and topological insights but even applications to differential equations. See, e.g., [28] where it is pointed out that by embedding $\mathcal{R}^3$ in a Clifford algebra, the square root of the Laplacian is a first-order, elliptic differential operator (and also for the Laplacian in $\mathcal{R}^n$).

3 The Four Color Problem

An especially interesting potential application for geometric algebra, including the octonions, can be found in the celebrated Four Color Conjecture, which asserts that every planar map can have its regions "colored" using four distinct labels so that regions which share a common boundary edge receive distinct colors. We give a brief review of the problem in terms of an equivalent planar graph formulation. See [4] and [30] for further detail.

A *graph* is a symmetric irreflexive relation (called *adjacency*) on a finite set of *nodes* (or "vertices"). Related (unordered) pairs of distinct points are called *edges* (or "lines") and the two vertices that constitute an edge are called its endpoints. The number of nodes is called the *order*. A graph is *complete* if every two distinct nodes are adjacent.

A graph is planar if it can be represented as a subset of the plane such that each edge corresponds to a simple curve, each vertex is a point which is an endpoint of any curve corresponding to an edge which includes the node, two curves intersect only at common endpoints of the corresponding edges, and no point corresponding to a node belongs to the interior of any curve. Intuitively, a graph is planar if it can be drawn in the plane without any edge crossings. A particular crossing-free drawing is called a plane graph.

A *coloring* of a graph is an assignment of labels to the nodes such that no two adjacent nodes get the same color; a coloring is an r-coloring if the set of labels does not exceed r in cardinality. For example, the complete graph of order n has an n-coloring but no k-coloring for $k < n$.

The Four Color Conjecture asserts that every planar graph can be 4-colored. This was proved in 1976 (after being unsolved for about a century) but the proof is lengthy and depends on computer calculations which are not humanly checkable. The Four Color Problem is (now) to find a simple proof.

It is convenient to reformulate in terms of the edges. This protean aspect is characteristic of the problem; see [30] and [19].

The *degree* of a node is the number of edges which include it as an endpoint. Two edges are called adjacent if they have a common endpoint. Call G an

(n, m)-graph if it has n nodes and m edges. A graph is *regular* if all degrees are equal; a *cubic* graph is a regular graph of degree 3. A 1-path is the trivial graph. For $n > 1$ an *n-path* is an $(n, n-1)$-graph in which it is possible to order the nodes and edges in an alternating sequence such that the two nodes flanking each edge are its endpoints and the first node and last node of the sequence are distinct. The path is said to join its first and last node. For $n > 2$ an *n-cycle* is an (n, n)-graph with a similar alternating sequence as above except that the first and last node coincide.

A graph is *connected* if any two nodes are joined by a path. A *tree* is a connected graph with no cycles. It is easy to show that every tree is planar. The vertices of degree 1 are called endpoints and the edges which include them are *leaves*. Every tree has a 2-coloring. If a tree has at least two nodes, then it has at least two endpoints. A tree is called cubic if all of its nodes have degree either 3 or 1; it is *rooted* if one of the leaves is selected as the "root." A plane tree has a uniquely determined cyclic order on its leaves (obtained by travelling around the boundary of a regular neighborhood in the plane). By selecting a root and an orientation of the plane (clockwise or counterclockwise), we obtain a unique linear order on the leaves which begins at the root.

Given two rooted plane trees with the same number of endpoints there is a natural correspondence between the leaves which identifies their roots, using the clockwise order for one tree and the counterclockwise order for the other tree. If the corresponding pairs of leaves are identified, then the resulting graph is planar. We call this the *fusion* of the two rooted plane trees.

An *edge k-coloring* is an assignment of at most k labels to the edges so that no two adjacent edges get the same color. A cubic tree has an edge 3-coloring. Using a theorem of Whitney, the Four Color Conjecture is equivalent to the following assertion: Given two rooted cubic plane trees of the same size, there exists an edge 3-coloring of the two trees such that corresponding leaves receive the same color [30, pp. 102-110].

An edge 3-coloring of a rooted cubic plane tree is completely determined by a choice of clockwise or counterclockwise orientations ($+$ or $-$, resp.) at each of the cubic nodes; choice of a color at the root and any fixed cyclic order on the three colors. We call the sequence of $+/-$ vertex orientations the *state*. One takes any given color at the root and propagates it out to all of the edges using the state at each vertex and the cyclic color order to decide whether to color the two edges neighboring an edge of color 1 with edges of colors 2,3 (or 3,2).

4 Role of associativity

By interpreting the three colors algebraically, the Four Color Conjecture can be stated in terms of various interpretations of associativity.

One can think of a rooted cubic plane tree as a *deterministic machine* which takes an input sequence, presented to its leaves, and uses a binary operation to multiply the inputs at each of the cubic nodes. More formally, there is an orientation of the graph as a directed graph (each edge is assigned a sense or direction so that it points from one endpoint to the other). This is done by orienting each edge so that it is traversed in the positive sense when traveling from a leaf to the root. Hence, each cubic node has indegree 2 and outdegree 1.

Rooted cubic plane trees amount to systems of parentheses for the evaluation of a binary operation. Evaluating a product can depend on the choice of the tree when the operation is not associative. The topology of the tree corresponds to the position of the parentheses needed to evaluate the sequence of inputs. If T and σ denote the tree and the sequence, then we write $T(\sigma)$ for the resulting algebraic output.

Kaufmann showed [22] that the Four Color Conjecture is equivalent to a restricted form of associativity for cross-product. Cross-product is the multiplication defined on the linear space $\mathcal{R}^3$ as in elementary calculus and physics. If the standard orthonormal basis is denoted i, j, k, then the following hold: $i \times j = k = -j \times i$, $i \times i = 0$, and the other relations resulting from each of the cyclic rotations of the colors in the order i, j, k. Note that cross-product is not associative since, e.g., $i(ij) = ik = -j \neq 0 = (ii)j$.

In the following theorem, by an argument attributed to G. Bergmann [21], it suffices to prove that for any two rooted cubic plane trees of the same size, there exists at least one sequence on which both both trees give a nonzero value for in that case, the cross-product multiplications never give a value different from the value which would be given by a corresponding tree-machine operating via quaternion multiplication (interpreting i, j, k, now, as the three unit pure quaternions). Since the quaternions are associative, the computed values at the roots must be identical.

Theorem 1 *The Four Color Conjecture is equivalent to the claim that for any positive integer n and any two rooted cubic plane trees T, T' with n cubic nodes, there exists a sequence σ in $\{i, j, k\}^n$ such that $T(\sigma)$ and $T'(\sigma)$ are equal and nonzero.*

Another interesting formulation of associativity is due to Kauffman and

Saleur [23]. They showed that the Four Color Conjecture is equivalent to the existence of an algebraic property for the Temperley-Lieb algebra.

The Temperley-Lieb algebras arose in the context of statistical mechanics but they are also related to spinor theory and many other things; see [21]. A T-L algebra can be constructed by taking certain elementary topological objects (tangles) and then constructing an algebra with the objects as generators with topologically motivated relations. The number of generators (i.e., the dimension) is always a Catalan number and this provides a link to parenthesis systems since the number of rooted cubic plane trees is also a Catalan number. For more on Catalan numbers, see Stanley [33]; many such Catalan classes are connected by natural bijections.

5 Quantum machines

If we reverse the digraph orientation, then a tree provides a unique directed path from root to each leaf. First, we make this into a machine and then complexify it.

Suppose that n is a positive integer and T is a rooted cubic plane tree with $n+1$ leaves. There are n cubic nodes, which have a unique linear ordering with respect to the tree-topology and planar orientation already chosen. Let an input consist of a pure unit quaternion and a "control" sequence in $\{-1, +1\}^n$. Each node takes its unique input and produces as output the next two unit quaternions with respect to the cyclic order i, j, k, reading its outputs in counterclockwise or clockwise order, resp., as the node receives a control value of $+1$ or -1. Thus, the effect is that the tree converts its input sequence (of length $n + 1$) into an output (also of length $n + 1$) which is a sequence of quaternion values on the leaves. One could just as well use the three standard unit vectors in $\mathcal{R}^3$ as input choices, with a comparable alteration of output.

To complexify this, we replace the choice of $+1$ or -1 by a linear combination of both, using complex weights. Hence, the resulting values appearing on the leaves will be complex linear combinations of quaternions (or of unit 3-vectors).

It is convenient to reinterpret in terms of *qubits*, which are generators of the complex vector space $V = \mathcal{C}^2$ usually written $|+>= (0, 1)$ and $|->= (1, 0)$. The quaternion basis i, j, k is encoded $|01>, |10>, |11>$, resp. Now the quantum tree machine defined as above is producing a map from $\mathcal{V}_{n+1} = V^{\otimes n+1}$ to itself, which we denote by T. The composition $T' \circ T$ is defined on the tensor basis and so extends to a linear map.

Theorem 2 *The Four Color Conjecture is equivalent to the claim that for* *any two rooted cubic plane trees T, T' of the same order, $T' \circ T$ is nonzero.*

6 Quantum algebra

Let $\mathcal{D}_n$ denote the digraph whose vertices are the rooted cubic plane trees with arcs corresponding to those pairs of parenthesis systems that differ by a single associative switch: i.e., $(x(yz)) \to ((xy)z)$, etc. One can visualize this using triangulations of an n-gon. Adjacent triangulations agree everywhere except for a single diagonal which is "flipped" to become the other diagonal of a fixed quadrilateral. Note that it is Whitney duality which relates the triangulations and trees (faces correspond to vertices in the dual while edges cross dual-edges); hence, the notion of flipping has a meaning for trees as well.

The underlying graph $\mathcal{T}_n$ of $\mathcal{D}_n$ has been called the "rotation graph" [32] and is the 1-skeleton of the associahedron (or Staheff polytope, also discovered by Mac Lane, see, e.g., Ziegler [38]). We are interested in the set $\mathcal{P}_n$ of all directed paths in $\mathcal{D}_n$ joining T to T'; this is known to be nonempty for any two vertices T, T'.

In [18] we give two isomorphisms c and c' from $V^{\otimes 2}$ to itself which formalize the action of an arc in the rotation digraph. Let τ be the endomorphism of V which reverses the generators by interchanging the two topological factors; i.e., $\tau(|+>) = |->$ and $\tau(|->) = |+>$. We define $c : V \otimes V \to V \otimes V$ by $c|b1, b2 >= |\tau(b_2), \tau(b_1)>$. and $c' : V \otimes V \to V \otimes V$ by $c'|b1, b2 >= |\tau(b_1), \tau(b_2)>$. The two isomorphisms differ in that c involves an extra switch of coordinates which means it satisfies the Yang-Baxter equation; see, e.g., Kassel [20, p. 167].

If $a = (T_1, T_2)$ is an arc of $\mathcal{D}_n$ we will define an *arc operator* which is an endomorphism of the n-fold tensor product $\mathcal{V}$. In each tree, T_1, T_2, the vertices are indexed according to the depth-first, clockwise order so a defines a permutation π_a on the n indices which reflects the topological reindexing which results from the edge annihilation/creation event.

Using c, we define a linear automorphism $\mathcal{F}_a$ of $\mathcal{V}$ which corresponds to the edge-flip. Suppose the two vertices at either end of the removed edge e are indexed by, say, $r < s$. Let the ends of the added edge be indexed by $t < u$. Now we define $\mathcal{F}_a$ on $(b_1 \otimes \cdots \otimes b_n)$ where each b_i is $+$ or $-$, which is the canonical tensor-product basis, by $(b_1 \otimes \cdots \otimes b_n) \mapsto (b'_1 \otimes \cdots \otimes b'_n)$, where $b'_t = \tau b_s$, $b'_u = \tau b_r$ and for $j \neq t, u$, $b'_j = b_k$ for $k = \sigma_a^{-1}(j)$.

The effect of $\mathcal{F}_a$ is to interweave c with the identity according to the topological reindexing, and one can plainly do the same thing with c'. Define $\mathcal{F}'_a$ for any a on the path analogously with $\mathcal{F}_a$ by replacing c with c'; so

$\mathcal{F}'_a(b_1 \otimes \cdots \otimes b_n) = (b'_1 \otimes \cdots \otimes b'_n)$, where $b'_t = \tau b_r$, $b'_u = \tau b_s$ and for $j \neq t, u$, $b'_j = b_k$ for $k = \sigma_a^{-1}(j)$. We can now define the arc operator ϕ_a corresponding to a as the *difference* of these two endomorphisms; $\phi_a := \mathcal{F}_a - \mathcal{F}'_a$ for any edge a in $\mathcal{T}_n$. The following is straightforward [18].

Lemma 1 *If e is an edge in some tree T in $\mathcal{T}_n$ with $r < s$ the indices of its two endpoints and a is the arc in the graph in $\mathcal{D}_n$ corresponding to flipping the edge, then $\phi_a(b_1 \otimes \cdots \otimes b_n) = 0$ if and only if $b_r = b_s$.*

Let us write a path P in $\mathcal{D}_n$ joining a pair of trees T, T' as a sequence of arcs $a_1, \ldots, a_r$, ($r = 0$ if $T = T'$). Then P induces the endomorphism $\Phi_P = (\mathcal{F}_{a_r} \circ \cdots \circ \mathcal{F}_{a_1}) - (\mathcal{F}'_{a_r} \circ \cdots \circ \mathcal{F}'_{a_1})$ of $\mathcal{V}_n$. If $r = 0$, let this be the zero map and if $r = 1$, note that Φ_P is just ϕ_{a_1}. Also, for any path $a_1, \ldots, a_n$, a *partial path* is a path of the form $a_1, \ldots, a_s$, where $0 \leq s \leq r$.

The following conjecture implies the Four Color Conjecture:

Conjecture 1 *For any two rooted cubic plane trees T, T' with n cubic nodes, there exists a path P in $\mathcal{D}_n$ joining T and T' such that for every partial path P' of P, $\Phi_{P'}$ is not a monomorphism.*

This would mean that there is a path P from T to T' such that for every partial path P' some tensor basis element of T is mapped to zero by $\Phi_{P'}$ so the path successively induces edge 3-colorings of the fusion of T with each of the trees occurring along the path. This does not imply, however, that a single choice of tensor basis element in T suffices.

7 Properties of the octonions

Looking for a new proof of the Four Color Conjecture, or its strong form above, one is naturally tempted to consider the infinite case. This ought to allow techniques of functional analysis to be used. Further,in the infinite-dimensional case, one could use an octonionic, rather than complex or quaternionic Hilbert space.

The octonions should have just the right properties as a substrate for the study of subtle properties of associativity in tensor product algebras. It is well-known that octaves form an *alternative* algebra; associativity holds when two of the three terms are equal; see, e.g., Schafer [31]. However, Albuquerque and Majid [1] have proved that the octonions are associative up to a natural transformation. Their work uses ideas from quantum algebra and from Mac Lane's theory of coherence in categories [20, p. 291].

The introduction of a thoroughly octaval viewpoint into the *topos* itself ought to have very interesting consequences for the enterprise of building a categorical model of continuum mechanics (see Lawvere and Schanuel [26].

One might expect to find hyperbolic geometry because the infinite rooted cubic plane tree has a natural hyperbolic realization. This is also supported by the case of the two extremal trees (corresponding to left-most or right-most parenthesizations, resp.). If these two trees, say L_n and R_n, are considered in dual form, the result of the fusion process becomes the cube of a path, P_n^3, and we remarked on its hyperbolic nature in [17]. Further, hyperbolic geometry was used by Sleator, Tarjan and Thurston [32] to majorize the diameter of $\mathcal{T}_n$.

The *cube* P_n^3 of the path [15] is the graph with nodes $\{1, \ldots, n\}$ in which nodes are adjacent if and only if they differ by at most 3. For any given scale s, $0 < s < 1$, there is a linear embedding of P_n into the plane which extends to a linear embedding of P_n^3 where the image of 1 is the point $(1, 0)$ in $\mathcal{R}^2$ and the image of $k + 1$ is $s\omega$ times the image of k, where ω is a third root of unity in the complex plane [17].

Time as a separable real axis can be formalized as a path or cycle graph (as done for a chain of atoms in [2, pp. 73-77]). In [17] we studied properties of the path cube and their consequences for a model of time.

However, if instead of replacing one axis, we replace all four by corresponding "screw" axes, what is the corresponding geometry? Suppose we start with a four-dimensional model and then replace each axis with a complex plane, but using the spiral embedding of some fixed scale s. Is there an interesting (e.g., highly connected) spatial subgraph with nodes at integral octonions for which the intersection with the axial screws are just the path-cubes? Every graph embeds linearly in only 3 dimensions so with 8 dimensions, one has a great deal of room.

Note that already the existence of an order-3 mapping along each axis implies that there must be a *triality* on the whole space. Triality is to duality as three is to two. E. Cartan defined it in terms of a linear automorphism with cube (i.e., three-fold composition with itself) equal to the identity and cyclically permuting three subspaces in a direct-sum decomposition of the vector space. See Crumeyrolle [10]. There is a specific automorphism of the group $Spin(8)$ which exhibits triality [11]. A three-fold symmetry in the Four Color Conjecture arises by cyclically permuting the colors. The idea of triality also arose in graph theory regarding a question of N. Hartsfield.

Finally, some interesting questions remain regarding optimization. Since the octonions are not associative, given a sequence of numbers, their product depends on how the numbers are associated. If one chooses, uniformly at random in the interval $[0.5, 2.0]$ a sequence $r_1, r_2, \ldots, r_n$ and then add an octonion chosen uniformly at random from the ball of radius ε in octonion space to form a sequence $c_1, c_2, \ldots, c_n$ of octonions, we can ask for the largest or the expected value of any product of the sequence, taken over all possible ways to associate the elements or even allowing reorderings. One could also begin by choosing the sequence of octonions subject to the property that $((\ldots(c_1c_2)c_3)\ldots c_{n-1})c_n) = 1$.

Optimization questions can also be asked regarding the norm of partial sums of zero-mean samples of unit vectors. An upper bound for the plane is $\sqrt{5}$ but for octonions the best upper bound known is approximately 147.

8 Remarks

According to Theorem 2, the Four Color Conjecture is equivalent to the propagation of twisting vectors in a network composed of two rooted cubic plane trees. If we consider gravity as the tendency to propagate twisting vectors, then the four color theorem amounts to the assertion of the existence of a planar gravitational field. Simultaneously, our tree-model also displays the archetype of a quantum wave, which spreads out over all possibilities but collapses into an actual coloring.

The rooted cubic plane trees may be viewed as a kind of artificial life where the coloring problem is equivalent to the ability of a tree to find a suitable sequence of $+/-$ states for its nodes so that it can control the movement of another tree (mate or predate). If the tree A-life is capable of carrying out a quantum computation, it can always make such a choice.

If the Conjecture holds, then it provides a mathematical model of quantum gravity in the sense of being able to transmit inertial force. So perhaps the physical model could be accessed biologically.

9 Appendix: Definition of the octonions

The octonions, also known as the octaves or Cayley-Graves numbers, are an algebraic structure defined on the 8-dimensional real vector space such that two octaves can be added, multiplied and divided, except that multiplication is neither commutative or associative. Otherwise, all the other expected properties hold such as distributivity. The octaves have extremely interesting algebraic, combinatorial and geometric properties; see, e.g., Coxeter [8], [7], [9] or Schafer [31] for further details.

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