# Math 035 Final Examination, Spring 2005

Name: Solution.

Instructor:

Attention:

1) All cell phone must be switched off and kept in your bag,
Or you can leave your phone at the teacher's desk
if you are afraid of missing important calls.
2) Exact numbers, such as $1/3$, $\sqrt{2}$ and $\sin 1$, rather than their approximations, such as 0.333, 1.414... etc., are preferred as answers.
3) Must show some detail rather than just show the final answer.
4) Calculators are not allowed.

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| Page #of pb |
| 1 (9) |
| 2 (19) |
| 3 (22) |
| 4 (10) |
| 5 (10) |
| 6 (10) |
| 7 (10) |
| 8 (10) |
| Total (100) |

(1) Find the following limits.

a) (3pts) $\lim_{x\to 1} \frac{x}{2+x}$.

$$= \frac{1}{3}$$

b) (3pts) $\lim_{x\to 2} \frac{x^2+3x-10}{x-2}$.

$$= \lim_{x\to 2} \frac{(x-2)(x+5)}{(x-2)} = \lim_{x\to 2} x+5 = 7$$

c) (3pts) $\lim_{x\to\infty} \frac{\sqrt{x^2+3x-10}}{x-2}$.

$$= \lim_{x\to\infty} \frac{|x|\sqrt{1+3/x-10/x^2}}{x(1-2/x)} \overset{x>0}{=} \lim_{x\to\infty} \frac{x\sqrt{1+3/x-10/x^2}}{x(1-2/x)}$$

$$= \frac{\sqrt{\lim_{x\to\infty} 1 + 3\lim_{x\to\infty}\frac{1}{x} - 10\lim_{x\to\infty} 1/x^2}}{\lim_{x\to\infty} 1 - 2\lim_{x\to\infty} 1/x} = 1$$

d) (3pts) $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$. This limit does not exist since

$$\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1^+} x+1 = 2$$

$$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{(x+1)(x-1)}{-(x-1)} = -\lim_{x\to 1^-} x+1 = -2$$

e) (5pts) $\lim_{x\to 0^+} \sin(x)^{\frac{1}{\ln(x)}}$.

First, we consider $\lim_{x\to 0^+} \ln\left[\sin(x)^{\frac{1}{\ln(x)}}\right] = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln x} \overset{\text{l'Hôpital}}{=} \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{1/x}$

$$= \lim_{x\to 0^+} \cos x \cdot \frac{1}{\lim_{x\to 0^+} \frac{\sin x}{x}} = 1\cdot 1 = 1$$

$$\therefore \lim_{x\to 0^+} \sin(x)^{\frac{1}{\ln(x)}} = \lim_{x\to 0^+} e^{\ln\left[\sin(x)^{\frac{1}{\ln(x)}}\right]} = e^{\lim_{x\to 0^+} \ln\left[\sin(x)^{\frac{1}{\ln(x)}}\right]} = e^{1} = e$$

f) (3pts) $\lim_{t\to 1} \int_1^t f(x)dx$

(assuming $f$ is continuous for the level of this course) then $F(t) = \int_1^t f(x)dx$ is differentiable and therefore continuous. Hence

$$\lim_{t\to 1} \int_1^t f(x)dx = 0.$$

(2) Find $F'(x)$ where $F(x)$ is given below.

a) (4pts) $F(x) = x^2 + e^x + x^e + e^\pi + \ln x + \ln 2$.

$$F'(x) = 2x + e^x + e x^{e-1} + \frac{1}{x}.$$

b) (4pts) $F(x) = (x^2+2x-7)^7(\sin x)^8$.

$$F'(x) = 7(x^2+2x-7)^6\cdot(2x+2)(\sin x)^8 + 8(\sin x)^7\cdot(\cos x)(x^2+2x-7)^7.$$

$$= (\sin x)^7(x^2+2x-7)^6\left[7\sin x\,(2x+2) + 8\cos x\,(x^2+2x-7)\right]$$

c) (4pts) $F(x) = \tan^{-1}(x + \sqrt{x})$.

$$F'(x) = \frac{1}{1+(x+\sqrt{x})^2} \cdot \left(1 - \frac{1}{2\sqrt{x}}\right)$$

d) (4pts) $F(x) = \dfrac{x}{1 + \ln(x)}$

$$F'(x) = \frac{1\cdot(1+\ln x) - x(0 + 1/x)}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}$$

e) (5pts) $F(x) = x^x$.

Let $y = x^x$ then $\ln y = \ln(x^x) = x\ln x$

$$\downarrow \frac{d}{dx}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\ln x + x\cdot 1/x = 1 + \ln x$$

$$\therefore \quad \frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x)$$

g) (4pts) $F(x) = \int_1^x \left(e^{t^2 \sin t}\right) dt$.

$$F'(x) \overset{FTC}{=} e^{x^2\cdot\sin x}$$

(3) (5pts) State the definition of $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ provided the limit exists

(4) (10pts) The equation of a curve is

$$x^3 + y^3 = 6xy.$$

a) Show that the point (3, 3) is on the curve.
b) Find the equation of the tangent line to the curve at the point (3, 3).
c) For what value(s) of $x$ (if any) is the tangent line horizontal?

a) Letting $x=3$, $y=3$ in the equation we found

$$3^3+3^3 = 6\cdot 3\cdot 3 \qquad (LHS = 54 = RHS)$$

b) Differentiate $x^3+y^3 = 6xy$ implicitly with respect to $x$ we found

$$3x^2 + 3y^2\frac{dy}{dx} = 6\cdot\left(1\cdot y + x\cdot\frac{dy}{dx}\right)$$

Solving for $\frac{dy}{dx}$ : $\frac{dy}{dx}(3y^2-6x) = 6y-3x^2.$

$$\therefore \frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}$$

$$\left.\frac{dy}{dx}\right|_{x=y=3} = \frac{2\cdot 3-3^2}{3^2-2\cdot 3} = \frac{-3}{3} = -1$$

equation for the tangent line at $(3,3)$ : slope $= -1$ point: $(3,3)$

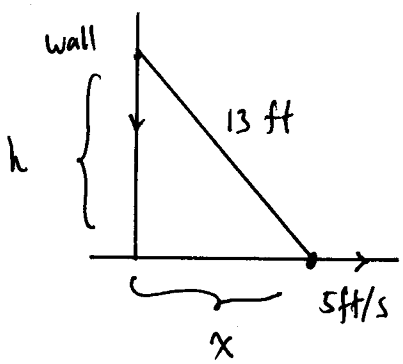
eqn.: $y-3 = -1(x-3)$

c) The tangent line is horizontal if $\frac{dy}{dx}=0 \iff 2y-x^2=0$ or $x^2=2y$

Letting $y=x^2/2$ in $x^3+y^3=6xy$ we found : $x^3+\frac{x^6}{8} = 3x^3$

$\iff x^3(16-x^3)=0 \iff x=0$ or $x=16^{1/3}$

(5) (10pts) A 13-ft ladder is leaning against a wall. Its base is sliding away from the wall at 5 ft/second. How fast is the top of the ladder sliding down when the base is 12 ft away from the wall?




Find $\frac{dh}{dt}$ when $x = 12$ ft.

Given that $\frac{dx}{dt} = 5$ ft/s.

1) Relationship between $x$ and $h$:

$$x^2 + h^2 = 13^2.$$

2) Find $\frac{dh}{dt}$:

$$x^2 + h^2 = 13^2$$

$\downarrow \frac{d}{dt}$ both sides implicitly

$$2x \frac{dx}{dt} + 2h \frac{dh}{dt} = 0$$

$$\therefore \quad \frac{dh}{dt} = -\frac{x}{h} \frac{dx}{dt}.$$

3) When $x = 12$ ft.

if you don't have a calculator

$$h = \sqrt{13^2 - 12^2} = \sqrt{(13+12)(13-12)} = \sqrt{25} = 5$$

$$\therefore \quad \frac{dh}{dt} = \frac{-12}{5} \cdot 5 \text{ ft/s} = -12 \text{ ft/s}.$$

(6) (10pts) The ~~derivation~~ derivative of a function $y = f(x)$ is given as $f'(x) = (x-1)^2(x-2)$.
Find the following for the function $y = f(x)$

(a) critical points of $f(x)$,

i) $f'(x) = 0 \iff (x-1)^2(x-2) = 0 \iff x = 1$ or $x = 2$

ii) $f'(x)$ undefined in the domain of $f$ : none.

$\therefore$ Critical pts: at $x = 1$ and $x = 2$.

(b) intervals over which $f$ is decreasing,

| | | 1 | | 2 | |
|---|---|---|---|---|---|
| $f'$ | $-$ | | $-$ | | $+$ |
| | $f'(0)<0$ | | $f'(1.345)<0$ | | $f'(1000)>0$ |

$f$ is decreasing on $(-\infty, 2)$

(c) intervals over which $f$ is concave upward, and downward.

First, we find $f''(x) = 2(x-1)(x-2) + (x-1)^2 = (x-1)\left[2(x-2)+(x-1)\right] = (x-1)(3x-5)$

$\therefore f''(x) = 0 \iff x = 1$ or $x = 5/3$

$\therefore$ Concave up on $(-\infty, 1)$ and $(5/3, \infty)$

Concave down on $(1, 5/3)$

| | | 1 | | 5/3 | |
|---|---|---|---|---|---|
| $f''$ | $+$ | | $-$ | | $+$ |

(d) $x$-coordinates of inflection points of $f(x)$,

$x = 1$, $x = 5/3$

(e) $x$-coordinates of local maximum points of $f(x)$

There are no local maximum

(f) $x$-coordinates of the absolute maximum points of $f(x)$ over the interval $[-3, 3]$.

We need to compare $f(1)$, $f(2)$ $f(-3)$ and $f(3)$

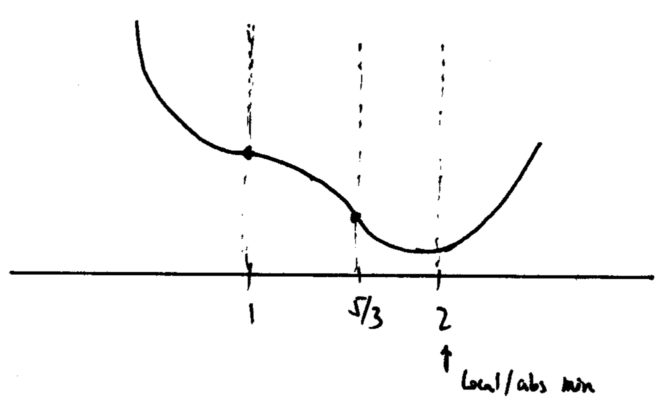
Since $f$ is decreasing on $(-\infty, 2)$ we infer : $f(-3) > f(1) > f(2)$

$f$ is increasing on $(2, \infty)$ " " $f(2) < f(3)$

$\therefore$ abs max on $[-3, 3]$ occurs either at $x = -3$ or $x = 3$ (or both). There isn't sufficient data to determine which one is (or both are) abs max.

(g) Roughly sketch the curve $y = f(x)$ based on information obtained in (a-~~f~~e).

one possibility :



(7) (10pts) Compute the following integrals:

a) $\int \frac{\sqrt{x}+x^4}{x}dx.$

$$= \int \frac{\sqrt{x}}{x}\,dx + \int \frac{x^4}{x}\,dx = \int x^{-1/2}\,dx + \int x^3\,dx = 2x^{1/2} + \frac{x^4}{4} + C$$

b) $\int_0^\pi (\sin x + \cos(2x))\,dx.$

$$= \int_0^{\pi} \sin x\,dx + \int_0^{\pi} \cos(2x)\,dx$$

$u = 2x \quad du = 2dx$

$$= -\cos x \Big|_0^{\pi} + \frac{1}{2}\int_0^{2\pi} \cos(u)\,du$$

$$= -\cos x \Big|_0^{\pi} + \frac{1}{2}\sin u \Big|_0^{2\pi}$$

$$= -\cos\pi + \cos 0 + \frac{1}{2}(\sin 2\pi - \sin 0)$$

$$= 1 + 1 = 2$$

c) $\int \frac{x^2}{\sqrt{1+x^3}}dx.$

$$= \frac{1}{3}\int \frac{3x^2}{\sqrt{1+x^3}}\,dx$$

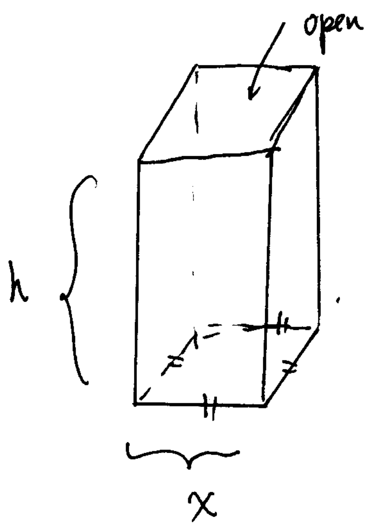
$u = 1+x^3$

$du = 3x^2\,dx$

$$= \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-1/2}\,du = \frac{1}{3}\cdot 2u^{1/2} + C$$

$$= \frac{2}{3}(1+x^3)^{1/2} + C$$

(8) (10pts) An open box with a square base is to be made of cardboard with a volume of $1 ft^3$. Find the dimension of the box that uses the least amount of material.




Constraint: $\underbrace{x^2 h}_{\text{volume}} = 1$

Minimize the surface area of the box.

Surface area: $x^2 + 4xh$

Using the constraint we obtain $h = 1/x^2$ and using this substitution in the formula for the surface area we obtain:

$$S(x) = x^2 + 4x \cdot \frac{1}{x^2} = x^2 + \frac{4}{x}$$

Minimize $S(x)$ on $(0, \infty)$

1) Critical points:

$S'(x) = 2x - \frac{4}{x^2}$ $\quad S'(x) = 0 \iff 2x - \frac{4}{x^2} = \frac{2x^3 - 4}{x^2} = 0$

$$2x^3 - 4 = 0 \iff x = 2^{1/3}.$$

$S'(x)$ undefined on $(0, \infty)$: none.

$\therefore$ Critical pt: $x = 2^{1/3}$.

2) Confirm that we have an abs min at $x = 2^{1/3}$ on $(0, \infty)$:

$S''(x) = 2 + \frac{8}{x^3} > 0$ since $x \in (0, \infty)$ $\therefore$ $S$ is concave up on $(0, \infty)$

We have at $x = 2^{1/3}$ the abs min of $S(x)$.

3) Dimensions: $x = 2^{1/3}$, $h = 1/x^2 = 1/2^{2/3}$. (all in ft).