Calculus I - Fall 2000 - Final Exam - 12/14/00

<u>Print your name:</u>

10 problems, 100 points, 2 hours 30 minutes.
Write your answers directly on this sheet.
Continue on back if necessary.
Support your answers and show all your work.
Closed notes, closed book.
Exact answers are required: For example, if the answer is $\sqrt{2}$, then 1.4142 is incorrect.
If the answer is $\ln(9)$, then 2.19722457733622 is incorrect, but $2\ln(3)$ is fine.

1. (8) Find the following limits:

$$\lim_{x\to 3} \frac{\sqrt{3x}-3}{2x-6} = \frac{1}{4} \qquad \lim_{x\to 1} x^{\frac{98}{1-x}} = e^{-98}$$

2. (9) a) State the definition of the derivative $f'(x)$. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if the limit exists

b) Let $f(x) = \frac{3}{2x}$. Use the definition to show that $f'(x) = \frac{-3}{2x^2}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{2(x+h)} - \frac{3}{2x}}{h} = \lim_{h\to 0} \frac{3x-3(x+h)}{2hx(x+h)} = \lim_{h\to 0} \frac{-3h}{2hx(x+h)} = \frac{-3}{2x^2}$$

3. (15) Find $\frac{dy}{dx}$:

$y = \sqrt{x}\cdot\tan x$ $\qquad$ $y = \frac{\ln(\sin x)}{x^2}$ $\qquad$ ~~$y = \frac{\ln(\sin x)}{x^2}$~~

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\tan x + \sqrt{x}\sec^2 x$ $\qquad$ $y' = \frac{x\cos x - 2\ln(\sin x)\cdot\sin x}{x^3\sin(x)}$

$y = e^{\arcsin x}$ $\qquad$ $y^x = x^2$ $\qquad$ $y = \int_{2x}^{4x} \frac{1}{t+\cos t}\,dt$

$y' = \frac{e^{\arcsin x}}{\sqrt{1-x^2}}$

$\ln y + \frac{x}{y}y' = \frac{2}{x}$

or $y' = \frac{y}{x}\left(\frac{2}{x} - \ln y\right)$

$y' = \frac{4}{4x+\cos(4x)} - \frac{2}{2x+\cos(2x)}$

4. (9) Find the slope of the tangent line of the curve given by the parametric equation

$y = t^2 + t + 1$, $x = t^2 - t$ at the point where $x = \frac{3}{4}$, $y = \frac{3}{4}$. *You will have to find t.*

$g(t)$ $f(t)$

$g'(t) = 2t+1 \quad f'(t) = 2t-1 \qquad \frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{2t+1}{2t-1}$. t for which $\begin{cases} x = t^2 - t = 3/4 \\ y = t^2+t+1 = 3/4 \end{cases}$ is $t = -1/2$

$\therefore \left.\frac{dy}{dx}\right|_{t=-1/2} = 0$

5. (5) Suppose the differentiable function f is defined on the interval [0,2], with f(1)=2,
f '(1) = 10. Use linear approximation to find an approximate value for f(1.1).

$f(1.1) \cong f(1) + f'(1)(1.1-1) = 2 + 2 \cdot 1/10 = 11/5$

6. (10) Two cars, an Audi A and a Buick B, are approaching an intersection of two roads running east-west and north-south, respectively. Car A is travelling at 56 mph and car B is travelling at 20 mph. If A is 5 miles from the intersection and B is 12 miles away, at what rate are the two cars approaching each other?

$\frac{1}{\sqrt{5^2+12^2}}(-56 \cdot 5 + 20 \cdot 12) = -40/13$ mi/h

7. (10) A cylindrical can must have a volume of 32π cubic inches. What should be its radius and height in order to minimize the cost of the material (ignore leftovers) if the circular top and bottom each cost two cents per square inch and the lateral surface costs one cent per square inch?

Volume of a cylinder: $V = \pi r^2 \cdot h$

Lateral surface area of a cylinder: $A = 2\pi r \cdot h$

radius = 2 in   height = 8 in

8. (12) Let $g(x) = \frac{x + \ln x}{x}$ for $x > 0$. Then $g'(x) = \frac{1 - \ln x}{x^2}$ and

$g''(x) = \frac{2\ln x - 3}{3}$.

Fill in the blanks:

The interval(s) on which g increases: $(0, e)$

The interval(s) on which g is concave up: $(e^{3/2}, \infty)$

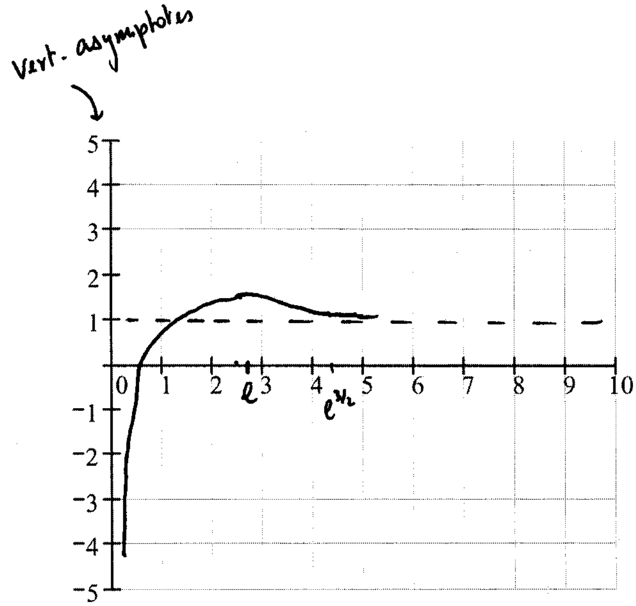
$\lim_{x\to\infty} g(x) =$ 1

$\lim_{x\to 0^+} g(x) =$ $-\infty$

Sketch the graph of g in the coordinate system to the right, showing asymptotes, local maxima and minima, and concavity.

Use the scale.

Indicate horizontal asymptotes with dotted lines.



9. (12) Find the following integrals:

$\int_3^9 \sqrt{x}\,dx$ $= 18 - 2\sqrt{3}$

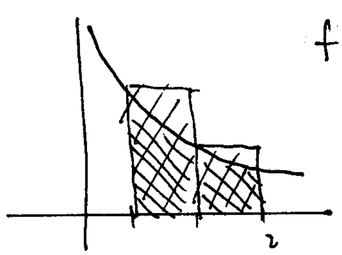
$\int 5t^3 e^{-t^4}\,dt$ $= -\frac{5}{4} e^{-t^4} + C$

$\int \left(\frac{1}{x} + \sec(x)^2\right) dx$ $= \ln x + \tan(x) + C$

10. (10) a)Write down the Riemann sum approximation for $\int_1^2 \frac{1}{t}\,dt$, using four equal subintervals and left hand endpoints as sample points, and compute its value.

$\left(\frac{1}{1} + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) \cdot \frac{1}{4}$

b) If 1000 subintervals are used instead of four, again using left hand endpoints as sample points, is the result smaller or larger than the result from a)?
Explain your answer.



$f(t) = 1/t$ is decreasing. using left hand end point means the estimated area is more than the actual area.

More (sub) intervals means more accurate estimate

∴ the result must be smaller than the one in a)