Extension of a Theorem of Whitney*

Paul C. Kainen (Washington, DC)
Shannon Overbay (Spokane, WA)

Abstract

It is shown that every planar graph with no separating triangles is a subgraph of a Hamiltonian planar graph; that is, Whitney's theorem holds without the assumption of a triangulation.

Key words. Hamiltonian planar graph, book thickness, separating triangle.

1 Main result

For k a positive integer, let B_k be the union of k closed half-planes (the *pages*) intersecting in a line L (the *spine*) which is the boundary of each of the pages. A *k-page book embedding* of a graph $G = (V, E)$ is an embedding of G into B_k with vertices mapped to the spine and with edges intersecting the spine only at their endpoints [14, p. 97].

The *book thickness* (also called "pagenumber") of G is the least number of pages in which G has a book embedding. A graph G has book thickness at most 2 if and only if G is a subgraph of a planar Hamiltonian graph [4].

Book thickness has been used as a model for complexity in computer science (e.g., [6], [9]), traffic flow [15], and RNA folding [12]; relations to other invariants have been studied [7], [15], [10].

A *block* is a maximal 2-connected subgraph. It is well-known that for any graph, the genus of the graph is the sum of the genuses of its blocks [3]. For book thickness, it is not difficult to show that maximum replaces sum:

*to appear in Appl. Math. Lett.

The book thickness of a graph is the maximum of the book thicknesses of its blocks. This result completes the proof of Theorem 1 below.

Whitney [18] proved that every planar triangulation with no separating triangles is Hamiltonian. His result can be extended to non-triangulations.

Theorem 1 *Every plane graph with no separating triangle is a subgraph of a Hamiltonian plane graph.*

The approach here, based on Overbay's thesis [17], uses connectivity. To *stellate* a face of a plane graph one adds a new vertex in the face and joins it by a homeomorphic copy of a star graph to each vertex in the boundary of the face.

Since there exist maximal planar graphs which are not Hamiltonian, this is the strongest possible extension. Note that one can't just add edges without possibly creating triangles that separate. Similarly, adding a stellating vertex in the interior of a face can produce a separating triangle. By adding a somewhat more complex subgraph into the faces, an alternative procedure will result in a triangulation with no separating triangles, but this requires additional vertices and edges.

Proof.

(1) *Every* 3*-connected plane graph with no separating triangles is a subgraph of a Hamiltonian graph.*

Stellate the nontriangular faces. Any separating triangle would need to use one of the new stellating vertices and so the two other vertices would be a separating pair for the original graph. Hence, the resulting triangulation has no separating triangles, which suffices by Whitney's theorem.

(2) *Every* 2*-connected plane graph with no separating triangles is contained in a* 3*-connected plane graph with no separating triangles.*

This is proved by induction on the number k of separating 2-sets. Let G be a 2-connected plane graph. If $k = 0$, then G is already 3-connected. Let $k \geq 1$ and choose $A = \{u, v\}$ a separating 2-set so that $G - A = G_1 \cup \ldots \cup G_n$ where the G_j denote connected components listed in the clockwise order of the edges joining them to v. One then adds vertices and edges in a specific way ([16] and [17]) creating a 2-connected plane graph G' with no separating triangles where $G \subset G'$ and G' has fewer separating 2-sets than G. □

The extension of Whitney's Theorem includes previous results: Explicit proofs were given in [6] that "X-trees" and "square-grids" have book thickness at most 2; these graphs are planar and have no separating triangles. Planar bipartite graphs were shown to be 2-page embeddable in [11], see also [9]; and [1, p. 28] noted the 2-page embeddability of triangle-free planar graphs.

2 Applications

Barnette asked if every cubic 3-connected planar bipartite graph is Hamiltonian; see, e.g., [13]. By Theorem 1, such a graph is a subgraph of a Hamiltonian planar graph.

Two graphs are homeomorphic if they have isomorphic subdivisions. It is well known that every graph is homeomorphic to a graph of book thickness at most three [2], [4]. For planar graphs, two pages suffice up to homeomorphism.

Corollary 2 *A graph is planar if and only if it is homeomorphic to a graph of book thickness at most two.*

Proof. Subdivide at least one edge in each separating triangle. □

Corollary 3 *Every planar graph is homeomorphic to the union of two outerplanar graphs.*

This is weaker than the conjecture of Chartrand, Geller, and Hedetniemi [5] that every planar graph has a two-fold edge-partition such that each subset induces an outerplanar subgraph. See also [8].

References

[1] D. Archdeacon, Topological graph theory: a survey, *Congr. Numer.* **115** (1996) 5-54, http://www.emba.uvm.edu/ archdeac/papers/survey.ps

[2] G. Atneosen, **On the embeddability of compacta in n-books: intrinsic and extrinsic properies**, Ph.D. Dissertation, Michigan State University, 1968.

[3] J. Battle, F. Harary, Y. Kodama, and J. W. T. Youngs, Additivity of the genus of a graph, *Bull. Amer. Math. Soc.* **68** (1962), 569-571.

[4] F. Bernhart and P. C. Kainen, The book thickness of a graph, *J. Combin. Theory* Ser. B **27** (1979) no. 3, 320–331.

[5] G. Chartrand, D. P. Geller, and S. Hedetniemi, Graphs with forbidden subgraphs, *J. Combin. Theory* Ser. B **10** (1971) 12-41.

[6] F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg, Embedding graphs in books: A layout problem with applications to VLSI design, *SIAM J. on Alg. and Disc. Methods* **8** (1987) no. 1, 33–58.

[7] A. M. Dean and J. P. Hutchinson, Relations among embedding parameters for graphs, in **Graph Theory, Combinatorics, and Applications**, vol. I, Wiley-Interscience Publ., New York, 1991, pp. 287–296.

[8] G. Ding, B. Oporowski, D. P. Sanders, and D. Vertigan, Surfaces, tree-width, clique-minors, and partitions, *J. Combin. Theory* Ser. B **79** (2000), no. 2, 221–246.

[9] V. Dujmovic and D. R. Wood, On linear layouts of graphs, *Discrete Mathematics and Theoretical Computer Science* **6** (2004) 339–358.

[10] M.B. Dillencourt, D. Eppstein, and D. S. Hirschberg. Geometric thickness of complete graphs. *Journal of Graph Algorithms and Applications* **4** 5–17 (2000). http://citeseer.ist.psu.edu/dillencourt00geometric.html

[11] H. P. de Fraysseix, P. Ossana de Mendez, and J. Pach, A left-first search algorithm for planar graphs, *Discrete Comput. Geom.* **13** (1995) 459–468.

[12] P. M. Gleiss and P. F. Stadler, Relevant cycles in biopolymers and random graphs, Tech. Rep. 99-07-042, Santa Fe Institute, Santa

Fe, NM, 1999, 4th Slovene International Conference in Graph Theory, Bled, June 28 – July 8, 1999.

[13] D. Holton, B. Manvel, B. D. McKay, Hamiltonian cycles in cubic 3-connected bipartite planar graphs, *J. Combin. Theory Ser. B* **38** (1985) 279-297.

[14] P. C. Kainen, Some recent results in topological graph theory, in **Graphs and combinatorics**, R. A. Bari and F. Harary, Eds., Springer Lecture Notes in Math. **406**, Berlin, 1974, pp. 76–108.

[15] P. C. Kainen, The book thickness of a graph, II, *Congr. Numer.* **71** (1990) 127-132.

[16] P. C. Kainen and S. Overbay, Book embeddings of graphs and a theorem of Whitney, Tech. Report GUGU-2/25/03, http://www.georgetown.edu/faculty/kainen/pbip3.pdf.

[17] S. Overbay, Generalized book embedings, Ph. D. Dissertation, Colorado State University, Fort Collins, CO, 1998.

[18] H. Whitney, A theorem on graphs, *Ann. of Math.* **32** (1931) 378–390.

Department of Mathematics, Georgetown University
Washington, DC 20057 U.S.A.
kainen@georgetown.edu (corresponding author)

Department of Mathematics, Gonzaga University
Spokane, WA 99258 U.S.A.
overbay@gonzaga.edu