Circular layouts for crossing-free matchings

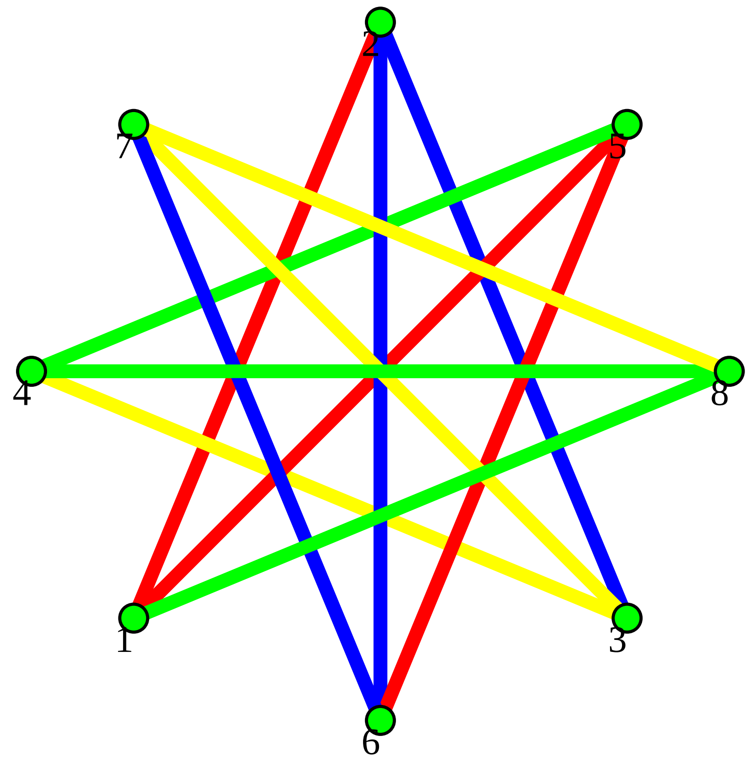
Paul C. Kainen
Department of Mathematics and Statistics
Georgetown University

Some of this material was presented at Knots in Washington XXIX, Dec. 2009. See: *On book embeddings with degree-1 pages*, submitted for publication.

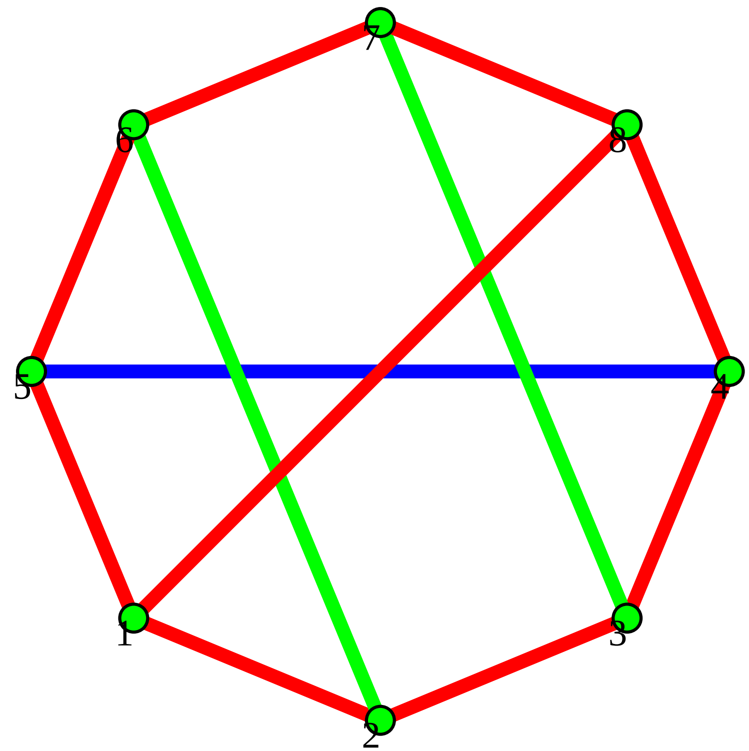
Let $G = (V, E)$, ω cyclic order on $V(G)$. An edge decomposition

$$\mathcal{E} : E = E_1 + E_2 + \cdots + E_k$$

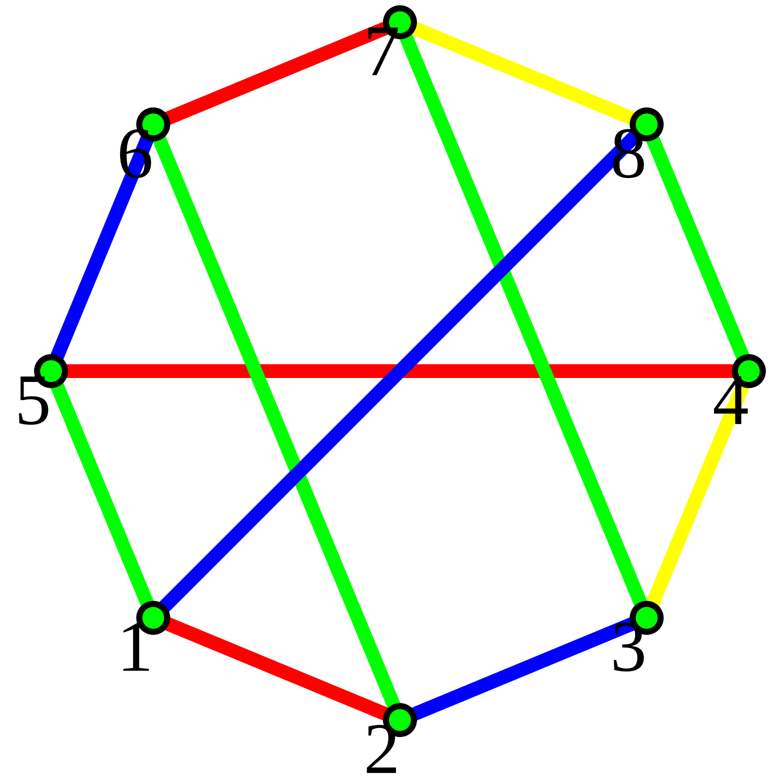
is a *book embedding* of (G, ω) if the *pages* $G(E_i)$ are outerplane wrt ω, all i. Let $bt(G, \omega)$ be least number of pages in any book embedding of (G, ω). For G_0 below, $bt(G_0, \{7, 4, 1, 6, ...\}) = 4$.



Let $bt(G) := min_\omega bt(G, \omega)$. For the graph G_0 above $bt(G_0) = bt(G_0, \{1, 2, 3, 4, 8, 7, 6, 5\}) = 3$.



A book embedding $(\mathcal{E}, \omega)$ of G is *matching* if the pages have maximum degree 1. In fact, $mbt(G_0) = 4$,



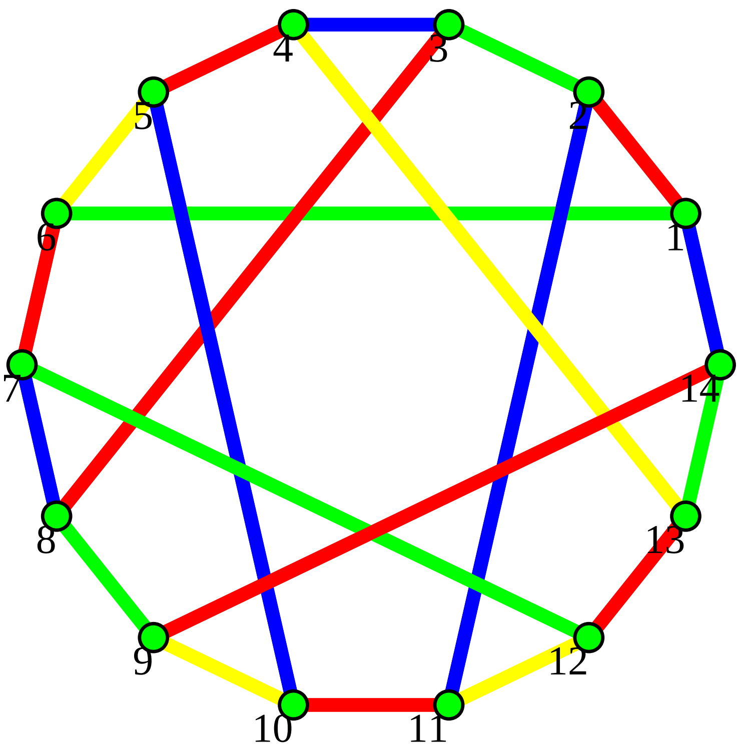
where mbt, etc., denotes the invariants with degree-1 pages.

By definition, $mbt(G) \geq \chi'(G) \geq \Delta(G)$, where χ' is *chromatic index* (least number of colors needed to keep all adjacent edges differently colored) and $\Delta(G)$ is max degree. Vizing showed that $\chi' \leq 1 + \Delta$, and $\chi' = \Delta$ when G is bipartite. Call G *dispersible* if $mbt(G) = \Delta(G)$.

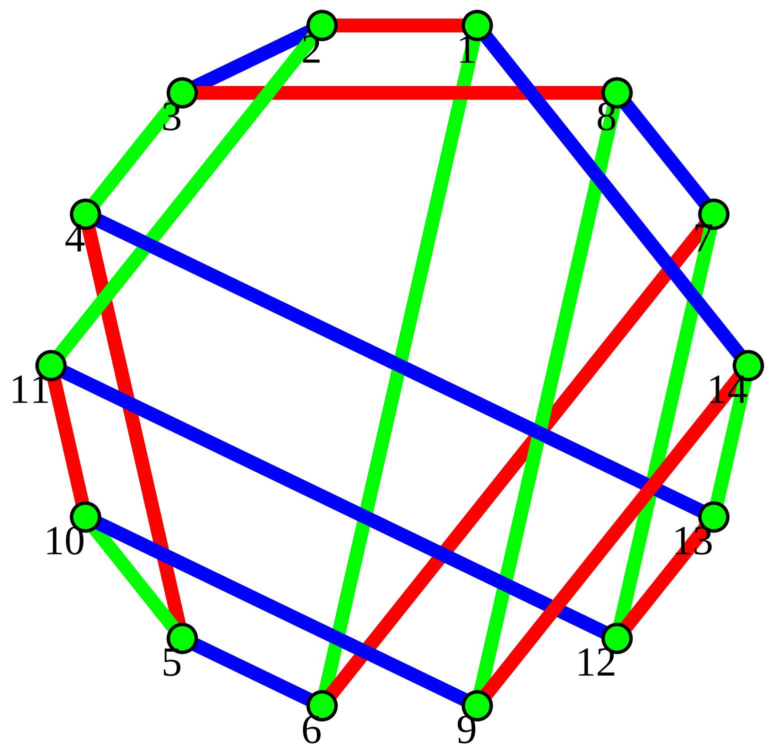
Conjecture (Bernhart and Kainen, 1979):
Every bipartite graph is dispersible.

For regular graphs, bipartiteness is necessary for dispersibility (Overbay, 1998). The Heawood graph (bipartite, cubic with 14 vertices) satisfies the conjecture. Of $100,000$ vertex orderings tested, exactly three had $mbt = 3$.

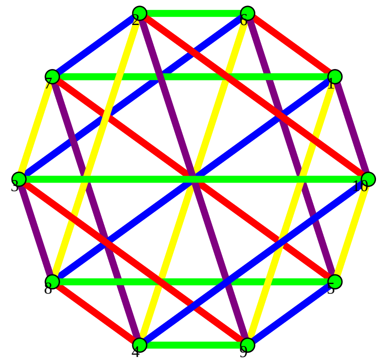
Here is the Heawood graph H with "naive" order, with $mbt = 4$.



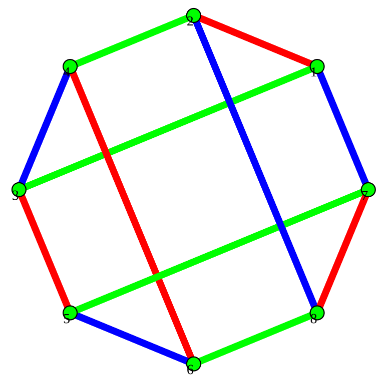
Here is a vertex order for H with $mbt = 3$.



The dispersability conjecture also holds for regular complete bipartite graphs $K_{p,p}$



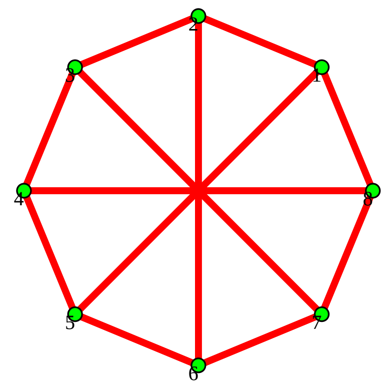
and for hypercubes Q_d



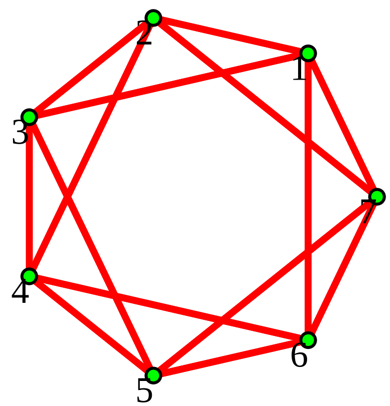
So far, the evidence supports the following conjecture:

Matching book thickness equals chromatic index for all regular graphs.

As test cases, we consider $c(n,k) := (V,E)$, where $V = \{1,\ldots,n\}$ and $E = \{ij : d(i,j) \in \{1,k\}\}$, $2 \leq k \leq n/2$, where $d(i,j)$ is distance along C_n. For instance, $c(8,4)$ is

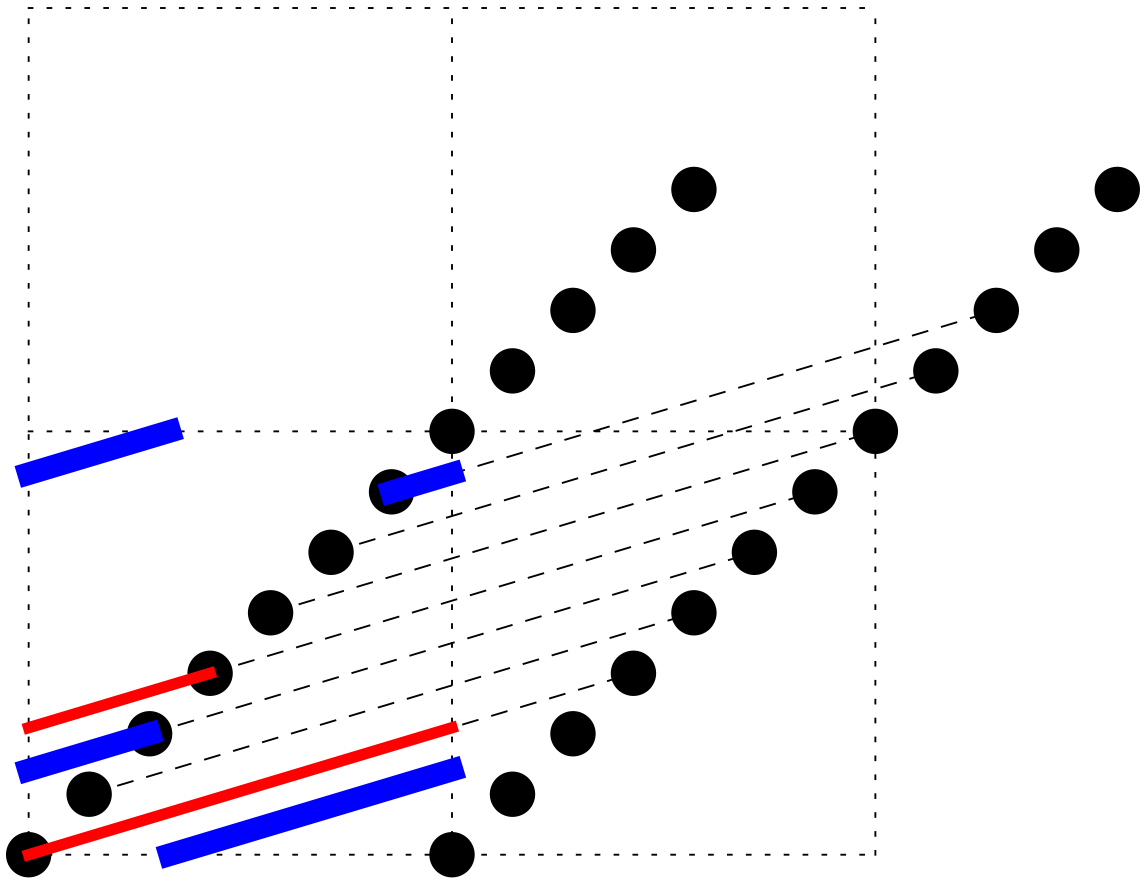


while $c(7,2)$ is



$c(n, k)$ is bipartite iff n even and k odd; $c(n, k)$ is planar if n is even and $k = 2$. Also, $c(n, k)$ is toroidal for all n, k (n should be 7 not 10 in the caption; cycle edges would lie on main diagonal).

C(n,k) is toroidal; n=10, k=3



Theorem: For $n = 2k + r$, $k \geq 2$ and $0 \leq r \leq 3$,

$$mbt(c(n,k)) = mbt\Big(c(n,k), cycPrm(n,k)\Big) = \Delta(c(n,k)) + 1 - b,$$

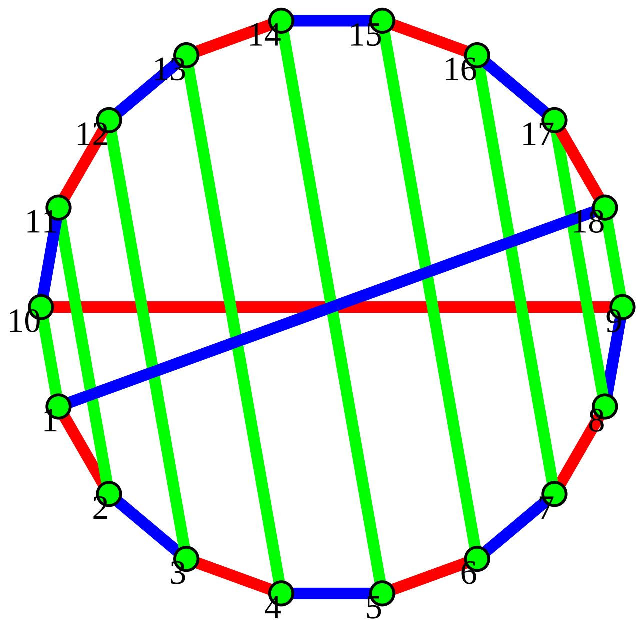
where $b = 1$ if $c(n,k)$ is bipartite, $b = 0$ otherwise and $cycPrm(n,k)$ is the vertex order $\{k, k-1, \ldots, 1, k+1, k+2, \ldots, n\}$.

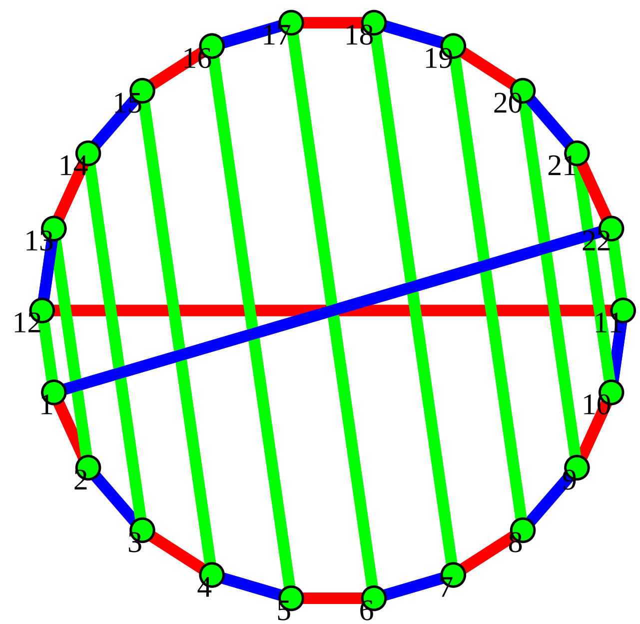
When $b = 0$, the extra page contains a *fixed* number of edges depending on the residue class of k (mod 4), while the first Δ pages have number of edges increasing with k. Thus, these matching book embeddings are *almost* Δ-page.

One could also consider $mbt_{max}(G) := \max_\omega mbt(G, \omega)$. But $mbt_{max}(c(2k,k)) \geq mbt(c(2k,k), \{1, 2, \ldots, 2k\}) = k$.

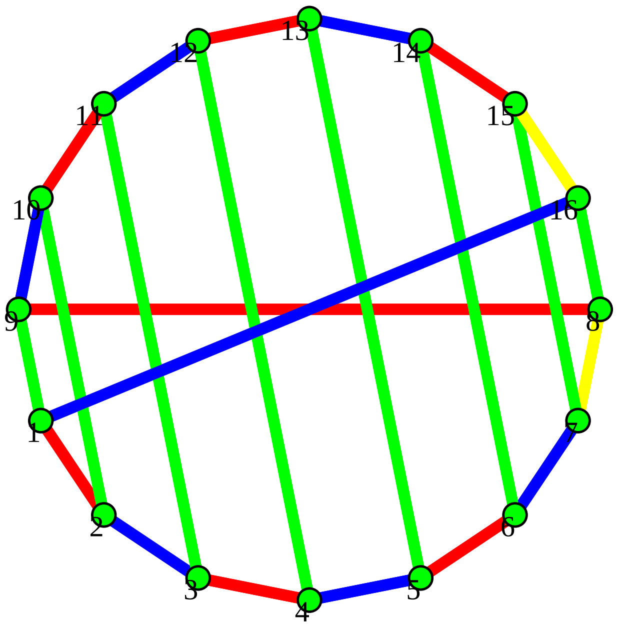
The following slides show that a regular layout into pages holds for the class of circulants given in the theorem above. For $C(n, 2)$ similar results hold with the fifth page requiring 2 edges when n is odd, 1 edge when $n \equiv 0 \pmod 4$, and 3 edges when $n \equiv 2 \pmod 4$.

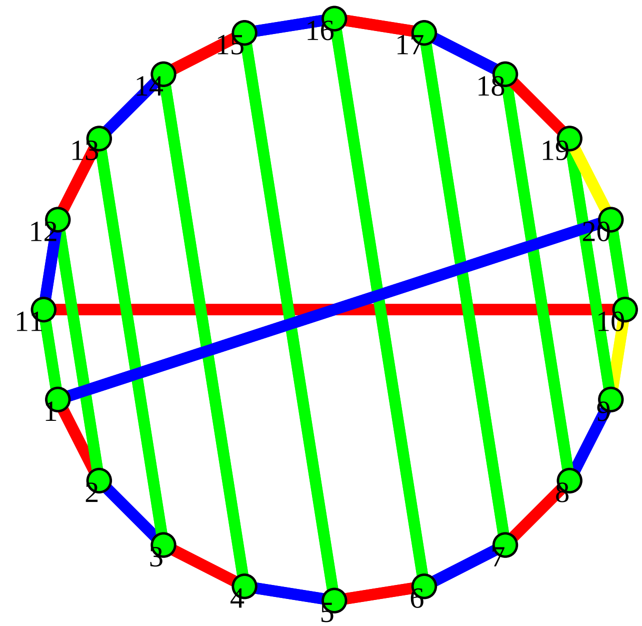
$c(2k,k)$, k odd; bipartite and 3-page



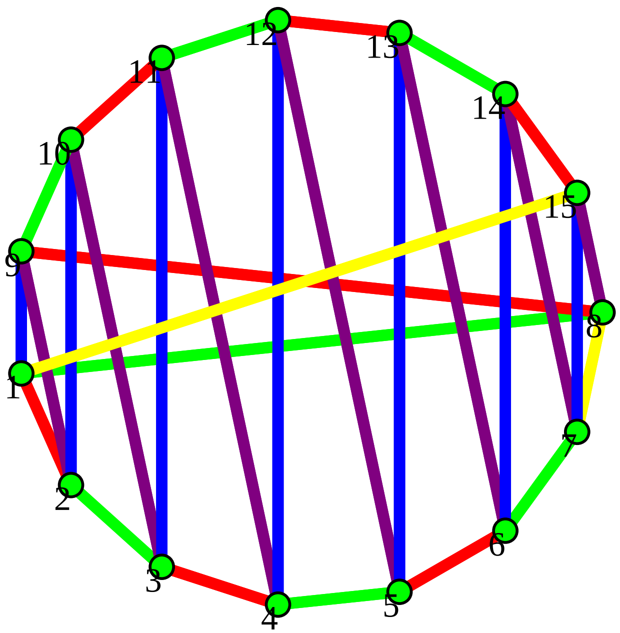


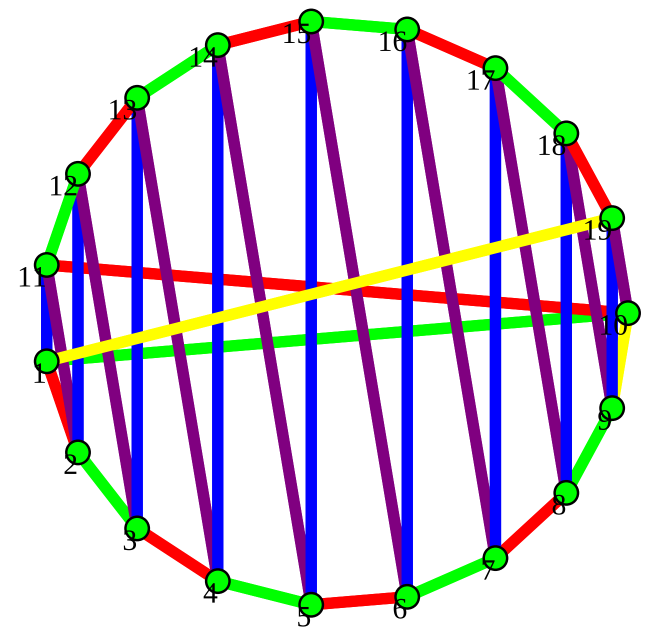
$c(2k, k)$, k even; not bipartite and 4-page with only two edges on the extra page.



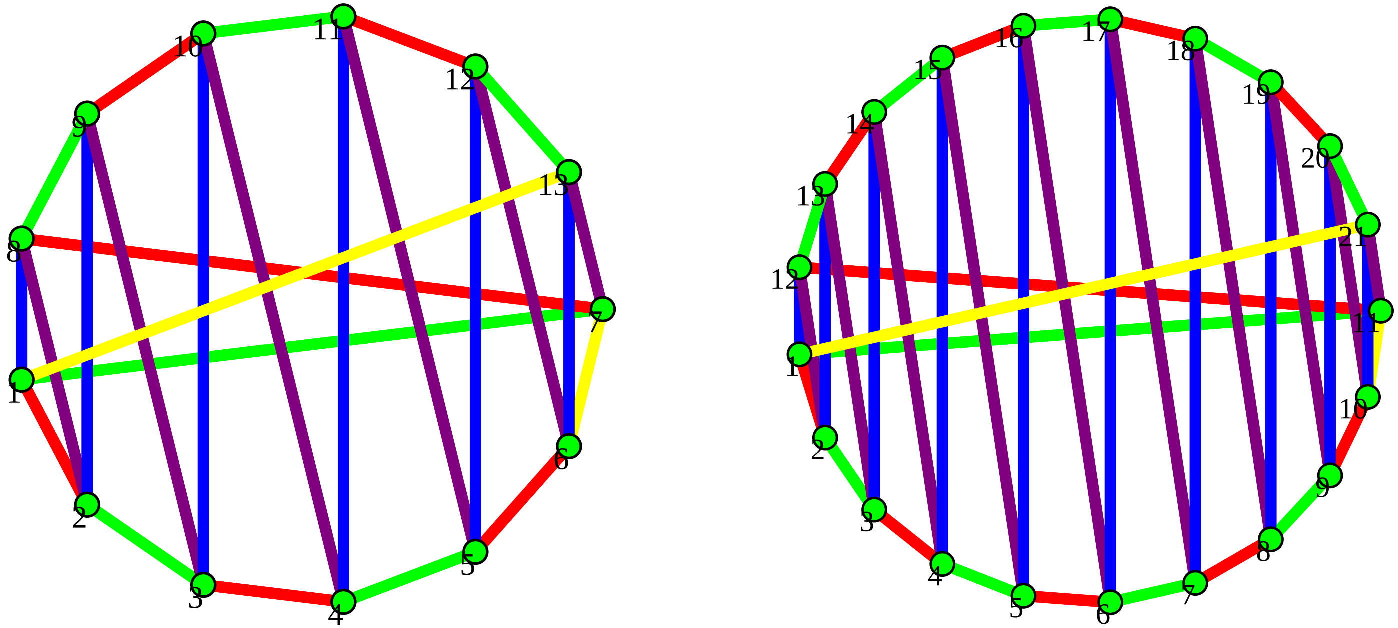


$c(2k+1,k)$, k odd; not bipartite and 5-page with only two edges on the extra page.

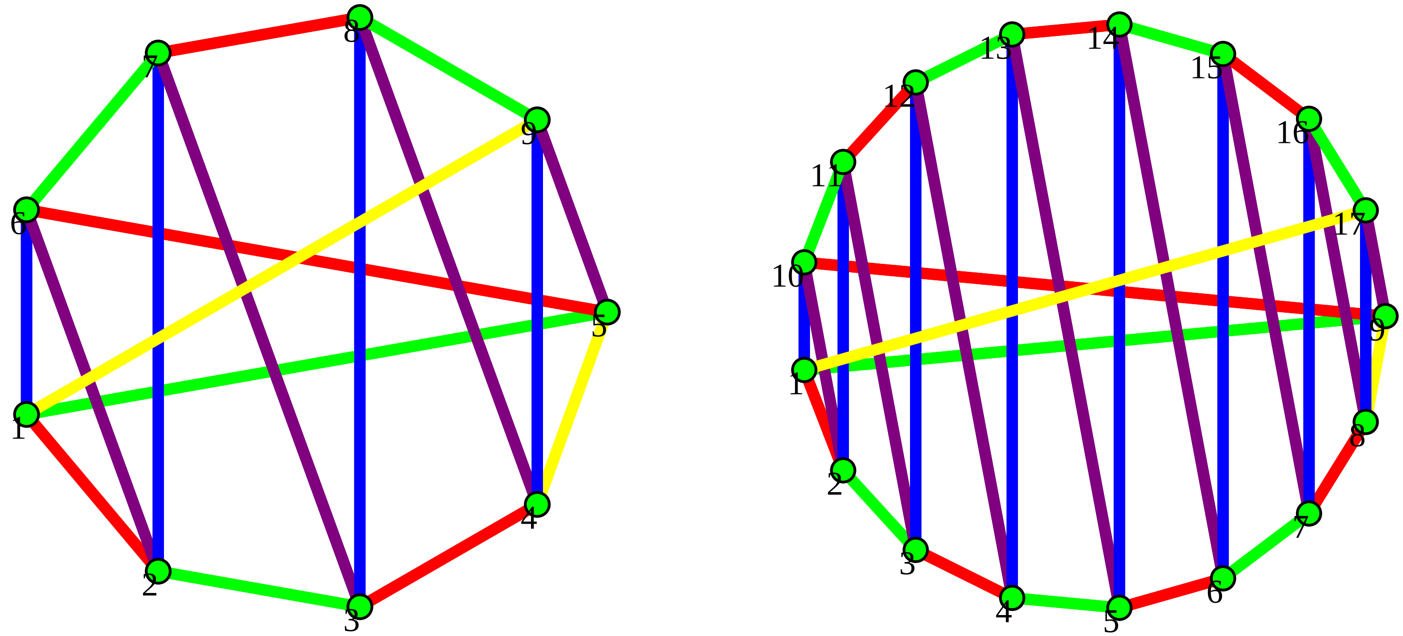




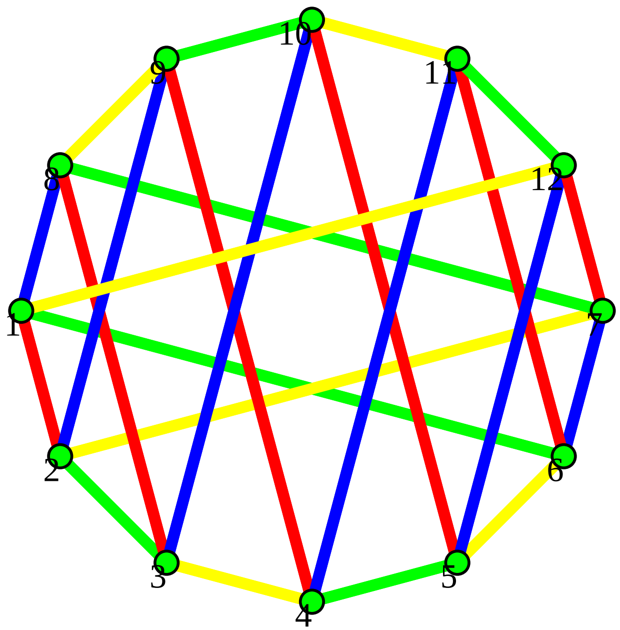
$c(2k+1,k)$, $k \equiv 2 \pmod 4$; not bipartite and 5-page with only two edges on the extra page.

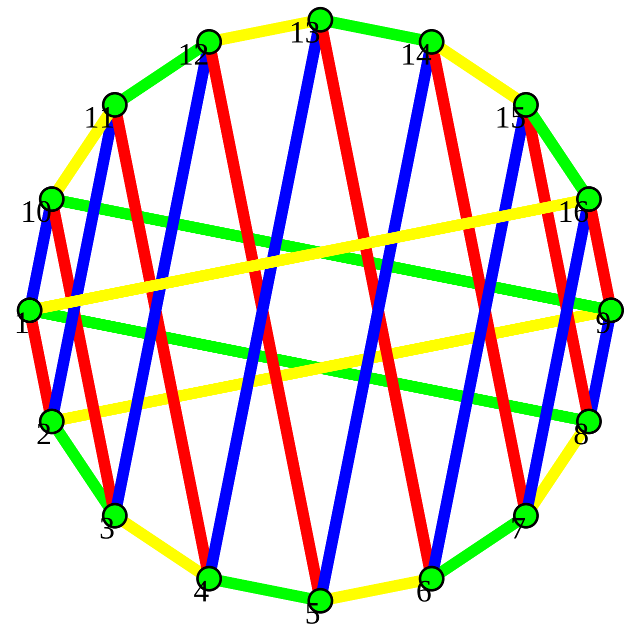


$c(2k+1,k)$, $k \equiv 0 \pmod 4$; not bipartite and 5-page with only two edges on the extra page.

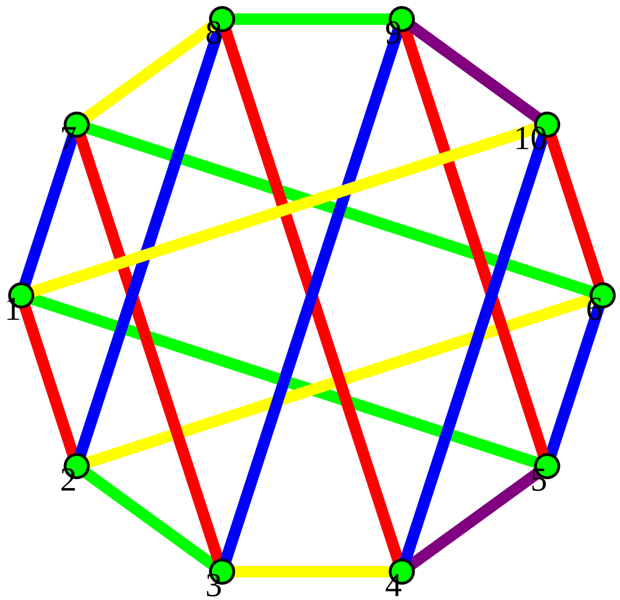


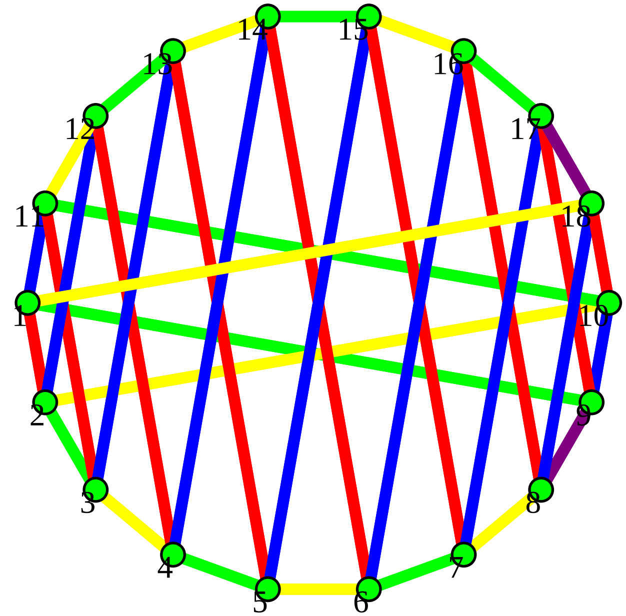
$c(2k+2,k)$, k odd; bipartite and 4-page.



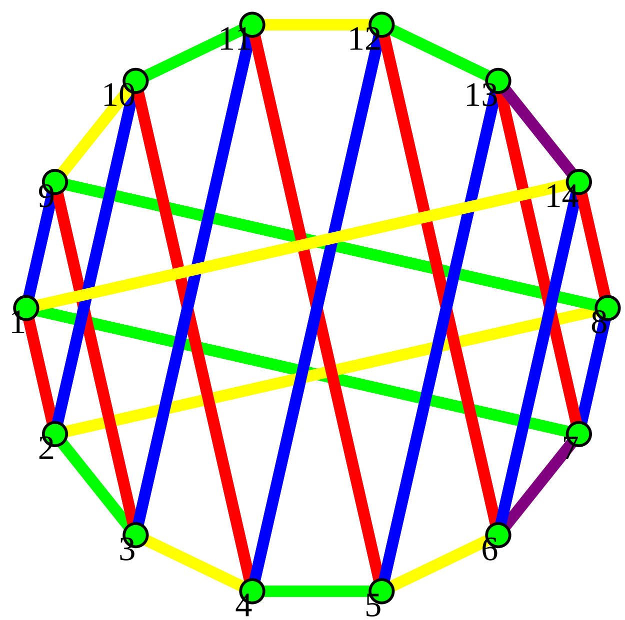


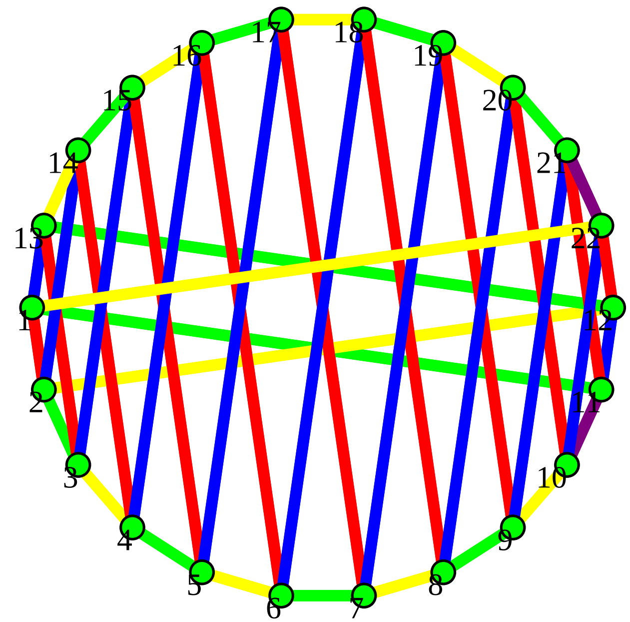
$c(2k+2,k)$, $k \equiv 0 \pmod 4$; not bipartite and 5-page with two edges on the extra page.



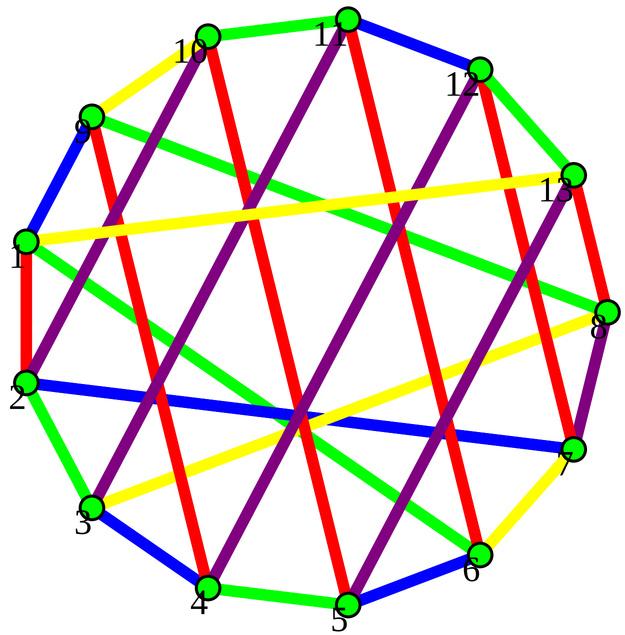


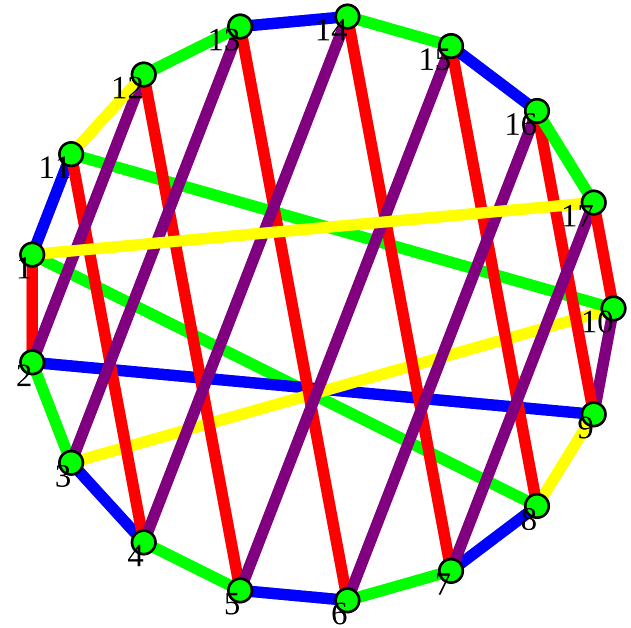
$c(2k+2,k)$, $k \equiv 2 \pmod 4$; not bipartite and 5-page with two edges on the extra page.



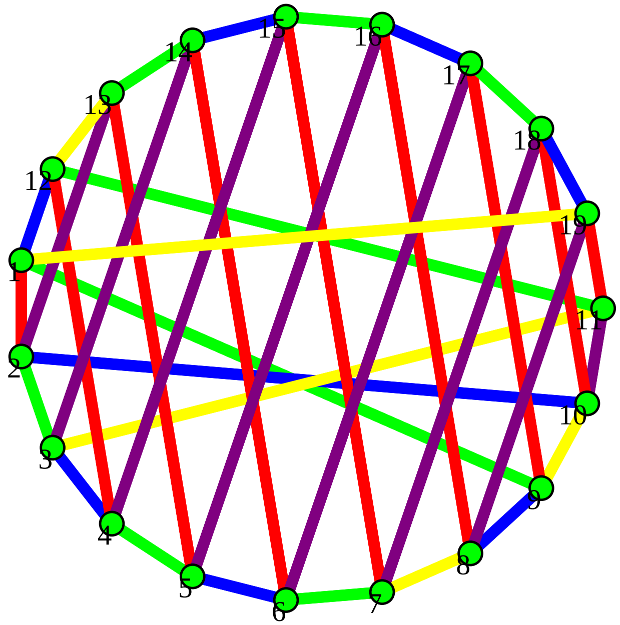


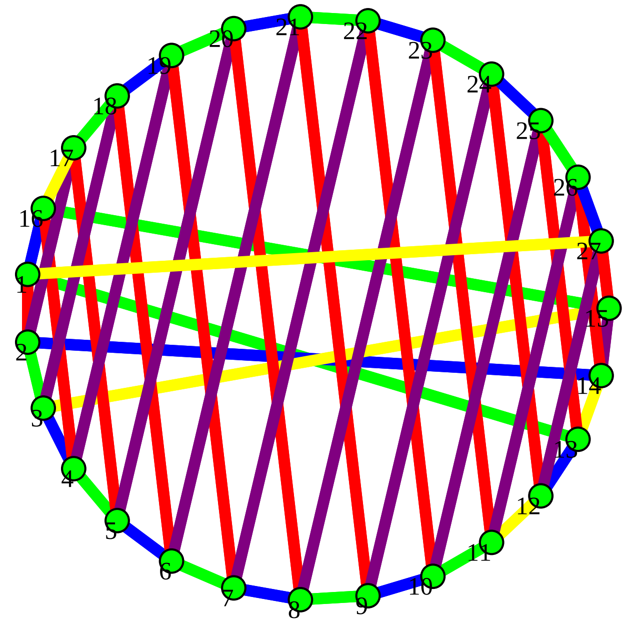
$c(2k+3, k)$, k odd; not bipartite and 5-page with four edges on the extra page.



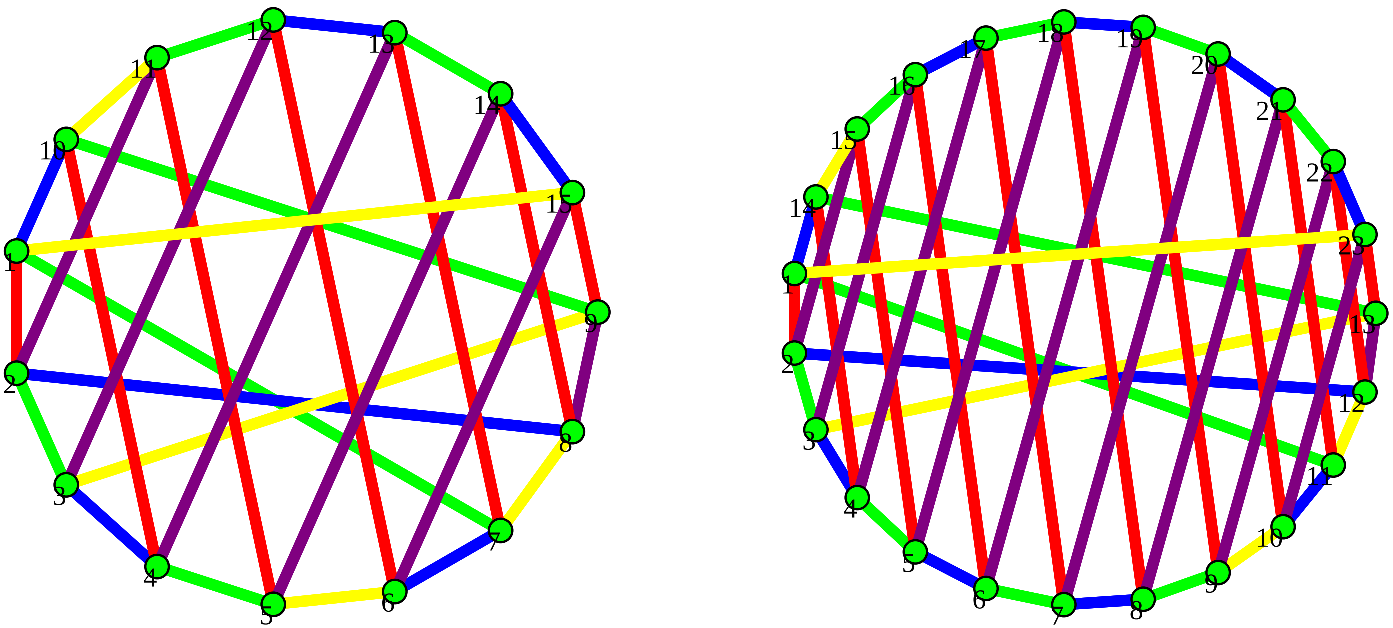


$c(2k+3, k)$, $k \equiv 0 \pmod 4$; not bipartite and 5-page with five edges on the extra page.





$c(2k+3,k)$, $k \equiv 2 \pmod 4$; not bipartite and 5-page with five edges on the extra page.



References

Bernhart, F. R. & Kainen, P. C., On the book thickness of a graph, *J. Comb. Th.*, Ser. B, **27** (1979) no. 3, 320–331.

Kainen, P. C. Some recent results in topological graph theory, in **Graphs and combinatorics** (Proc. GWU Conf., 1973), Bari, R. A., Harary, F., Eds., Springer Lecture Notes in Math. 406, Berlin, 1974, pp. 76–108.

Kainen, P. C., Thickness and coarseness of graphs, *Abh. Math. Sem. U. Hamburg* **39** (1973) 88–95.

Kainen, P. C., The book thickness of a graph, II, *Congr. Num.* **71** (1990) 127–132.

Overbay, S., Generalized book embeddings, Ph.D. Dissertation, Colorado State University, Fort Collins, Co, 1998.