Solving General Equilibrium Models with Incomplete Markets and Many Assets

June 23, 2006

Martin D. D. Evans[1]
Georgetown University and NBER
Department of Economics
Washington DC 20057
Tel: (202) 338-2991
evansm1@georgetown.edu

Viktoria Hnatkovska
Georgetown University
Department of Economics
Washington DC 20057

vh6@georgetown.edu

Abstract

This paper presents a new numerical method for solving general equilibrium models with many assets. The method can be applied to models where there are heterogeneous agents, time-varying investment opportunity sets, and incomplete markets. It also can be used to study models where time-varying risk premia and optimal portfolio choices arise endogenously. We illustrate how the method is used by solving one– and two-sector versions of a two–country general equilibrium model with production and dynamic portfolio choice. We check the accuracy of our method by comparing the numerical solution to the one-sector model against its known analytic properties. We then apply the method to the two-sector model where no analytic solution is available. In both models the standard accuracy tests confirm the effectiveness of our method.

JEL Classification: C68; D52; G11.
Keywords: Portfolio Choice; Incomplete Markets; Time-Varying Risk Premia; Perturbation Methods; Projection Methods.

[1] Financial support from the National Science Foundation is gratefully acknowledged.

Introduction

This paper presents a new numerical method for solving general equilibrium models with many assets. The method can be applied to models where there are heterogeneous agents, time-varying investment opportunity sets, and incomplete markets. In this paper we illustrate how the method is used by solving one– and two-sector versions of a two–country general equilibrium model with production. Formal accuracy tests confirm that our technique provides a very accurate solution to both models.

Equilibrium in the one-sector model we study is characterized by complete risk-sharing, constant portfolio rules and constant risk premia. This model can be solved analytically, so we use it to provide a simple check of the accuracy of our numerical procedure. Solving for the equilibrium in the two-sector model is much more complex because markets are incomplete. In this environment portfolio decisions affect real allocation decisions, and vice-versa. This complex interaction between the real and financial sides of the economy cannot be addressed by existing solution methods. Our method allows for these interactions. Furthermore, it allows us to find the dynamic portfolio rules and time-varying risk premia that arise in equilibrium as a consequence of market incompleteness. Our ability to compute an accurate numerical solution to a model with these complex features means that our method has many important applications.

Our technique combines a hybrid projection-perturbation method with continuous-time approximations. In so doing, we contribute to the literature along several dimensions. First, relative to the finance literature, our method delivers optimal portfolios in a discrete-time general equilibrium setting in which returns are endogenously determined. It also enables us to characterize the dynamics of returns and the stochastic investment opportunity set as functions of macroeconomic state variables.[2] Second, relative to the macroeconomics literature, portfolio decisions are derived without assuming complete asset markets or constant returns to scale in production.[3]

Our solution method also relates to the literature on numerical methods. First, it builds on the perturbation methods developed and applied in Judd and Guu (1993, 1997), Judd (1998) and further discussed in Collard and Juillard (2001), Jin and Judd (2002), Schmitt-Grohe and Uribe (2004) among others. These methods extend solution techniques relying on linearizations by allowing for second- and higher-order terms in the approximation of the policy functions. Second, our method builds upon the projection technique introduced by Judd (1992). This method parametrizes the decision rules using basis functions and chooses the optimal rules that minimize some residual function. Unfortunately, these methods can only be used in applications that omit a key feature of models with portfolio choice: namely, the conditional heteroskedasticity of the state vector that captures the time-varying nature of the investment opportunity set. Continuous time approximations of Campbell, et. al. (2003) can handle such endogenous heteroskedasticity, however

[2] A number of approximate solution methods have been developed in partial equilibrium frameworks. Kogan and Uppal (2000) approximate portfolio and consumption allocations around the solution for a log-investor. Barberis (2000), Brennan, Schwartz, and Lagnado (1997) use discrete-state approximations. Brandt, Goyal, and Santa-Clara (2001) solve for portfolio policies by applying dynamic programming to an approximated simulated model. Brandt and Santa-Clara (2004) expand the asset space to include asset portfolios and then solve for the optimal portfolio choice in the resulting static model.

[3] Solutions to portfolio problems with complete markets are developed in Heathcote and Perri (2004), Serrat (2001), Kollmann (2005), Baxter, Jermann and King (1998), Uppal (1993), Engel and Matsumoto (2004). Pesenti and van Wincoop (1996) analyze equilibrium portfolios in a partial equilibrium setting with incomplete markets.

they are developed only in partial equilibrium. We extend their approach to a general equilibrium setting.

The paper is structured as follows. Section 1 presents the one–sector version of the model we use to illustrate our solution method. Section 2 describes the method in detail. Section 3 presents and compares the numerical solution of the model to its analytic counterpart. A formal assessment of the method's accuracy is provided in Section 4. Section 5 presents the two–sector version of the model and examines its equilibrium properties. Section 6 concludes.

1 The One-Sector Model

This section describes the one–sector version of the model we employ to illustrate our solution method. It is a standard international asset pricing model with portfolio choice and builds upon Danthine and Donaldson's (1994) formulation of an asset pricing model with production. We consider a frictionless production world economy consisting of two symmetric countries, called home (H) and foreign (F). Each country is populated by a continuum of identical households who consume and invest in different assets, and firms who produce a single good that is freely traded between the two countries. Firms are perfectly competitive and issue equity that is traded on the world stock market.

1.1 Firms

Our firms are infinitely lived. They issue equity claims to the stream of their dividends, and households can use this equity for their saving needs. Each firm owns capital and undertakes independent investment decisions. A representative firm in the H country starts period t with the stock of firm-specific capital K_t and an equity liability A_t equal to 1. Period$-t$ production is $Z_t K_t^{\theta}$, with $\theta > 0$. Z_t denotes the state of productivity. The output produced by firms in the F country is given by an identical production function using firm-specific foreign capital, $\hat{K}_t$, and productivity, $\hat{Z}_t$. (Hereafter we use "ˆ" to denote foreign variables.) The goods produced by H and F firms are identical and can be costlessly transported between countries. Under these conditions, the law of one price must prevail to eliminate arbitrage opportunities.

At the beginning of period t, each firm observes the productivity realization, produces output, and uses the proceeds to finance investment I_t and to pay dividends to the shareholders. We assume that firms allocate output to maximize the value of the firm to its shareholders every period. Let P_t denote the ex-dividend price of a share in the representative H firm at the start of period t, and let D_t be the dividend per share paid at t. With $A_t = 1$, the value of the firm at the start of period t is $P_t + D_t$, and the optimization problem it faces can be summarized as

$$\max_{I_t} \left(D_t + P_t \right), \tag{1}$$

subject to

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t, & (2) \\ D_t &= Z_t K_t^{\theta} - I_t, & (3) \end{aligned}$$

where $\delta > 0$ is the depreciation rate on physical capital. The representative firm in the F country solves an analogous problem; that is to say they choose investment $\hat{I}_t$ to maximize $\hat{D}_t + \hat{P}_t$, where $\hat{P}_t$ is the ex-dividend price of a share and $\hat{D}_t$ is the dividend per share paid at t. As before, we normalize the equity liability $\hat{A}_t$ to 1.

Let $z_t \equiv [\ln Z_t, \ln \hat{Z}_t]'$ denote the state of productivity in period t. We assume that z_t follows an AR(1) process:

$$z_t = a z_{t-1} + S_e^{1/2} e_t,$$

where a is a 2×2 matrix and e_t is a 2×1 vector of i.i.d. mean zero, unit variance shocks. $S_e^{1/2}$ is a 2×2 matrix of scaling parameters.

1.2 Households

Each country is populated by a continuum of households who have identical preferences. The preferences of households in the H country are defined in terms of H consumption C_t, and are given by

$$\mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \ln C_{t+i}, \tag{4}$$

where $0 < \beta < 1$ is the discount factor. $\mathbb{E}_t$ denotes expectations conditioned on information at the start of period t. Preferences for households in country F are similarly defined in terms of foreign consumption, $\hat{C}_t$.

Households in our economy can save by holding domestic equity shares, international bonds and equity issued by foreign firms. The budget constraint of the representative H household can be written as

$$W_{t+1} = R^{\mathrm{W}}_{t+1} \left(W_t - C_t \right), \tag{5}$$

where W_t is financial wealth, and R^{W}_{t+1} is the (gross) return on wealth between period t and $t+1$. This return depends on how the household allocates wealth across the available array of financial assets, and on the realized return on those assets. In particular,

$$R^{\mathrm{W}}_{t+1} = R_t + \alpha^{\mathrm{H}}_t (R^{\mathrm{H}}_{t+1} - R_t) + \alpha^{\mathrm{F}}_t (R^{\mathrm{F}}_{t+1} - R_t), \tag{6}$$

where R_t is the return on bonds, and R^{H}_{t+1} and R^{F}_{t+1} are the returns on H and F equity. The fraction of wealth that H country households hold in H and F equity are α^{H}_t and α^{F}_t respectively.

The budget constraint for F households is similarly defined as

$$\hat{W}_{t+1} = \hat{R}^{\mathrm{W}}_{t+1} (\hat{W}_t - \hat{C}_t),$$

with

$$\hat{R}^{\mathrm{W}}_{t+1} = R_t + \hat{\alpha}^{\mathrm{H}}_t (R^{\mathrm{H}}_{t+1} - R_t) + \hat{\alpha}^{\mathrm{F}}_t (R^{\mathrm{F}}_{t+1} - R_t),$$

where $\hat{\alpha}^{\mathrm{H}}_t$ and $\hat{\alpha}^{\mathrm{F}}_t$ denote the shares of wealth allocated by F households into H and F country equities.

Households in country H choose how much to consume and how much wealth to allocate into the equity of H and F firms to maximize expected utility (4) subject to (5) and (6) given current equity prices and the interest rate on bonds. This problem can be recursively expressed as:

$$\mathcal{Q}\left(W_t\right)=\max_{\left\{C_t,\alpha_t^{\text{H}},\alpha_t^{\text{F}}\right\}}\left\{\ln C_t+\beta\mathbb{E}_t\left[\mathcal{Q}\left(R_{t+1}^{\text{W}}\left(W_t-C_t\right)\right)\right]\right\}, \tag{7}$$

with $C_t\geq 0$ and $W_t>0$. $\mathcal{Q}(.)$ denotes the household's value function. The optimization problem facing F households is analogous.

1.3 Equilibrium

This section summarizes the conditions characterizing the equilibrium in our model. The first order conditions for the representative H household's problem in (7) are

$$1 = \mathbb{E}_t\left[M_{t+1}R_{t+1}^{\text{H}}\right], \tag{8a}$$

$$1 = \mathbb{E}_t\left[M_{t+1}R_t\right], \tag{8b}$$

$$1 = \mathbb{E}_t\left[M_{t+1}R_{t+1}^{\text{F}}\right], \tag{8c}$$

where $M_{t+1}\equiv\beta\left(\partial U/\partial C_{t+1}\right)/\left(\partial U/\partial C_t\right)$ is the discounted intertemporal marginal rate of substitution (IMRS) between consumption in period t and period $t+1$. The returns on equity issued by H and F firms are defined as

$$R_{t+1}^{\text{H}}=\left(P_{t+1}+D_{t+1}\right)/P_t \qquad \text{and} \qquad R_{t+1}^{\text{F}}=\left(\hat{P}_{t+1}+\hat{D}_{t+1}\right)/\hat{P}_t.$$

With these definitions, the Euler equation in (8a) can be rewritten as $P_t=\mathbb{E}_t\left[M_{t+1}\left(P_{t+1}+D_{t+1}\right)\right].$ Using this expression to substitute for P_t in the H firm's investment problem (1)-(3) gives the following recursive formulation:

$$\begin{aligned}\mathcal{V}(K_t,Z_t) &= \max_{I_t}\left(D_t+P_t\right),\\ &= \max_{I_t}\left(D_t+\beta\mathbb{E}_t\left[M_{t+1}(D_{t+1}+P_{t+1})\right]\right),\\ &= \max_{I_t}\left(Z_tK_t^{\theta}-I_t+\beta\mathbb{E}_t\left[M_{t+1}\mathcal{V}(K_{t+1},Z_{t+1})\right]\right),\end{aligned} \tag{9}$$

where $\mathcal{V}(.)$ denotes the value of the firm. The first order condition associated with this optimization problem is

$$1=\mathbb{E}_t\left[M_{t+1}R_{t+1}^{k}\right],$$

where $R_{t+1}^{k}\equiv\theta Z_{t+1}\left(K_{t+1}\right)^{\theta-1}+(1-\delta)$ is the return on capital. This condition determines the optimal investment of H firms and thus implicitly identifies the level of dividends in period t, D_t, via equation (3). The first order conditions for firms in country F take an analogous form.

It is worth noting that our model has equity home bias built in as firms use the IMRS of domestic agents, (e.g. M_{t+1} in the case of H firms) to value the dividend steam in (9). Although the array of assets available to households is sufficient for complete risk-sharing in this version of the model, in the two–sector version we present below markets are incomplete. As a result, the IMRS for H and F households will differ and households in the two countries will generally prefer different dividend streams. In principle, this formulation of how firms choose investment/dividends can induce home bias in household equity holdings.

Solving for the equilibrium in this economy requires finding equity prices $\{P_t,\, \hat{P}_t\}$, and the interest rate R_t, such that markets clear when households follow optimal consumption, savings and portfolio strategies, and firms make optimal investment decisions. Under the assumption that bonds are in zero net supply, market clearing in the bond market requires that

$$0 = B_t + \hat{B}_t. \tag{10}$$

The goods market clears globally. In particular, since H and F firms produce a single good that can be costlessly transported between countries, the market clearing condition is

$$C_t + \hat{C}_t = Y_t - I_t + \hat{Y}_t - \hat{I}_t = D_t + \hat{D}_t. \tag{11}$$

The market clearing conditions in the H and F equity markets are

$$1 = A_t^{\text{H}} + \hat{A}_t^{\text{H}} \qquad \text{and} \qquad 1 = A_t^{\text{F}} + \hat{A}_t^{\text{F}}. \tag{12}$$

where A_t^i $(\hat{A}_t^i)$ denotes the number of shares of equity issued by $i = \{\text{H},\, \text{F}\}$ firms held by H (F) households. These share holdings are related to the portfolio shares by the identities, $P_t A_t^{\text{H}} \equiv \alpha_t^{\text{H}} (W_t - C_t)$ and $\hat{P}_t A_t^{\text{F}} \equiv \alpha_t^{\text{F}} (W_t - C_t)$. The share holdings of F households are $\hat{A}_t^{\text{H}}$ and $\hat{A}_t^{\text{F}}$ with $P_t \hat{A}_t^{\text{H}} \equiv \hat{\alpha}_t^{\text{H}} (\hat{W}_t - \hat{C}_t)$ and $\hat{P}_t \hat{A}_t^{\text{F}} \equiv \hat{\alpha}_t^{\text{F}} (\hat{W}_t - \hat{C}_t)$.

2 Solution Method

2.1 Overview

Our solution method relies on several existing numerical methods for functional problems. In particular, it borrows some of the features of perturbation and projection methods, and amends them with the continuous time approximations. We next provide a brief overview of these techniques and highlight the novelty of our approach.

The goal of both the perturbation and projection methods consists of solving a system of equations that characterize the equilibrium of the model. This system has the general form

$$0 = \mathbb{E}_t f\left(Y_{t+1}, Y_t, \mathcal{X}_{t+1}, \mathcal{X}_t, \sigma \varepsilon_{t+1}\right), \tag{13}$$

where f is a known function. $\mathcal{X}_t$ is a vector of variables that describe the state of the economy at time t. In our illustrative model, $\mathcal{X}_t$ contains the state of productivity, capital stocks and households' wealth. Y_t is a vector of non-predetermined variables at time t. It includes consumption, dividends, asset prices, etc. ε_t

are innovations with the scaling parameter (matrix) σ. In our model the disturbance terms in ε_t are i.i.d. productivity shocks with mean zero and unitary variance; $\sigma\varepsilon_t = S_e^{1/2} e_t$.

Both methods start by conjecturing the form of the solution to the problem summarized in (13). In particular, the optimal policy functions characterizing competitive equilibria of the model can be presented as $Y_t = \mathcal{G}(\mathcal{X}_t, \sigma)$ and $\mathcal{X}_{t+1} = \mathcal{H}\left(\mathcal{X}_t, \sigma\varepsilon_{t+1}\right)$. Substituting these conjectures into (13) we get

$$\begin{aligned} 0 &= \mathbb{E}_t f\left(\mathcal{G}(\mathcal{H}\left(\mathcal{X}_t, \sigma\varepsilon_{t+1}\right), \sigma), \mathcal{G}(\mathcal{X}_t, \sigma), \mathcal{H}\left(\mathcal{X}_t, \sigma\varepsilon_{t+1}\right), \mathcal{X}_t, \sigma\varepsilon_{t+1}\right) \\ &\equiv \mathcal{F}\left(\mathcal{X}_t, \sigma\varepsilon_{t+1}\right). \end{aligned} \tag{14}$$

Even though the true form of $\mathcal{G}(.)$ and $\mathcal{H}(.)$ are unknown, we can write their approximations as

$$\widehat{\mathcal{G}}(\mathcal{X}_t, \sigma) = \sum\nolimits_i \psi_i \varphi_i\left(\mathcal{X}_t, \sigma\right), \tag{15}$$

$$\widehat{\mathcal{H}}(\mathcal{X}_t, \sigma\varepsilon_{t+1}) = \sum\nolimits_i \delta_i \varphi_i\left(\mathcal{X}_t, \sigma\right), \tag{16}$$

where $\{\psi_i\}$ and $\{\delta_i\}$ are parameter vectors, and $\varphi_i\left(\mathcal{X}_t, \sigma\right)$ are some elementary functions. The form of the approximations varies across the methods which we discuss next.

Perturbation Methods

The perturbation method has become a popular tool in solving stochastic general equilibrium models because it balances accuracy and computational time. The method, as developed by Collard and Juillard (2001), Jin and Judd (2002), and Schmitt-Grohe and Uribe (2004), consists of approximating $\mathcal{G}(\mathcal{X}_t, \sigma)$ and $\mathcal{H}(\mathcal{X}_t, \sigma\varepsilon_{t+1})$ around a non-stochastic steady state $\mathcal{X}^*$ and a set of perturbation parameters. The steady state values for $\mathcal{X}$ are obtained from the steady state equation $0 = f\left(Y^*, Y^*, \mathcal{X}^*, \mathcal{X}^*, 0\right)$. The standard deviations of the innovations to productivity in σ are usually used as perturbation parameters. Policy functions are approximated using n-th. order Taylor series expansions, which implies that the elementary functions $\varphi_i\left(\mathcal{X}_t, \sigma\right)$ in (15) and (16) are n-th. order polynomials of $\mathcal{X}_t$ and σ.

To obtain the unknown coefficients ψ and δ, the perturbation method relies on the Implicit Function Theorem. In particular, first the system of equilibrium conditions $\mathcal{F}\left(\mathcal{X}_t, \sigma\varepsilon_{t+1}\right) = 0$ is approximated up to the n-th. order around the steady state and $\sigma = 0$. Since the equilibrium conditions in $\mathcal{F}\left(.\right)$ must be equal to zero for any $\mathcal{X}$ or σ, the system of derivatives of $\mathcal{F}\left(.\right)$ up to any order must be equal to zero as well. Namely,

$$\mathcal{F}_{x^{n_x}\sigma^{n_\sigma}}\left(\mathcal{X}_t, \sigma\right) = 0 \qquad \forall \mathcal{X}_t, \sigma, n_x, n_\sigma \tag{17}$$

where $\mathcal{F}_{x^{n_x}\sigma^{n_\sigma}}\left(\mathcal{X}_t, \sigma\right)$ represents the n_x-th. derivative of $\mathcal{F}\left(.\right)$ with respect to $\mathcal{X}_t$, and the n_σ-th. derivative with respect to σ. The task of finding the coefficients in (15) and (16) now reduces to solving the system of restrictions in (17) evaluated at the steady state and $\sigma = 0$.

Projection Methods

The projection method was introduced in Economics by Judd (1992). In its general formulation, the technique consists of choosing basis functions over the space of continuous functions and using them to approximate $\mathcal{G}(\mathcal{X}_t, \sigma)$ and $\mathcal{H}(\mathcal{X}_t, \sigma\varepsilon_{t+1})$. In most applications, families of orthogonal polynomials, like Chebyshev's polynomials, are used to form $\varphi_i(\mathcal{X}_t, \sigma)$.[4] Given the chosen order of approximation, n, the problem of solving the model translates into finding the coefficient vectors ψ and δ that minimize a residual function. This function is formed by replacing the true policy functions in (14) by their approximate counterparts, $\widehat{\mathcal{G}}(\mathcal{X}_t, \sigma)$ and $\widehat{\mathcal{H}}(\mathcal{X}_t, \sigma\varepsilon_{t+1})$:

$$\mathcal{R}(\mathcal{X}_t, \sigma\varepsilon_{t+1}; \widehat{\mathcal{G}}, \widehat{\mathcal{H}}, \psi, \delta) = \mathbb{E}_t f\left(\widehat{\mathcal{G}}(\widehat{\mathcal{H}}(\mathcal{X}_t, \sigma\varepsilon_{t+1}), \sigma), \widehat{\mathcal{G}}(\mathcal{X}_t, \sigma), \widehat{\mathcal{H}}(\mathcal{X}_t, \sigma\varepsilon_{t+1}), \mathcal{X}_t, \sigma\varepsilon_{t+1}\right).$$

The optimal values of ψ and δ minimize this residual function, given some weighting function which determines the size of approximation error. This procedure creates the "projection", whose name varies depending on the choice of the weight. The least squares projection chooses coefficients ψ and δ to minimize the norm of the residual function. Other alternatives include the Galerkin method and the Collocation method (Judd 1992, 1998). To obtain the conditional expectations that appear in the residual function as well as the integrals necessary to calculate the projections, the method relies on numerical integration techniques.

Continuous Time Approximations

The use of continuous time approximations is advocated by Campbell (1994), while their application to portfolio choice problems is described in Campbell, Chan and Viceira (2003), CCV hereafter. The technique uses a continuous time approximation for portfolio returns which preserves the multiplicative nature of portfolio weighting. CCV approximations hold exactly in continuous time when asset prices follow diffusions and remain very accurate in discrete time for short time intervals. The method is applicable to partial equilibrium models in which the process for returns is exogenous. It consists of taking Taylor series expansions of the model equilibrium conditions, which in combination with approximate budget constraints and exogenous processes for returns give a system of linear-quadratic equations for portfolio shares and consumption in terms of state variables. An iterative numerical procedure can then be used to solve this system for optimal consumption and portfolio rules.

Our Method

Our method draws on all three approaches outlined above. It combines a hybrid projection-perturbation method with the continuous time approximations in a way that supplements the strengths of each approach. Our reasoning follows that of Judd (1998). Each method has important advantages. In solving the problem

[4] Chebyshev's polynomial method belongs to the family of the spectral projection methods for which the basis functions are nonzero almost everywhere. Finite element projection methods use basis functions that are nonzero within a small support. For a comparison of the two see Aruoba et.al. (2005).

summarized in (14), the perturbation method fixes the ψ and δ coefficients in (15)-(16) at the values implied by the derivatives in (17). At the same time, the $\varphi_i(\mathcal{X}_t, \sigma)$ functions are flexible, chosen to reflect the problem at hand. Projection methods, on the other hand, choose values for $\{\psi, \delta\}$ optimally, given a choice for the $\varphi_i(\mathcal{X}_t, \sigma)$ functions. Thus a researcher who uses either method faces a trade-off between having $\varphi_i(\mathcal{X}_t, \sigma)$ or $\{\psi, \delta\}$ set a priori. Our method allows for flexibility in both the $\varphi_i(\mathcal{X}_t, \sigma)$ functions and the $\{\psi, \delta\}$ coefficients. Specifically, we start by solving a perturbation problem, which provides us with approximate policy functions for some vector of coefficients. Given the $\varphi_i(\mathcal{X}_t, \sigma)$ functions derived in the first step, we then solve for the coefficients that minimize a residual function. The remainder of this section presents our procedure in general terms.

The set of equations characterizing the equilibrium of our model can be written in a general form as

$$\begin{aligned} 0 &= \mathbb{E}_t f\left(Y_{t+1}, Y_t, \mathcal{X}_{t+1}, \mathcal{X}_t, \mathcal{S}^{1/2}(\mathcal{X}_t)\,\varepsilon_{t+1}\right), \\ \mathcal{X}_{t+1} &= \mathcal{H}\left(\mathcal{X}_t, \mathcal{S}^{1/2}(\mathcal{X}_t)\,\varepsilon_{t+1}\right), \end{aligned} \tag{18}$$

where $\mathcal{S}^{1/2}(\mathcal{X}_t)$ is a lower triangular matrix or Cholesky factor. As before, $\mathcal{X}_t$ is a vector of variables that describe the state of the economy at time t, while Y_t is a vector of non-predetermined variables at time t. The function $f(.)$ denotes the equations characterizing the equilibrium, while $\mathcal{H}(.,.)$ determines how past states affect the current state. ε_{t+1} is a vector of mean zero i.i.d. innovations with unit variances. The vector of shocks driving the equilibrium dynamics of the model is $U_{t+1} \equiv \mathcal{S}^{1/2}(\mathcal{X}_t)\,\varepsilon_{t+1}$. This vector includes exogenous shocks, like the productivity shocks, and innovations to endogenous variables, like the shocks to households' wealth. The shocks have a conditional mean of zero and a conditional covariance equal to $\mathcal{S}(\mathcal{X}_t)$, a function of the current state vector $\mathcal{X}_t$:

$$\begin{aligned} \mathbb{E}(U_{t+1}|\mathcal{X}_t) &= 0, \\ \mathbb{E}\left(U_{t+1}U_{t+1}'|\mathcal{X}_t\right) &= \mathcal{S}^{1/2}(\mathcal{X}_t)\,\mathcal{S}^{1/2}(\mathcal{X}_t)' = \mathcal{S}(\mathcal{X}_t). \end{aligned} \tag{19}$$

An important aspect of our formulation is that it explicitly allows for the possibility that shocks driving the equilibrium dynamics are conditionally heteroskedastic. By contrast, standard perturbation methods assume that U_{t+1} follows an i.i.d. process, in which case $\mathcal{S}(\mathcal{X}_t)$ would be a constant matrix. As we shall see, it is not possible to characterize the equilibrium of a model with portfolio choice and incomplete markets in this way. Conditional heteroskedasticity arises as an inherent feature of such models, and must be accounted for in any solution technique.

Given our formulation in (18) and (19), a solution to the model is characterized by a decision rule for the non-predetermined variables

$$Y_t = \mathcal{G}(\mathcal{X}_t, \mathcal{S}(\mathcal{X}_t)), \tag{20}$$

that satisfies the equilibrium conditions in (18):

$$0 = \mathbb{E}_t f\left(\mathcal{G}\left(\mathcal{H}\left(\mathcal{X}_t, \mathcal{S}^{1/2}(\mathcal{X}_t)\,\varepsilon_{t+1}\right), \mathcal{S}\left(\mathcal{H}\left(\mathcal{X}_t, \mathcal{S}^{1/2}(\mathcal{X}_t)\,\varepsilon_{t+1}\right)\right)\right),\right.$$
$$\left.\mathcal{G}\left(\mathcal{X}_t, \mathcal{S}\left(\mathcal{X}_t\right)\right), \mathcal{H}\left(\mathcal{X}_t, \mathcal{S}^{1/2}(\mathcal{X}_t)\,\varepsilon_{t+1}\right), \mathcal{X}_t, \mathcal{S}^{1/2}(\mathcal{X}_t)\,\varepsilon_{t+1}\right).$$

Or in a more compact notation,

$$0 = \mathcal{F}(\mathcal{X}_t).$$

The first step in our method follows the perturbation procedure by approximating the policy functions as

$$\begin{aligned}
\widehat{\mathcal{G}} &= \sum\nolimits_i \psi_i \varphi_i\left(\mathcal{X}_t\right), \\
\widehat{\mathcal{H}} &= \sum\nolimits_i \delta_i \varphi_i\left(\mathcal{X}_t\right), \\
\widehat{\mathcal{S}} &= \sum\nolimits_i s_i \varphi_i\left(\mathcal{X}_t\right),
\end{aligned}$$

for some unknown coefficient sequences $\{\psi_i\}$, $\{\delta_i\}$, and $\{s_i\}$. $\varphi_i\left(\mathcal{X}_t\right)$ are ordinary polynomials in $\mathcal{X}_t$. Next we approximate f, as $\widehat{f}$. The equations associated with the real side of the economy are approximated using Taylor series expansions, while those pertinent to the portfolio side are approximated using continuous time expansions of CCV. We denote the derivatives in these expansions as $\{\varsigma_i\}$.

Next we apply the projection method that uses $\varphi_i\left(\mathcal{X}_t\right)$ as basis functions. Substituting $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$, and $\widehat{\mathcal{S}}$ into $\widehat{f}$ and taking expectations analytically gives us an approximation for $\mathcal{F}$:

$$\widehat{\mathcal{F}}\left(\mathcal{X}_t; \widehat{\mathcal{G}}, \widehat{\mathcal{H}}, \widehat{\mathcal{S}}, \zeta, \psi, \delta, s\right) = \sum\nolimits_i \zeta_i \varphi_i\left(\mathcal{X}_t\right),$$

where $\{\zeta_i\}$ are functions of $\{\varsigma_i\}$, $\{\psi_i\}$, $\{\delta_i\}$, and $\{s_i\}$. This is our residual function $\mathcal{R}(\mathcal{X}_t; \widehat{\mathcal{G}}, \widehat{\mathcal{H}}, \widehat{\mathcal{S}}, \zeta, \psi, \delta, s)$. The coefficient vectors ζ, ψ, δ, and s are found my minimizing this function in conjunction with a weighting matrix.

Two features of our method deserve special note. The first concerns our treatment of expectations. In particular, the system of equilibrium conditions $\mathcal{F}\left(.\right)$ often involves computing conditional expectations of highly nonlinear functions (e.g., Euler equations). In most cases, such calculations require integration and are challenging. The projection method addresses the problem by using numerical integration based on quadrature methods. Our method instead approximates $\mathcal{F}\left(.\right)$ by combining Taylor series expansions with continuous time approximations, and then calculates the conditional expectations of the approximated functions analytically. This is a great computational saving, and makes solving relatively complex models feasible [see, for example, Evans and Hnatkovska (2005) and Hnatkovska (2005)].

The second feature concerns the function $\mathcal{S}\left(\mathcal{X}_t\right)$, which identifies the covariance of the shocks driving the state vector $\mathcal{X}_t$. We need to accommodate conditional heteroskedasticity in the dynamics of $\mathcal{X}_t$ because it arises from the structure of models that incorporate portfolio choice with incomplete markets. This is true even when the exogenous shocks to the economy (e.g. productivity shocks) are homoskedastic. When markets are incomplete we need to track the distribution of wealth across the economy to identify the IMRS for each household and hence compute their portfolio choices.[5] This means that $\mathcal{X}_t$ must include the wealth of individual households; elements that will be conditionally heteroskedastic when optimal portfolio shares vary with the state of the economy (see below). The $\mathcal{S}(\mathcal{X}_t)$ function is therefore necessary to represent the general equilibrium implications of time-varying portfolio choice when markets are incomplete.[6] An accurate characterization of the portfolio choice problem facing households also requires the $\mathcal{S}(\mathcal{X}_t)$ function because it identifies how the second and higher-order conditional moments of all the variables vary with $\mathcal{X}_t$. These moments allow us to compute the equilibrium risk premia and optimal portfolio shares as functions of $\mathcal{X}_t$. The ability to solve an incomplete markets' model with endogenous time-varying risk premia and portfolios means that our method has many important applications.

The subsections that follow describe each step of our solution method in detail.

2.2 Log-Approximations

To further clarify why our formulation in (18) and (19) allows for conditional heteroskedasticity in the dynamics of the state vector, we return to the model. In particular, let us focus on the log-approximated equations arising from the households' first order conditions and budget constraint. Hereafter we use lower-case letters to denote the log transformation of the corresponding variable, measured as a deviation from its steady state level or initial value.

Following CCV we use a first-order log-approximation to households' budget constraints. In the case of H households it is given by

$$\begin{aligned}\Delta w_{t+1} &= \ln\left(1 - C_t/W_t\right) + r^{\mathrm{W}}_{t+1}, \\ &= \kappa - \tfrac{\mu}{1-\mu}\left(c_t - w_t\right) + r^{\mathrm{W}}_{t+1},\end{aligned} \tag{21}$$

where μ is the steady state consumption-wealth ratio and $\kappa \equiv \ln(1-\mu)$. In our model, households have log preferences so the optimal consumption-wealth ratio is a constant equal to $1-\beta$. In this case $c_t - w_t = 0$ and $\kappa = \ln\beta$. r^{W}_{t+1} is the log return on optimally invested wealth which CCV approximate as

$$r^{\mathrm{W}}_{t+1} = r_t + \boldsymbol{\alpha}_t' er_{t+1} + \tfrac{1}{2}\boldsymbol{\alpha}_t'\left(\operatorname{diag}\left(\Theta_t\right) - \Theta_t \boldsymbol{\alpha}_t\right), \tag{22}$$

where $\boldsymbol{\alpha}_t' \equiv \begin{bmatrix} \alpha^{\mathrm{H}}_t & \alpha^{\mathrm{F}}_t \end{bmatrix}$ is the vector of portfolio shares, $er_{t+1}' \equiv \begin{bmatrix} r^{\mathrm{H}}_{t+1} - r_t & r^{\mathrm{F}}_{t+1} - r_t \end{bmatrix}$ is a vector of excess equity returns, and Θ_t is the conditional covariance of er_{t+1}. The approximation error associated with this

[5] When markets are complete we can solve the portfolio problems using the common IMRS that can be identified from real allocations alone (i.e. without the distribution of wealth). See, for example, Kollmann (2005).

[6] The $\mathcal{S}(\mathcal{X}_t)$ function also impacts on real allocation decisions, like consumption and investment. So in general there will be a feedback between the portfolio decisions of households and the real economy. By contrast, when markets are complete equilibrium allocations can be computed separately from household's financial decisions (see, for example, Obstfeld and Rogoff 1996, pp. 302).

expression disappears in the limit where asset prices follow continuous–time diffusion processes.

Next, we turn to the first-order conditions in (8). Using the standard log-normal approximation, we obtain

$$\mathbb{E}_t r^{\chi}_{t+1} - r_t + \tfrac{1}{2}\mathbb{V}_t\left(r^{\chi}_{t+1}\right) = -\mathbb{CV}_t\left(m_{t+1}, r^{\chi}_{t+1}\right), \tag{23a}$$

$$r_t = -\mathbb{E}_t m_{t+1} - \tfrac{1}{2}\mathbb{V}_t(m_{t+1}), \tag{23b}$$

where r^{χ}_{t+1} is the log return for equity $\chi = \{\text{H}, \text{F}\}$, and $m_{t+1} \equiv \ln M_{t+1}$ is the log IMRS. $\mathbb{V}_t(.)$ and $\mathbb{CV}_t(.,.)$ denote the variance and covariance conditioned on period$-t$ information. With log utility $m_{t+1} = \ln\beta - \Delta c_{t+1} = \ln\beta - \Delta w_{t+1}$, so (23a) can be rewritten in vector form as

$$\mathbb{E}_t er_{t+1} = \Theta_t \boldsymbol{\alpha}_t - \tfrac{1}{2}\text{diag}\left(\Theta_t\right). \tag{24}$$

Combining this expression with (21) and (22) gives

$$\Delta w_{t+1} = \kappa - \tfrac{1-\mu}{\mu}\left(c_t - w_t\right) + r_t + \tfrac{1}{2}\boldsymbol{\alpha}'_t\Theta_t\boldsymbol{\alpha}_t + \boldsymbol{\alpha}'_t\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right). \tag{25}$$

Equation (25) provides us with a log-approximate version of the H household's budget constraint. It shows that the growth in household wealth between t and $t+1$ depends upon the consumption/wealth ratio in period t (a constant in the case of log utility), the period-t risk free rate, r_t, portfolio shares, $\boldsymbol{\alpha}_t$, the variance-covariance matrix of excess returns, Θ_t, and the unexpected return on assets held between t and $t+1$, $\boldsymbol{\alpha}'_t\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right)$. Notice that the susceptibility of wealth in $t+1$ to unexpected returns depends on the period-t portfolio choices, $\boldsymbol{\alpha}_t$. Consequently, the volatility of wealth depends endogenously on the portfolio choices made by households and the equilibrium behavior of returns. In an equilibrium where returns have an i.i.d. distribution, $\boldsymbol{\alpha}_t$ will be constant,[7] and wealth will be conditionally homoskedastic. Of course in a general equilibrium setting the properties of returns are themselves determined endogenously, so there is no guarantee that optimally chosen portfolio shares or the second moments of returns will be constant. Consequently, if wealth is an element in the state vector $\mathcal{X}_t$, our solution method needs to allow for the presence of conditional heteroskedasticity in $\mathcal{X}_t$. Our (approximate) solution for the $\mathcal{S}\left(\mathcal{X}_t\right)$ function reveals the extent to which heteroskedasticity arises in the dynamics of wealth given the optimal choice of $\boldsymbol{\alpha}_t$ for a general distribution of equilibrium returns.[8]

Of course, standard perturbation methods can still be used to solve models where the equilibrium dynamics of the wealth *must* be homoskedastic, or where wealth can be excluded from $\mathcal{X}_t$. The latter case occurs when the array of assets available for trading allows for perfect risk-sharing so that markets are complete. When markets are incomplete, by contrast, it is not possible to characterize the equilibrium dynamics of the

[7] This can be seen in our case of log utility by inspecting (24). Clearly, if $\mathbb{E}_t er_{t+1}$ and Θ_t are constant because returns are i.i.d., $\boldsymbol{\alpha}_t$ must also be constant.

[8] It is worth emphasizing that heteroskedasticity does not arise here because we are dealing with a log-approximated version of the household's budget constraint. It is an inherent feature of the household's budget constraint because portfolio choices affect the susceptibility of future wealth to the unexpected returns on individual assets (see equations 5 and 6 above). The log-approximation in (25) simply illustrates the point in a particularly clear way.

economy without including household wealth in the state vector $\mathcal{X}_t$.[9] As a consequence, in this setting it is necessary to allow for conditional heteroskedasticity in the dynamics of the state variables as our formulation in (18) and (19) does.

Equation (24) implicitly identifies the optimal choice of the H households' portfolio shares, $\boldsymbol{\alpha}_t$. This equation was derived from the households' first-order conditions under the assumption that the joint conditional distribution of log returns is approximately normal. Notice that the approximation method does not require an assumption about the portfolio shares chosen in the steady state. By contrast, standard perturbation methods consider Taylor series approximations to the model's equilibrium conditions with respect to decision variables around the value they take in the non-stochastic steady state. As Judd and Guu (2000) point out, this method is inapplicable when the steady-state value of the decision variable is indeterminate. This is an important observation when solving a model involving portfolio choice. In the non-stochastic steady state assets are perfect substitutes in household portfolios because returns are identical, so the optimal choice of portfolio is indeterminate.

While the steady state portfolio shares are absent from equation (24), the problem of indeterminacy still arises in our one-sector model. In particular, we have to take a stand on the steady state distribution of asset holdings when log-approximating the market clearing conditions: Consider, for example, the market clearing condition for H equity in (12). Combining this condition with the portfolio share definitions, and the fact that the consumption-wealth ratio for all households is equal to $1-\beta$, we obtain

$$\frac{P_t}{\beta W_t} = \alpha_t^{\text{H}} + \hat{\alpha}_t^{\text{H}} \frac{\hat{W}_t}{W_t}.$$

We consider a second-order Taylor series approximation to this expression around the steady state values for $P_t/\beta W_t$ and a particular value for $\hat{W}_t/W_t$. Specifically, we parameterize the value of $\hat{W}/W$ and then work out its implications for the value of $P/\beta W$.[10] This is particularly simple in the case where wealth is assumed to be equally distributed (i.e. $\hat{W}/W = 1$). Here symmetry and market clearing in the goods market require that $D = C = (1-\beta)W$. It follows that $P/\beta W = [(1-\beta)/\beta]\,(P/D) = 1$ because the Euler equation for stock returns implies that the steady state value of P/D equals $\beta/(1-\beta)$. In this case, the second-order log-approximation embedding goods market clearing becomes

$$1 + p_t - w_t + \tfrac{1}{2}\left(p_t - w_t\right)^2 = \alpha_t^{\text{H}} + \hat{\alpha}_t^{\text{H}} \left(1 + \hat{w}_t - w_t + \tfrac{1}{2}\left(\hat{w}_t - w_t\right)^2\right).$$

Log-approximations implied by the other market clearing conditions are similarly obtained. Specifically, when initial wealth is assumed to be equally distributed, market clearing in F equity, bonds and goods imply,

[9] Specifically, we need to track m_{t+1} and $\hat{m}_{t+1}$, which for standard utility specifications are functions of Δc_{t+1} and $\Delta \hat{c}_{t+1}$ respectively. We therefore need to solve the consumption/savings problem of each household to identify Δc_{t+1} and $\Delta \hat{c}_{t+1}$, a task that necessitates the inclusion of household wealth in the state vector.

[10] Our approach of parametrizing the initial wealth distribution across agents is an alternative to the Judd and Guu (2000) bifurcation procedure for dealing with portfolio indeterminacy.

correspondingly, that

$$\begin{aligned}
1+\hat{p}_t-\hat{w}_t+\tfrac{1}{2}\left(\hat{p}_t-\hat{w}_t\right)^2 &= \hat{\alpha}^{\mathrm{F}}_t+\alpha^{\mathrm{F}}_t\left(1+w_t-\hat{w}_t+\tfrac{1}{2}\left(w_t-\hat{w}_t\right)^2\right), \\
p_t+\hat{p}_t &= d_t+\hat{d}_t, \\
c_t+\hat{c}_t &= d_t+\hat{d}_t.
\end{aligned} \tag{26}$$

This approach to the indeterminacy problem has another important advantage. In cases where wealth must be an element of $\mathcal{X}_t$, its presence introduces a nonstationary unit root component in the $\mathcal{X}_t$ process: shocks to returns will generally have permanent effects on wealth.[11] As we show below, our procedure accommodates the presence of a unit root by characterizing the equilibrium dynamics of the model in a neighborhood of the initial state, $\mathcal{X}_0$. To study the equilibrium properties of the model we must therefore specify the elements of $\mathcal{X}_0$. Thus, choosing the initial distribution of wealth not only provides a way to resolve indeterminacy concerning portfolio shares in the non-stochastic steady state, it also allows us to analyze the equilibrium dynamics of a model that is inherently nonstationary. Of course, we do have to keep in mind that the accuracy of the equilibrium dynamics provided by our solution will deteriorate as $\mathcal{X}_t$ moves further from $\mathcal{X}_0$.

The remaining equations characterizing the model's equilibrium are log-approximated in a standard way. Optimal investment by H and F firms requires that

$$\mathbb{E}_t r^k_{t+1}-r_t+\tfrac{1}{2}\mathbb{V}_t\left(r^k_{t+1}\right) = \mathbb{CV}_t\left(r^k_{t+1},w_{t+1}\right), \tag{27a}$$

$$\mathbb{E}_t \hat{r}^k_{t+1}-r_t+\tfrac{1}{2}\mathbb{V}_t\left(\hat{r}^k_{t+1}\right) = \mathbb{CV}_t\left(\hat{r}^k_{t+1},\hat{w}_{t+1}\right), \tag{27b}$$

where r^k_{t+1} and $\hat{r}^k_{t+1}$ are the log returns on capital approximated by

$$r^k_{t+1}\cong\psi z_{t+1}-(1-\theta)\psi k_{t+1} \qquad \text{and} \qquad \hat{r}^k_{t+1}\cong\psi\hat{z}_{t+1}-(1-\theta)\psi\hat{k}_{t+1}, \tag{28}$$

with $\psi\equiv 1-\beta(1-\delta)<1$. The dynamics of the H and F capital stock are approximated by

$$k_{t+1}\cong\frac{1}{\beta}k_t+\frac{\psi}{\beta\theta}z_t-\left(\frac{\psi}{\theta\beta}-\delta\right)d_t \qquad \text{and} \qquad \hat{k}_{t+1}\cong\frac{1}{\beta}\hat{k}_t+\frac{\psi}{\beta\theta}\hat{z}_t-\left(\frac{\psi}{\theta\beta}-\delta\right)\hat{d}_t. \tag{29}$$

Finally, we turn to the relationship between the price of equity, dividends and returns. As in Campbell and Shiller (1989), we relate the log return on equity to log dividends and the log price of equity by

$$r^{\mathrm{H}}_{t+1}=\rho p_{t+1}+(1-\rho)d_{t+1}-p_t \qquad \text{and} \qquad r^{\mathrm{F}}_{t+1}=\hat{\rho}\hat{p}_{t+1}+(1-\hat{\rho})\hat{d}_{t+1}-\hat{p}_t, \tag{30}$$

with $\rho\equiv 1/(1+\exp(\overline{d-p}))$ and $\hat{\rho}\equiv 1/(1+\exp(\overline{\hat{d}-\hat{p}}))$ where $\overline{d-p}$ and $\overline{\hat{d}-\hat{p}}$ are the average log

[11] For example, when households have log preferences the first two terms on the right in (25) are constant. Under these circumstances, a positive unexpected return will permenantly raise wealth unless the household finds it optimal to adjust their future portfolio shares so that $\boldsymbol{\alpha}'_{t+i}\Theta_{t+i}\boldsymbol{\alpha}_{t+i}$ falls and/or r_{t+i} falls by a compensating amount.

dividend-price ratios in the H and F countries. In the non-stochastic steady state $\rho = \hat{\rho} = \beta$. Making this substitution, iterating forward, taking conditional expectations, and imposing $\lim_{j\to\infty} \mathbb{E}_t \beta^j p_{t+j} = 0$ and $\lim_{j\to\infty} \mathbb{E}_t \beta^j \hat{p}_{t+j} = 0$, we obtain

$$p_t = \sum_{i=0}^{\infty} \beta^i \left\{(1-\beta)\mathbb{E}_t d_{t+1+i} - \mathbb{E}_t r^{\mathrm{H}}_{t+1+i}\right\}, \tag{31a}$$

$$\hat{p}_t = \sum_{i=0}^{\infty} \beta^i \left\{(1-\beta)\mathbb{E}_t \hat{d}_{t+1+i} - \mathbb{E}_t r^{\mathrm{F}}_{t+1+i}\right\}. \tag{31b}$$

These approximations show how log equity prices are related to expected future dividends and returns.

2.3 State Variables Dynamics

The key step in our solution procedure is deriving a general yet tractable set of equations that describe the equilibrium dynamics of the state variables. One problem we immediately face in this regard is the dimensionality of the state vector. As we noted above, the distribution of wealth plays an integral role in determining equilibrium prices and returns when markets are incomplete, so household wealth needs to be included in the state vector. In models with a continuum of heterogenous households it is obviously impossible to track the wealth of individuals, so moments of the wealth distribution need to be included in the state vector. The question of how many moments to include is not easily answered.

Dimensionality is still a problem when heterogeneity across households is limited. In our model there are only two types of households, so it suffices to keep track of H and F households' wealth. The dimensionality problem occurs under these circumstances because uncertainty enters multiplicatively into the dynamics of wealth. (Recall that portfolio shares determine the susceptibility of wealth to unexpected return shocks.) If wealth is part of the state vector, $\mathcal{X}_t$, and both portfolio shares and realized returns depend on $\mathcal{X}_t$, the level of wealth will depend on the elements in $\mathcal{X}_t\mathcal{X}_t'$. This means that the equilibrium dynamics of wealth will in general depend on the behavior of the levels, squares and cross-products of the individual state variables. This dependence between the lower and higher moments of the state variables remains even after log-approximation. In equation (25) we see that H household wealth depends on the quadratic form for portfolio shares, which are themselves functions of the state vector, including wealth. As a result, the state vector needs to be expanded to include squares and cross-products. Of course a similar logic applies to the equilibrium behavior of squares and cross-products involving wealth. So by induction, a complete characterization of the equilibrium wealth dynamics could easily require an infinite number of elements in $\mathcal{X}$. Our solution procedure uses a finite subset of state variables $X \subset \mathcal{X}$ that provides a good approximation to the equilibrium dynamics.

We will use the model presented in Section 1 to illustrate our procedure. Let $x_t \equiv [z_t, k_t, \hat{k}_t, w_t, \hat{w}_t]'$ where $k_t \equiv \ln(K_t/K)$, $\hat{k}_t \equiv \ln\left(\hat{K}_t/K\right)$, $w_t \equiv \ln(W_t/W_0)$ and $\hat{w}_t \equiv \ln(\hat{W}_t/\hat{W}_0)$. More generally, x_t will be an $l\times 1$ vector that contains the variables that make up the state vector. We will approximate the equilibrium dynamics of the model with the vectors

$$X_t = \begin{bmatrix} 1 \\ x_t \\ \tilde{x}_t \end{bmatrix} \qquad \text{and} \qquad U_t = \begin{bmatrix} 0 \\ u_t \\ \tilde{u}_t \end{bmatrix},$$

where $\tilde{x}_t \equiv vec\,(x_t x_t')$. The shock vector U_t is partitioned conformably with X_t and both vectors contain $\mathfrak{L} = 1 + l + l^2$ elements.

To determine the dynamics of X_t, we first conjecture that x_t follows

$$x_{t+1} = \Phi_0 + (I - \Phi_1)x_t + \Phi_2\tilde{x}_t + u_{t+1}, \tag{32}$$

where Φ_0 is the $l \times 1$ vector of constants, Φ_1 is the $l \times l$ matrix of autoregressive coefficients and Φ_2 is the $l \times l^2$ matrix of coefficients on the second-order terms. u_{t+1} is a vector of innovations with zero conditional mean, and conditional covariance that is a function of X_t :

$$\begin{aligned} \mathbb{E}\left(u_{t+1}|x_t\right) &= 0, \\ \mathbb{E}\left(u_{t+1}u_{t+1}'|x_t\right) &= \Omega(X_t) = \Omega_0 + \Omega_1 x_t x_t' \Omega_1'. \end{aligned}$$

Below we shall use the vectorized conditional variance of u_t which we write as

$$vec\,(\Omega(X_t)) = \begin{bmatrix} \Sigma_0 & 0 & \Sigma_1 \end{bmatrix} \begin{bmatrix} 1 \\ x_t \\ \tilde{x}_t \end{bmatrix} = \Sigma X_t. \tag{33}$$

The next step is to derive an equation describing the dynamics of $\tilde{x}_t$ consistent with (32) and (33). For this purpose we consider the continuous time analogue to (32) and derive the dynamics of $\tilde{x}_{t+1}$ via Ito's lemma. As the Appendix shows, the resulting process can be approximated in discrete time by

$$\tilde{x}_{t+1} = \tfrac{1}{2}D\Sigma_0 + (\Phi_0 \otimes I) + (I \otimes \Phi_0)\,x_t + \left(I - (\Phi_1 \otimes I) - (I \otimes \Phi_1) + \tfrac{1}{2}D\Sigma_1\right)\tilde{x}_t + \tilde{u}_{t+1} \tag{34}$$

where

$$\tilde{u}_{t+1} = \left[(I \otimes x_t) + (x_t \otimes I)\right] u_{t+1},$$

$$D = \left[\mathbb{U}\left(\frac{\partial x}{\partial x'} \otimes I\right) + \left(\frac{\partial x}{\partial x'} \otimes I\right)\right], \qquad \text{and} \qquad \mathbb{U} = \sum_r \sum_s E_{rs} \otimes E_{r,s}'.$$

$E_{r,s}$ is the elementary matrix which has a unity at the $(r,s)^{th}$ position and zero elsewhere. Equation (34) approximates the dynamics of $\tilde{x}_{t+1}$ because it ignores the role played by cubic and higher order terms involving the elements of x_t. In this sense, (34) represents a second–order approximation to the dynamics of the second–order terms in the state vector.[12] Notice that the variance of u_{t+1} affects the dynamics of $\tilde{x}_{t+1}$ via the D matrix and that $\tilde{u}_{t+1}$ will generally be conditionally heteroskedastic.

We can now combine (32) and (34) into a single equation:

$$\begin{bmatrix} 1 \\ x_{t+1} \\ \tilde{x}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \Phi_0 & I-\Phi_1 & \Phi_2 \\ \frac{1}{2}D\Sigma_0 & (\Phi_0 \otimes I)+(I \otimes \Phi_0) & I-(\Phi_1 \otimes I)-(I \otimes \Phi_1)+\frac{1}{2}D\Sigma_1 \end{bmatrix} \begin{bmatrix} 1 \\ x_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} 0 \\ u_{t+1} \\ \tilde{u}_{t+1} \end{bmatrix},$$

or more compactly

$$X_{t+1} = \mathbb{A}X_t + U_{t+1}, \tag{35}$$

with $\mathbb{E}\left(U_{t+1}|X_t\right) = 0$. We also need to determine the conditional covariance of the U_{t+1} vector. In the Appendix we show that

$$\mathbb{E}\left(U_{t+1}U'_{t+1}|X_t\right) \equiv \mathcal{S}\left(X_t\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega\left(X_t\right) & \Gamma\left(X_t\right) \\ 0 & \Gamma\left(X_t\right)' & \Psi(X_t) \end{pmatrix}, \tag{36}$$

where

$$\begin{aligned} vec\left(\Gamma\left(X_t\right)\right) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \tilde{x}_t, \\ vec\left(\Gamma\left(X_t\right)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2 \tilde{x}_t, \\ vec\left(\Psi(X_t)\right) &= \Psi_0 + \Psi_1 x_t + \Psi_2 \tilde{x}_t. \end{aligned}$$

The Γ_i, Λ_i and Ψ_i matrices are complicated functions of the parameters in (32) and (33); their precise form is given in the Appendix.

To this point we have shown how to approximate the dynamics of X_t given a conjecture concerning Φ_0, Φ_1, Φ_2, Σ_0 and Σ_1. We now turn to the issue of how these matrices are determined. For this purpose we make use of two further results. Let a_t and b_t be two generic endogenous variables related to the state vector by $a_t = \pi_a X_t$ and $b_t = \pi_b X_t$, where π_a and π_b are $1 \times \mathfrak{L}$ vectors. Our second-order approximation for the dynamics of X_t implies that

$$\mathbb{CV}_t\left(a_{t+1}, b_{t+1}\right) = \mathcal{A}\left(\pi_a, \pi_b\right)X_t, \tag{R1}$$

and

$$a_t b_t = \mathcal{B}\left(\pi_a, \pi_b\right)X_t. \tag{R2}$$

$\mathcal{A}(.,.)$ and $\mathcal{B}(.,.)$ are $1 \times \mathfrak{L}$ vectors with elements that depend on π_a, π_b and the parameters of the X_t process. The precise form of these vectors is also shown in the Appendix.

[12] One way to check the accuracy of this approximation is to derive a generalization of (34) involving third–order terms and then compute the contribution of these terms to the dynamics of x_t and $\tilde{x}_t$. Since the elements of x_t are measured in terms of percentage deviations from steady state or initial values, third order terms are unlikely to be significant. Nevertheless, as we note below, we are cognizant of the approximation error in (34) when examining the "solution" to a model.

To see how these results are used, we return to the model. The dynamics of the state vector depend upon households' portfolio choices, $\{\alpha_t^{\text{H}}, \alpha_t^{\text{F}}, \hat{\alpha}_t^{\text{H}}, \hat{\alpha}_t^{\text{F}}\}$, firms' dividend choices, $\{d_t, \hat{d}_t\}$, equilibrium equity prices, $\{p_t, \hat{p}_t\}$, and the risk free rate, r_t. Let us assume, for the present, that each of these non-predetermined variables is linearly related to the state. (We shall verify that this is indeed the case below.) In particular, let π_i be the $1 \times \mathfrak{L}$ row vector that relates variable i to the state X_t and let h_i be the $1 \times \mathfrak{L}$ vector that selects the i^{th} element out of X_t. We can now easily derive the restrictions on the dynamics of productivity, capital and wealth.

Recall that the first two rows of x_t comprise the vector of productivities that follow an exogenous AR(1) process. The corresponding elements of Φ_0, Φ_1, Φ_2, Σ_0 and Σ_1 are therefore entirely determined by the parameters of this process. The next elements in x_t are the log capital stocks. If equilibrium dividends satisfy $d_t = \pi_d X_t$ and $\hat{d}_t = \pi_{\hat{d}} X_t$, we can rewrite the log-approximated dynamics for k_t and $\hat{k}_t$ shown in (29) as

$$
\begin{aligned}
h_k X_{t+1} &= \left(\tfrac{1}{\beta} h_k + \tfrac{\psi}{\beta\theta} h_z - \left(\tfrac{\psi}{\theta\beta} - \delta\right) \pi_d\right) X_t, \\
h_{\hat{k}} X_{t+1} &= \left(\tfrac{1}{\beta} h_{\hat{k}} + \tfrac{\psi}{\beta\theta} h_{\hat{z}} - \left(\tfrac{\psi}{\theta\beta} - \delta\right) \pi_{\hat{d}}\right) X_t.
\end{aligned}
$$

Notice that these equations must hold for all realizations of X_t. So substituting for X_{t+1} with (35) and equating coefficients we obtain

$$
h_k \mathbb{A} = \tfrac{1}{\beta} h_k + \tfrac{\psi}{\beta\theta} h_z - \left(\tfrac{\psi}{\theta\beta} - \delta\right) \pi_d \quad \text{and} \quad h_{\hat{k}} \mathbb{A} = \tfrac{1}{\beta} h_{\hat{k}} + \tfrac{\psi}{\beta\theta} h_{\hat{z}} - \left(\tfrac{\psi}{\theta\beta} - \delta\right) \pi_{\hat{d}}.
$$

These equations place restrictions on the elements of Φ_0, Φ_1, and Φ_2. Furthermore, because k_{t+1} and $\hat{k}_{t+1}$ are solely functions of the period–t state, the corresponding element rows and columns of $\mathbb{E}(u_{t+1}u'_{t+1}|X_t) \equiv \Omega(X_t)$ are vectors of zeros. This observation puts restrictions on the elements of Σ_0 and Σ_1.

Deriving the equilibrium restrictions on the dynamics of wealth in (25) is a little more complicated and requires the use of R1 and R2. Our starting point is the approximation for log equity returns in (30) which we now write in terms of the state vector:

$$
r_{t+1}^{\text{H}} = \pi_{\text{H}} X_{t+1} - \pi_p X_t \qquad \text{and} \qquad r_{t+1}^{\text{F}} = \pi_{\text{F}} X_{t+1} - \pi_{\hat{p}} X_t,
$$

where $\pi_{\text{H}} \equiv \beta\pi_p + (1-\beta)\pi_d$ and $\pi_{\text{F}} \equiv \beta\pi_{\hat{p}} + (1-\beta)\pi_{\hat{d}}$. Notice that unexpected log returns are $r_{t+1}^{\chi} - \mathbb{E}_t r_{t+1}^{\chi} = \pi_\chi (X_{t+1} - \mathbb{E}_t X_{t+1})$ for $\chi = \{\text{H}, \text{F}\}$, so applying R1 we obtain

$$
\mathbb{V}_t(er_{t+1}) \equiv \Theta_t = \begin{bmatrix} \mathcal{A}(\pi_{\text{H}}, \pi_{\text{H}}) X_t & \mathcal{A}(\pi_{\text{H}}, \pi_{\text{F}}) X_t \\ \mathcal{A}(\pi_{\text{H}}, \pi_{\text{F}}) X_t & \mathcal{A}(\pi_{\text{F}}, \pi_{\text{F}}) X_t \end{bmatrix}.
$$

Now recall that our log-approximated version of the H household budget constraint contains a quadratic function of the portfolio shares and Θ_t. To evaluate this component, let us assume that the portfolio shares satisfy $\alpha_t^{\text{H}} = \pi_\alpha^{\text{H}} X_t$ and $\alpha_t^{\text{F}} = \pi_\alpha^{\text{F}} X_t$, so that

$$\boldsymbol{\alpha}_t' \Theta_t \boldsymbol{\alpha}_t = \left[\begin{array}{cc} \pi_\alpha^{\text{H}} X_t & \pi_\alpha^{\text{F}} X_t \end{array} \right] \left[\begin{array}{cc} \mathcal{A}(\pi_{\text{H}}, \pi_{\text{H}}) X_t & \mathcal{A}(\pi_{\text{H}}, \pi_{\text{F}}) X_t \\ \mathcal{A}(\pi_{\text{H}}, \pi_{\text{F}}) X_t & \mathcal{A}(\pi_{\text{F}}, \pi_{\text{F}}) X_t \end{array} \right] \left[\begin{array}{c} \pi_\alpha^{\text{H}} X_t \\ \pi_\alpha^{\text{F}} X_t \end{array} \right].$$

Applying R2 to the right hand side gives

$$\begin{aligned} \boldsymbol{\alpha}_t' \Theta_t \boldsymbol{\alpha}_t &= \mathcal{B}\left(\pi_\alpha^{\text{H}}, \mathcal{B}(\mathcal{A}(\pi_{\text{H}}, \pi_{\text{H}}), \pi_\alpha^{\text{H}}) + \mathcal{B}(\mathcal{A}(\pi_{\text{H}}, \pi_{\text{F}}), \pi_\alpha^{\text{F}})\right) X_t \\ &\quad + \mathcal{B}\left(\pi_\alpha^{\text{F}}, \mathcal{B}(\mathcal{A}(\pi_{\text{H}}, \pi_{\text{F}}), \pi_\alpha^{\text{H}}) + \mathcal{B}(\mathcal{A}(\pi_{\text{F}}, \pi_{\text{F}}), \pi_\alpha^{\text{F}})\right) X_t, \\ &= \Lambda X_t. \end{aligned}$$

According to (25), $\mathbb{E}_t w_{t+1} = w_t + \ln \beta + r_t + \frac{1}{2} \boldsymbol{\alpha}_t' \Theta_t \boldsymbol{\alpha}_t$, while the dynamics of the state vector in (35) imply that $\mathbb{E}_t w_{t+1} = h_w \mathbb{A} X_t$. Equating these moments for all possible values of X_t requires that

$$h_w \mathbb{A} = h_w + \ln \beta h_1 + \pi_r + \tfrac{1}{2} \Lambda.$$

This expression provides us with another set of restrictions on the elements of Φ_0, Φ_1, and Φ_2.

The model also places restrictions on the second moments of the state variables. For example, the restrictions on the variances and covariances of the productivity vector z_t are by assumption homoskedastic:

$$h_z vec(\Omega(X_t)) = vec\left(\mathbb{CV}_t(z_{t+1}, z_{t+1})\right) = vec(S_e).$$

Second moments involving the capital stock are also straightforward: they are all zero because capital is chosen one period in advance (see equation (2) above). Next, consider the moments involving wealth. First note that for any variable $a_t = \pi_a X_t$,

$$\mathbb{CV}_t(w_{t+1}, a_{t+1}) = \alpha_t^{\text{H}} \mathbb{CV}_t(r_{t+1}^{\text{H}}, a_{t+1}) + \alpha_t^{\text{F}} \mathbb{CV}_t(r_{t+1}^{\text{F}}, a_{t+1}).$$

Applying R1 and R2 to the right hand side, gives

$$\begin{aligned} \mathbb{CV}_t(w_{t+1}, a_{t+1}) &= \pi_\alpha^{\text{H}} X_t \mathcal{A}(\pi_{\text{H}}, \pi_a) X_t + \pi_\alpha^{\text{F}} X_t \mathcal{A}(\pi_{\text{F}}, \pi_a) X_t \\ &= \left(\mathcal{B}(\pi_\alpha^{\text{H}}, \mathcal{A}(\pi_{\text{H}}, \pi_a)) + \mathcal{B}(\pi_\alpha^{\text{F}}, \mathcal{A}(\pi_{\text{F}}, \pi_a))\right) X_t. \end{aligned}$$

Our conjecture for the conditional covariance of x_t in (33) implies that the second moments of wealth depend only on the constant and second order terms in X_t. This conjecture requires that

$$\begin{aligned} h_a vec(\Omega(X_t)) &\equiv h_a \left[\begin{array}{ccc} \Sigma_0 & 0 & \Sigma_1 \end{array} \right] \\ &= \mathcal{B}(\pi_\alpha^{\text{H}}, \mathcal{A}(\pi_{\text{H}}, \pi_a)) + \mathcal{B}(\pi_\alpha^{\text{F}}, \mathcal{A}(\pi_{\text{F}}, \pi_a)), \end{aligned}$$

where $h_a vec(\Omega(X_t)) = \mathbb{CV}_t(w_{t+1}, a_{t+1})$. For a we use the elements of $x_t \equiv [z_t, w_t, \hat{w}_t]'$. An analogous set of restrictions applies to the dynamics of F household wealth.

2.4 Non-Predetermined Variable Dynamics

To this point we have shown how the equilibrium conditions of the model impose restrictions on the dynamics of the state variables under the assumption that the vector of non-predetermined variables Y_t (i.e., $\alpha_t^{\text{H}}, \alpha_t^{\text{F}}, \hat{\alpha}_t^{\text{H}}, \hat{\alpha}_t^{\text{F}}, d_t, \hat{d}_t, p_t, \hat{p}_t$ and r_t) satisfies

$$Y_t = \Pi X_t,$$

for some matrix Π with rows π_i. We now turn to the question of how the elements of Π are determined from the equilibrium conditions and the dynamics of the state vector.

We begin with the restrictions on H equity prices. In particular, our aim is to derive a set of restrictions that will enable us to identify the elements of π_p where $p_t = \pi_p X_t$ in equilibrium. Our derivation starts with expected returns. Specifically, we note from the log–approximated first order conditions in (23a) with $\chi = \{\text{H}\}$ that

$$\begin{aligned}
\mathbb{E}_t r_{t+1}^{\text{H}} &= r_t + \mathbb{CV}_t(w_{t+1}, r_{t+1}^{\text{H}}) - \tfrac{1}{2}\mathbb{V}_t(r_{t+1}^{\text{H}}), \\
&= r_t + \left(\mathcal{B}(\pi_\alpha^{\text{H}}, \mathcal{A}(\pi_{\text{H}}, \pi_{\text{H}})) + \mathcal{B}(\pi_\alpha^{\text{F}}, \mathcal{A}(\pi_{\text{F}}, \pi_{\text{H}})) - \tfrac{1}{2}\mathcal{A}(\pi_{\text{H}}, \pi_{\text{H}})\right) X_t, \\
&= (\pi_r + \pi_{er}^{\text{H}}) X_t.
\end{aligned}$$

Combining this expression for expected returns with the assumed form for equilibrium dividends, the dynamics of the state vector, and (31a) gives

$$\begin{aligned}
p_t &= \sum_{i=0}^{\infty} \beta^i \left\{(1-\beta)\pi_d \mathbb{E}_t X_{t+1+i} - (\pi_r + \pi_{er}^{\text{H}})\, \mathbb{E}_t X_{t+i}\right\}, \\
&= \left[(1-\beta)\pi_d \mathbb{A} - (\pi_r + \pi_{er}^{\text{H}})\right] (I - \beta\mathbb{A})^{-1} X_t.
\end{aligned}$$

Thus, given our assumption about dividends, the risk free rate, and the optimality of portfolio choices we find that log equity prices satisfy $p_t = \pi_p X_t$ where

$$\pi_p = \left[(1-\beta)\pi_d \mathbb{A} - (\pi_r + \pi_{er}^{\text{H}})\right] (I - \beta\mathbb{A})^{-1}. \tag{37}$$

A similar exercise confirms that $\hat{p}_t = \pi_{\hat{p}} X_t$ where

$$\pi_{\hat{p}} = \left[(1-\beta)\pi_{\hat{d}} \mathbb{A} - (\pi_r + \pi_{er}^{\text{F}})\right] (I - \beta\mathbb{A})^{-1}. \tag{38}$$

The restrictions in (37) and (38) depend on the form of the dividend policies via the π_d and $\pi_{\hat{d}}$ vectors. These vectors are determined by the firms' first order conditions. In particular, using the fact that $\mathbb{V}_t\left(r_{t+1}^k\right) = \psi^2 \mathbb{V}_t\left(z_{t+1}\right)$ and $\mathbb{CV}_t\left(r_{t+1}^k, w_{t+1}\right) = \psi \mathbb{CV}_t\left(z_{t+1}, w_{t+1}\right)$ from (28), we can use R1 and R2 to write the log-approximated first order condition for H firm in (27a) as

$$\begin{aligned}
\mathbb{E}_t r_{t+1}^k &= r_t + \mathbb{CV}_t\left(r_{t+1}^k, w_{t+1}\right) - \tfrac{1}{2}\mathbb{V}_t\left(r_{t+1}^k\right), \\
&= \left[\pi_r + \psi\left(\mathcal{B}(\pi_\alpha^{\mathrm{H}}, \mathcal{A}(\pi_{\mathrm{H}}, h_z)) + \mathcal{B}(\pi_\alpha^{\mathrm{F}}, \mathcal{A}(\pi_{\mathrm{F}}, h_z))\right) - \tfrac{1}{2}\psi^2\mathcal{A}(h_z, h_z)\right] X_t.
\end{aligned}$$

At the same time, (28) and (29) imply that

$$\begin{aligned}
\mathbb{E}_t r_{t+1}^k &= \psi\mathbb{E}_t z_{t+1} - (1-\theta)\psi\left\{\tfrac{1}{\beta}k_t + \tfrac{\psi}{\beta\theta}z_t - \left(\tfrac{\psi}{\theta\beta} - \delta\right)d_t\right\}, \\
&= \left[\psi h_z\mathbb{A} - (1-\theta)\psi\left\{\tfrac{1}{\beta}h_k + \tfrac{\psi}{\beta\theta}h_z - \left(\tfrac{\psi}{\theta\beta} - \delta\right)\pi_d\right\}\right] X_t.
\end{aligned}$$

Combining these expressions and equating coefficients gives

$$\begin{aligned}
\pi_d = & \tfrac{\theta\beta}{(1-\theta)\psi(\psi-\delta\theta\beta)}\left\{\pi_r + \psi\left(\mathcal{B}(\pi_\alpha^{\mathrm{H}}, \mathcal{A}(\pi_{\mathrm{H}}, h_z)) + \mathcal{B}(\pi_\alpha^{\mathrm{F}}, \mathcal{A}(\pi_{\mathrm{F}}, h_z))\right) - \tfrac{1}{2}\psi^2\mathcal{A}(h_z, h_z) - \psi h_z\mathbb{A}\right\} \\
& + \frac{1}{(\psi - \delta\theta\beta)}\left\{\theta h_k + \psi h_z\right\}.
\end{aligned}$$

The first order condition for F firm gives an analogous expression for $\pi_{\hat{d}}$.

The behavior of the non-predetermined variables must also be consistent with market clearing. According to (26), market clearing in the bonds requires that $p_t + \hat{p}_t = d_t + \hat{d}_t$, a condition that implies

$$\pi_p + \pi_{\hat{p}} = \pi_d + \pi_{\hat{d}}.$$

In the case of the H and F equity markets we need

$$\begin{aligned}
1 + p_t - w_t + \tfrac{1}{2}\left(p_t - w_t\right)^2 &= \alpha_t^{\mathrm{H}} + \hat{\alpha}_t^{\mathrm{H}}\left(1 + \hat{w}_t - w_t + \tfrac{1}{2}\left(\hat{w}_t - w_t\right)^2\right), \\
1 + \hat{p}_t - \hat{w}_t + \tfrac{1}{2}\left(\hat{p}_t - \hat{w}_t\right)^2 &= \hat{\alpha}_t^{\mathrm{F}} + \alpha_t^{\mathrm{F}}\left(1 + w_t - \hat{w}_t + \tfrac{1}{2}\left(w_t - \hat{w}_t\right)^2\right).
\end{aligned}$$

Rewriting these equations in terms of X_t, applying R2, and equating coefficients gives

$$\begin{aligned}
h_1 + \pi_p - h_w + \tfrac{1}{2}\mathcal{B}\left(\pi_p - h_w, \pi_p - h_w\right) &= \pi_\alpha^{\mathrm{H}} + \mathcal{B}\left(\pi_{\hat{\alpha}}^{\mathrm{H}}, \left(h_1 + h_{\hat{w}} - h_w + \tfrac{1}{2}\mathcal{B}\left(h_{\hat{w}} - h_w, h_{\hat{w}} - h_w\right)\right)\right), \\
h_1 + \pi_{\hat{p}} - h_{\hat{w}} + \tfrac{1}{2}\mathcal{B}\left(\pi_{\hat{p}} - h_{\hat{w}}, \pi_{\hat{p}} - h_w\right) &= \pi_{\hat{\alpha}}^{\mathrm{F}} + \mathcal{B}\left(\pi_\alpha^{\mathrm{F}}, \left(h_1 + h_w - h_{\hat{w}} + \tfrac{1}{2}\mathcal{B}\left(h_w - h_{\hat{w}}, h_w - h_{\hat{w}}\right)\right)\right).
\end{aligned}$$

The remaining market clearing condition comes from the goods market. Walras Law makes this condition redundant when the restrictions implied by the other market clearing conditions are imposed, so there is no need to consider its implications directly.

2.5 Numerical Procedure

We have described how the log-approximated equations characterizing the equilibrium of the model are used to derive a set of restrictions on the behavior of the state vector and the non-predetermined variables. A solution to the model requires that we find values for all the parameters in process for X_t and Y_t that satisfy these restrictions given values for the exogenous taste and technology parameters. More formally, let Υ denote all the elements of the policy matrix Π, the coefficient matrices $\{\Phi_0, \Phi_1, \Phi_2\}$ and the second moment coefficients in Σ. Our objective is to find the value for Υ such that the residual function

$$\mathcal{R}(X_t; \widehat{\mathcal{G}}, \widehat{\mathcal{H}}, \widehat{\mathcal{S}}, \zeta, \psi, \delta, s) = \widehat{\mathcal{F}}\left(\Pi \mathbb{A} X_t + \Pi U_{t+1}, \Pi X_t, \mathbb{A} X_t + U_{t+1}, X_t, \widehat{\mathcal{S}}(X_t)\right)$$

reaches its minimum. For this purpose we choose Υ to minimize the least squares projection

$$||\widehat{\mathcal{F}}\left(\Pi \mathbb{A} X_t + \Pi U_{t+1}, \Pi X_t, \mathbb{A} X_t + U_{t+1}, X_t, \widehat{\mathcal{S}}(X_t)\right)||^2$$

where $||\ .\ ||$ denotes the Euclidean norm. As before $\widehat{\mathcal{F}}(.)$ consists of the approximate equilibrium conditions, including the restrictions on the second moments, implied by the model. The matrix $\mathbb{A}$ and function $\widehat{\mathcal{S}}(X_t)$ are specified in terms of Υ from equations (35) and (36).

3 Results

The one-sector model provides an environment in which we can assess the accuracy of our solution method. In particular, the structure of the model is sufficiently simple for us to analytically determine the equilibrium portfolio holdings of households. We can therefore compare these holdings to those implied by the numerical solution to the model.

The analytic solution to the model is based on the observation that the array of assets available to households (i.e., equity issued by H and F firms and risk free bonds) permits complete risk-sharing. We can see why this is so by returning to conditions determining the household portfolio choices. In particular, combining the log-approximated first order conditions with the budget constraint as shown in (24) under the assumption of log preferences, we obtain

$$\boldsymbol{\alpha}_t = \Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}(\Theta_t)) \qquad \text{and} \qquad \hat{\boldsymbol{\alpha}}_t = \Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}(\Theta_t)), \tag{39}$$

where, as before, $\boldsymbol{\alpha}'_t \equiv [\ \alpha_t^{\mathrm{H}} \quad \alpha_t^{\mathrm{F}}\ ]$, $\hat{\boldsymbol{\alpha}}'_t \equiv [\ \hat{\alpha}_t^{\mathrm{H}} \quad \hat{\alpha}_t^{\mathrm{F}}\ ]$, $er'_{t+1} \equiv [\ r_{t+1}^{\mathrm{H}} - r_t \quad r_{t+1}^{\mathrm{F}} - r_t\ ]$, and $\Theta_t \equiv \mathbb{V}_t(er_{t+1})$. The key point to note here is that all households face the same set of returns and have the same information. So the right hand side of both expressions in (39) are identical in equilibrium. H and F households will therefore find it optimal to hold the same portfolio shares. This has a number of implications if the initial distribution of wealth is equal. First, household wealth will be equalized across countries. Second, since households with log utility consume a constant fraction of wealth, consumption will also be equalized. This symmetry in household behavior together with the market clearing conditions implies that bond holdings are zero and wealth is equally split between H and F equities (i.e., $\alpha_t^{\mathrm{H}} = \hat{\alpha}_t^{\mathrm{H}} = \alpha_t^{\mathrm{F}} = \hat{\alpha}_t^{\mathrm{F}} = 1/2$). The symmetry in consumption also implies that $m_{t+1} = \hat{m}_{t+1}$ so risk sharing is complete.

Table 1 reports statistics on the simulated portfolio holdings of households computed from the numerical solution to the model. For this purpose we used the solution method described above to find the parameters of X_t and Y_t processes consistent with the log-approximated equilibrium conditions. These calculations were performed assuming a discount factor β equal to 0.99, the technology parameter θ equal to 0.36 and a depreciation rate for capital, δ, of 0.02. The log of H and F productivity, $\ln Z_t$ and $\ln \hat{Z}_t$, are assumed to follow independent AR(1) processes with the same autocorrelation coefficient, a_{ii}, $i = \{\text{H}, \text{F}\}$, equal to 0.95 and innovation variance S_e^{ii}, $i = \{\text{H}, \text{F}\}$, equal to 0.0001. Once the model is "solved", we simulate X_t over 500 quarters starting from an equal wealth distribution. We then discard the first 100 quarters from each simulation. The statistics we report in Table 1 are derived from 100 simulations and so are based on 10,000 years of simulated quarterly data in the neighborhood of the initial wealth distribution.[13]

Table 1: Simulated Portfolio Holdings (One-Sector Model)

	A_t^{H} (i)	A_t^{F} (ii)	B_t (iii)
			% GDP
mean	0.5000	0.5000	0.00%
stdev	0.0000	0.0000	0.25%
min	0.4999	0.4999	-1.52%
max	0.5001	0.5001	2.53%

Note: A_t^{H} and A_t^{F} correspond, respectively, to H household's holdings of equity issued by H and F firms. B_t refers to H household's bond holdings as a share of H GDP.

Columns (i), (ii) and (iii) report statistics on the asset holdings of H households computed from the model simulations. Theoretically speaking, we should see that $B_t = 0$ and $A_t^{\text{H}} = A_t^{\text{F}} = 0.5$. (Recall that the supply of H and F equity are both normalized to unity.) The simulation results conform closely to these predictions. The equity portfolio holdings show no variation and on average are exactly as theory predicts. Average bond holdings, measured as a share of model's GDP are similarly close to zero, but show a little more variation. Overall, simulations based on our solution technique appear to closely replicate the complete risk sharing allocation theory predicts.

4 Method Accuracy

To assess the performance of our solution method more formally we compute several tests of model accuracy. First, we evaluate the importance of the third and higher order terms omitted in the model solution. Second, we report the size of Euler equation errors. Third, we compute a summary measure of accuracy based on

[13] The innovations to equilibrium wealth are small enough to keep H and F wealth close to their initial levels over a span of 500 quarters so the approximation error in (32) remains very small.

the den Haan and Marcet (1994) χ^2 test. All the results in this section are based on 100 simulations of the model, each simulation being 3000 quarters long. Throughout we will use $|\ .\ |$ to denote the absolute value.

Higher-order Terms

Recall that when deriving the approximate dynamics of the state vector in equation (35) we ignored the impact of cubic and higher-order terms of x_t. In this way we abstracted from the role of skewness, kurtosis, and higher-order moments of returns for the portfolio decisions of households. We now evaluate the importance of these terms in model simulations. Using our previous notation (where we used x_t to denote linear states in deviations from steady state or initial distribution and $\tilde{x}_t$ to denote their quadratic transformation), we obtain the third-order terms as $vec(x_t\tilde{x}_t')$ and calculate their summary statistics. In particular, we obtain maximum, average, and standard deviation of long simulations for each element in $|vec(x_t\tilde{x}_t')|$, and then compute the distribution of those statistics across all the elements. Table 2 reports the 90th, 95th and 99th percentiles of the corresponding distributions. We find that 99% of the largest third-order terms in $|vec(x_t\tilde{x}_t')|$ are smaller than 5.58E-03. Among the average absolute third-order terms, 99% lie to the left of 1.18E-04, while the standard deviation of third-order terms exceeds 4.28E-03 only 1 % of the time.

Table 2. Accuracy: 3rd Order Terms (One-Sector Model)

	90% (i)	95% (ii)	99% (iii)
max	2.92E-03	3.58E-03	5.58E-03
mean	6.23E-05	8.30E-05	1.18E-04
stdev	2.51E-03	3.31E-03	4.28E-03

Note: max, mean and stdev refer to the corresponding summary statistic calculated for each element in the absolute vector of third-order terms, $|vec(x_t\tilde{x}_t')|$. 90%, 95%, and 99% stand for the respective percentiles of the distributions of these summary statistics across the cross-section of $|vec(x_t\tilde{x}_t')|$.

Euler Equation Errors

In order to assess the accuracy of model approximations Judd (1992) recommends using the size of the errors that households and firms make. Following this approach we use the simulated series for the model variables to derive these errors as

$$\begin{aligned}\xi_{t+1} \quad = \quad & f(\widehat{\mathcal{G}}\left(\widehat{\mathcal{H}}\left(X_t, \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}\right),\ \ \widehat{\mathcal{S}}\left(\widehat{\mathcal{H}}\left(X_t, \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}\right)\right)\right), \\ & \widehat{\mathcal{G}}\left(X_t, \widehat{\mathcal{S}}\left(X_t\right)\right), \widehat{\mathcal{H}}\left(X_t, \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}\right),\ \ X_t,\ \ \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}),\end{aligned} \tag{40}$$

where $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$, and $\widehat{\mathcal{S}}$ are the approximate decision rules and ξ_{t+1} is a $g \times 1$ vector. In our one-sector model this system corresponds to a set of $g = 4$ Euler equations in each country: two for equity, one for capital, and one for bonds. As an illustration, the Euler equation error for H households is given by

$$\xi_{t+1} = 1 - [M_{t+1} \otimes R^{\varkappa}],$$

where $R^{\varkappa} = \{R^{\mathrm{H}}_{t+1}, R^{\mathrm{F}}_{t+1}, R^{k}_{t+1}, R_t\}$ and $M_{t+1} = \beta W_t / W_{t+1}$. Note that ξ provides a scale-free measure of the error. An analogous sequence of Euler equation errors exists for F households. Table 3 reports the upper percentiles of the distribution of $|\xi|$. Columns (i) and (ii) show percentiles for the errors from H households' Euler equations for H and F equity; column (iii) reports errors for the firm's optimality condition; and column (iv) is for the bond Euler equation errors. The numbers reported in the table are comparable to those reported in the accuracy checks for standard growth models without portfolio choice [e.g., Arouba et al. (2005) and Pichler (2005)].

Table 3. Accuracy: EE Errors (One-Sector Model)

	A^{H}_t (i)	A^{F}_t (ii)	K_t (iii)	B_t (iv)
90$^{\text{th}}$ percentile	0.0026	0.0026	0.0029	0.0033
95$^{\text{th}}$ percentile	0.0031	0.0031	0.0035	0.0039
99$^{\text{th}}$ percentile	0.0040	0.0040	0.0046	0.0051

Note: A^{H}_t and A^{F}_t refer to the absolute errors from the Euler equations for H household's holdings of equity issued by H and F firms; K_t and B_t correspond to the absolute errors from capital and bond Euler equations at H.

The Den Haan and Marcet Test

While the summary statistics on Euler equation errors provide a quick assessment of model accuracy, one may want to construct a more formal metric that simultaneously tests the size of the errors from all the optimality conditions in the model. Den Haan and Marcet (1994) develop such a metric. Their test of approximation accuracy consists of checking whether Euler equation errors are orthogonal to any function of the state variables describing the information set in period t. Consider again the Euler equation errors ξ derived in (40) and let $\omega(X_t)$ denote any function that converts the $\mathfrak{L}-$dimensional vector of state variables X_t into a $q-$dimensional sequence of instrumental variables, $\omega : \mathbb{R}^{\mathfrak{L}} \longrightarrow \mathbb{R}^q$. Then, if households form their expectations rationally, the Euler equation errors must satisfy

$$\mathbb{E}[\xi_{t+1} \otimes \omega(X_t)] = 0. \tag{41}$$

The idea behind the test consists of evaluating how close is equation (41) to being satisfied for the simulated series X_t and for any function $\omega(.)$. In particular, let *bars* denote simulated data from the model, allowing us to calculate the sample analog of (41) as

$$B_T = \frac{1}{T}\sum^{T} \bar{\xi}_{t+1} \otimes \omega(\bar{X}_t),$$

where T is a simulated sample size. Den Haan and Marcet (1994) evaluate whether B_T is close to zero by constructing a test-statistic

$$J_T = TB_T' A_T^{-1} B_T, \tag{42}$$

where A_T is a consistent estimate of the matrix

$$\sum_{i=-\infty}^{\infty} \mathbb{E}\left[\left(\xi_{t+1} \otimes \omega(X_t)\right)\left(\xi_{t+1-i} \otimes \omega(X_{t-i})\right)'\right].$$

Under the null that the solution is accurate and if X_t is stationary and ergodic, den Haan and Marcet (1994) show that J_T converges in distribution to χ^2 with qg degrees of freedom. The solution is considered accurate if J_T is in the non-critical region of χ^2_{qg} distribution.

Table 4. Accuracy: den Haan and Marcet χ^2 Test (One-Sector Model)

	A_t^{H} and K_t (i)	A_t^{H} (ii)	A_t^{F} (iii)	K_t (iv)	B_t (v)
lower 5%	0.05	0.05	0.05	0.04	0.07
upper 5%	0.02	0.03	0.03	0.04	0.06

Note: A_t^{H} and A_t^{F} correspond to the percentiles of the χ^2 test statistics calculated based on the errors from H country Euler equation for H and F equity; K_t and B_t refer to the percentiles of the χ^2 test statistics for capital and bond Euler equations at H.

Implementing the test on our solution to the one-sector model requires care. In this particular case we know that equilibrium wealth, consumption growth and the two IMRS, m_{t+1} and $\hat{m}_{t+1}$, are perfectly collinear. As a consequence, the errors from the Euler equations for equities, bonds and capital are very highly correlated. This makes it impossible to invert matrix A_T that enters the test statistic in (42) accurately. Indeed, we find that the condition number for matrix A_T calculated based on the Euler equations within each country to be in excess of 10^6. We therefore focus on two Euler equation errors in our accuracy tests: one from households' H equity Euler equation; and one from the Euler equation for capital. The results are reported in column (i) of Table 4. In columns (ii)-(v) we also report tests for each Euler equation in the H country individually. In all cases the vector of instruments $\omega(X_t)$ consists of a constant, $\{Z_t, \hat{Z}_t, K_t, \hat{K}_t, \Delta W_t\}$, and the two lags of $\{K_t, \hat{K}_t, \Delta W_t\}$.[14] We obtain the matrix A_T from the standard GMM estimate that allows for heteroskedasticity but no serial correlation in the errors.

Following den Haan and Marcet (1994) we repeat the test N times for different realizations of the stochastic processes and compare the resulting distribution of J_T with its true distribution. Table 4 reports the percentage of realizations of J_T in the lower and upper 5% of a χ^2_{qg} distribution obtained from 100 repetitions of the test. In each repetition we used 3000 quarters from simulations of the approximate solution to the model. As the table shows, our method is extremely accurate. The distributions of J_T statistics for each Euler equation follow the true χ^2_{12} very closely as the simulated percentiles almost coincide with the true ones. This can be seen especially clearly from Figure 1 which plots the cdf of the χ^2_{24} distribution and the cdf of the test statistic calculated for the H equity and capital Euler equations jointly.

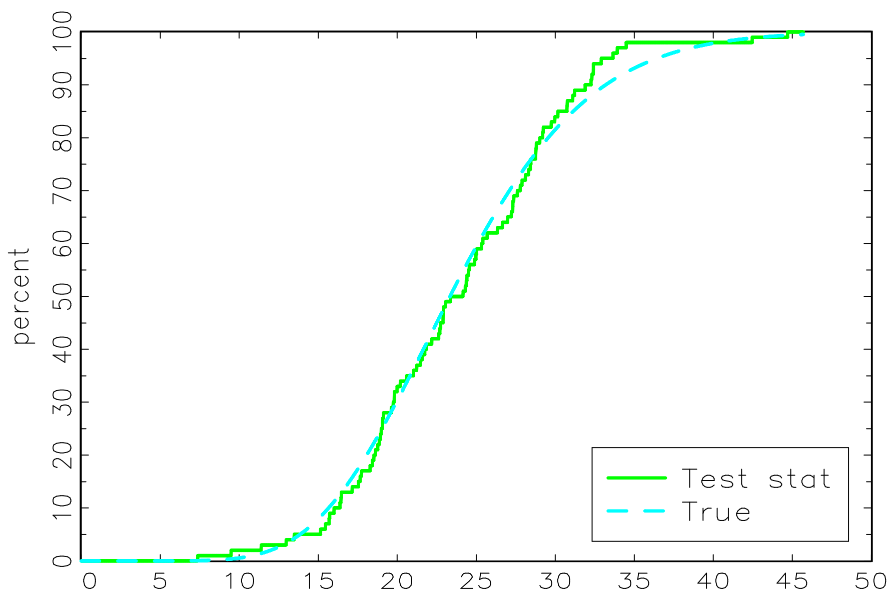

Figure 1. Distribution of the χ^2 test statistic

Note: Solid line "Test stat" corresponds to the cdf of the χ^2 test statistic obtained using simulated H equity and capital EE errors. Dashed line "True" plots the χ^2_{24} cdf.

5 The Two-Sector Model

The power of our solution procedure resides in its applicability to models with portfolio choice and incomplete markets. Analytic solutions are unavailable in these models and existing numerical solution methods are inapplicable. In this section we consider a two–sector extension of the model in which markets are incomplete. A detailed analysis of this model is provided in a companion paper, Evans and Hnatkovska (2005). In what

[14] We did not use lagged productivity shocks to reduce the collinearity across the set of instruments. We include the first difference of wealth to insure that our instruments are stationary.

follows we highlight the main differences of this model from the one-sector economy described earlier, and summarize its portfolio implications.

5.1 The Model

In this version of the model households in the two countries have preferences defined over the consumption of two goods: a tradable and nontradable. The preferences of a representative household in the H country are given by

$$\mathbb{E}_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}^{\mathrm{T}}, C_{t+i}^{\mathrm{N}}),$$

where $0 < \beta < 1$ is the discount factor, and $U(.)$ is a concave sub-utility function defined over the consumption of traded and non-traded goods, C_t^{T} and C_t^{N} :

$$U(C^{\mathrm{T}}, C^{\mathrm{N}}) = \frac{1}{\phi} \ln \left[\lambda_{\mathrm{T}}^{1-\phi} \left(C^{\mathrm{T}}\right)^{\phi} + \lambda_{\mathrm{N}}^{1-\phi} \left(C^{\mathrm{N}}\right)^{\phi} \right],$$

with $\phi < 1$. λ_{T} and λ_{N} are the weights the household assigns to tradable and nontradable consumption respectively. The elasticity of substitution between tradable and nontradable consumption is $(1-\phi)^{-1} > 0$. Preferences for households in country F are similarly defined in terms of foreign consumption of tradables and non-tradables, $\hat{C}_t^{\mathrm{T}}$ and $\hat{C}_t^{\mathrm{N}}$. Notice that preferences are not separable across the two consumption goods.

The menu of assets available to households now includes the equity issued by H and F firms producing tradable goods, risk free bonds, and the equity issued by domestic firms producing nontradable goods. Households are not permitted to hold the equity of foreign firms producing nontradable goods. With the new array of assets, the budget constraint for H households becomes

$$W_{t+1} = R_{t+1}^{\mathrm{W}} \left(W_t - C_t^{\mathrm{T}} - Q_t^{\mathrm{N}} C_t^{\mathrm{N}}\right),$$

where $$R_{t+1}^{\mathrm{W}} = R_t + \alpha_t^{\mathrm{H}}(R_{t+1}^{\mathrm{H}} - R_t) + \alpha_t^{\mathrm{F}}(R_{t+1}^{\mathrm{F}} - R_t) + \alpha_t^{\mathrm{N}}(R_{t+1}^{\mathrm{N}} - R_t).$$

Q_t^{N} is the relative price of H nontradables in terms of tradables, and R_{t+1}^{H} and R_{t+1}^{F}, as before, are the returns on equity issued by the firms producing traded goods at home and abroad:

$$R_{t+1}^{\mathrm{H}} = \left(P_{t+1}^{\mathrm{H}} + D_{t+1}^{\mathrm{H}}\right)/P_t^{\mathrm{H}} \qquad \text{and} \qquad R_{t+1}^{\mathrm{F}} = \left(P_{t+1}^{\mathrm{F}} + D_{t+1}^{\mathrm{F}}\right)/P_t^{\mathrm{F}}.$$

P_t^{H} (P_t^{F}) is the price of equity issued by H (F) country firms producing traded goods, while D_t^{H} (D_t^{F}) is the corresponding flow of dividends. R_{t+1}^{N} is the return on equity issued by domestic firms producing nontradables, measured in terms of tradables:

$$R_{t+1}^{\mathrm{N}} = \left\{\left(P_{t+1}^{\mathrm{N}} + D_{t+1}^{\mathrm{N}}\right)/P_t^{\mathrm{N}}\right\}\left\{Q_{t+1}^{\mathrm{N}}/Q_t^{\mathrm{N}}\right\},$$

where P_t^{N} is the price of equity issued by H firms producing nontradables and D_t^{N} is the flow of dividends, both measured in terms of nontradables. The budget constraint and returns on F household wealth are analogously defined.

The production side of the model remains unchanged aside from the addition of the nontradable sector in each country. For simplicity we assume that the production of nontradables requires no capital. Nontradable output in countries H and F is given by ηZ_t^{N} and $\eta \hat{Z}_t^{\text{N}}$, where $\eta > 0$ is a constant. Z_t^{N} and $\hat{Z}_t^{\text{N}}$ denote the period$-t$ state of nontradable productivity in countries H and F respectively. The productivity vector is now $z_t \equiv [\ln Z_t^{\text{T}}, \ln \hat{Z}_t^{\text{T}}, \ln Z_t^{\text{N}}, \ln \hat{Z}_t^{\text{N}}]'$. We continue to assume that z_t follows an AR(1) process:

$$z_t = a z_{t-1} + S_e^{1/2} e_t,$$

where e_t is a 4×1 vector of i.i.d. mean zero, unit variance shocks, and $S_e^{1/2}$ is the scaling matrix.

5.2 Equilibrium

As in a one-sector model, the equilibrium conditions comprise the first-order conditions of households and firms and the market clearing conditions. Since the production of nontradable output requires no capital, firms in this sector simply pass on their revenues to shareholders in the form of dividends. In the tradable sector, the first-order conditions governing dividends remain unchanged. Optimal household behavior now covers the choice between different consumption goods, and a wider array of financial assets. The first-order conditions for H households, in addition to (8), now include

$$\begin{aligned} Q_t^{\text{N}} &= \frac{\partial U / \partial C_t^{\text{N}}}{\partial U / \partial C_t^{\text{T}}}, \\ 1 &= \mathbb{E}_t \left[M_{t+1} R_{t+1}^{\text{N}} \right], \end{aligned}$$

where $M_{t+1} \equiv \beta(\partial U / \partial C_{t+1}^{\text{T}}) / (\partial U / \partial C_t^{\text{T}})$. The first order conditions for F households are expanded in an analogous manner.

Solving for an equilibrium now requires finding equity prices, $\{P_t^{\text{H}}, P_t^{\text{F}}, P_t^{\text{N}}, \hat{P}_t^{\text{N}}\}$, goods prices, $\{Q_t^{\text{N}}, \hat{Q}_t^{\text{N}}\}$, and the interest rate, R_t, such that markets clear when households follow optimal consumption, saving and portfolio strategies, and firms in the tradable sector make optimal investment decisions. As above, we assume that bonds are in zero net supply so that (10) continues to be the bond market clearing condition. Similarly, equation (11) is the market clearing condition in tradable goods market. Market clearing in the non-tradable sector of each country requires that

$$C_t^{\text{N}} = Y_t^{\text{N}} = D_t^{\text{N}}, \qquad \text{and} \qquad \hat{C}_t^{\text{N}} = \hat{Y}_t^{\text{N}} = \hat{D}_t^{\text{N}}.$$

As above, we normalize the number of outstanding shares issued by firms in each sector to unity so market clearing in the equity markets requires that

$$1 = A_t^{\mathrm{H}} + \hat{A}_t^{\mathrm{H}}, \qquad 1 = A_t^{\mathrm{F}} + \hat{A}_t^{\mathrm{F}},$$
$$1 = A_t^{\mathrm{N}}, \qquad 1 = \hat{A}_t^{\mathrm{N}}.$$

A_t^{N} and $\hat{A}_t^{\mathrm{N}}$ are the number of shares held by H and F households in domestic nontradable firms. Asset holdings are obtained from portfolio shares using the identities:

$$\begin{array}{lll} & \text{H households} & \text{F households} \\ \text{H tradable equity:} & A_t^{\mathrm{H}} = \alpha_t^{\mathrm{H}} W_t^{\mathrm{C}}/P_t^{\mathrm{H}}, & \hat{A}_t^{\mathrm{H}} = \hat{\alpha}_t^{\mathrm{H}} \hat{W}_t^{\mathrm{C}}/P_t^{\mathrm{H}}, \\ \text{F tradable equity:} & A_t^{\mathrm{F}} = \alpha_t^{\mathrm{F}} W_t^{\mathrm{C}}/P_t^{\mathrm{F}}, & \hat{A}_t^{\mathrm{F}} = \hat{\alpha}_t^{\mathrm{F}} \hat{W}_t^{\mathrm{C}}/P_t^{\mathrm{F}}, \\ \text{nontradable equity:} & A_t^{\mathrm{N}} = \alpha_t^{\mathrm{N}} W_t^{\mathrm{C}}/Q_t^{\mathrm{N}} P_t^{\mathrm{N}}, & \hat{A}_t^{\mathrm{N}} = \hat{\alpha}_t^{\mathrm{N}} \hat{W}_t^{\mathrm{C}}/\hat{Q}_t^{\mathrm{N}} \hat{P}_t^{\mathrm{N}}, \\ \text{bonds} & B_t = \alpha_t^{\mathrm{B}} W_t^{\mathrm{C}} R_t, & \hat{B}_t = \hat{\alpha}_t^{\mathrm{B}} \hat{W}_t^{\mathrm{C}} R_t, \end{array} \tag{43}$$

where $W_t^{\mathrm{C}} \equiv W_t - C_t^{\mathrm{T}} - Q_t^{\mathrm{N}} C_t^{\mathrm{N}}$ and $\hat{W}_t^{\mathrm{C}} \equiv \hat{W}_t - \hat{C}_t^{\mathrm{T}} - \hat{Q}_t^{\mathrm{N}} \hat{C}_t^{\mathrm{N}}$ denote period$-t$ wealth net of consumption expenditure with $\alpha_t^{\mathrm{B}} \equiv 1 - \alpha_t^{\mathrm{H}} - \alpha_t^{\mathrm{F}} - \alpha_t^{\mathrm{N}}$ and $\hat{\alpha}_t^{\mathrm{B}} \equiv 1 - \hat{\alpha}_t^{\mathrm{H}} - \hat{\alpha}_t^{\mathrm{F}} - \hat{\alpha}_t^{\mathrm{N}}$.

5.3 Results

Table 5 reports statistics on the simulated portfolio holdings of households computed from our solution to the two-sector model. The results are based on the same values for β, θ, δ, and S_e. In addition, we set the share parameters λ^{T} and $\hat{\lambda}^{\mathrm{T}}$ equal 0.5 and the elasticity of substitution $1/(1-\phi)$ equal to 0.74. The autocorrelation in nontradable and tradable productivity is set to 0.99 and 0.78 respectively. Innovations to productivity are assumed to be i.i.d. with variance equal to $S_e^{ii} = 0.0001$, $i = \{\mathrm{H}, \mathrm{F}, \mathrm{N}, \hat{\mathrm{N}}\}$. As above, the statistics are computed from model simulations covering 10,000 years of quarterly data.

Table 5: Simulated Portfolio Holdings (Two-Sector Model)

	A_t^{H} (i)	A_t^{F} (ii)	A_t^{N} (iii)	B_t (iv)
				% GDP
mean	0.5000	0.5000	1.0000	-0.23%
stdev	0.0019	0.0019	0.0000	12.19%
min	0.4918	0.4925	1.0000	-40.29%
max	0.5076	0.5077	1.0000	71.19%

Note: A_t^{H}, A_t^{F}, and A_t^{N} correspond, respectively, to H household's holdings of equity issued by H, F traded firms, and H nontraded firms. B_t refers to H household's bond holdings as a share of H GDP.

Columns (i) - (iv) report statistics on the asset holdings of H households computed from the model simulations. As in the one-sector model, households continue to diversify their holdings between the equity issued by H and F firms producing tradable goods. (Household holdings of equity issued by domestic firms producing nontradable goods must equal unity to clear the market.) While these holdings are split equally on average, they are far from constant. Both the standard deviation and range of the tradable equity holdings are orders of magnitude larger than the simulated holdings from the one-sector model. Differences between the one- and two-sector models are even more pronounced for bond holdings. In the two-sector model shocks to productivity in the nontradable sector affect H and F households differently and create incentives for international borrowing and lending. In equilibrium most of this activity takes place via trading in the bond market, so bond holdings display a good deal of volatility in our simulations.

In section 2 we highlighted one of the main novelties of our approach, the conditional heteroskedasticity of the state vector, $\mathcal{S}(X_t)$. We argued that this feature allowed us to account for time-variability in risk premium and households' optimal portfolio rules. To further emphasize this point, consider the implications of setting $\mathcal{S}(X_t)$ equal to a constant matrix. In this case our solution method simplifies to a standard projection-perturbation routine and produces time-invariant risk premia and portfolio shares. Specifically, let $\boldsymbol{\alpha}_t' \equiv [\ \alpha_t^{\mathrm{H}} \quad \alpha_t^{\mathrm{F}} \quad \alpha_t^{\mathrm{N}}\ ]$ and $\hat{\boldsymbol{\alpha}}_t' \equiv [\ \hat{\alpha}_t^{\mathrm{H}} \quad \hat{\alpha}_t^{\mathrm{F}} \quad \hat{\alpha}_t^{\mathrm{N}}\ ]$ denote the vector of portfolio shares for H and F households in the two-sector model. Then,

$$\boldsymbol{\alpha}_t = \Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}(\Theta_t)) \qquad \text{and} \qquad \hat{\boldsymbol{\alpha}}_t = \widehat{\Theta}_t^{-1}(\mathbb{E}_t \widehat{er}_{t+1} + \tfrac{1}{2}\mathrm{diag}(\widehat{\Theta}_t)), \tag{44}$$

with $\Theta_t = \mathbb{V}_t(er_{t+1})$ and $\widehat{\Theta}_t = \mathbb{V}_t(\widehat{er}_{t+1})$, where $er_{t+1}' \equiv [\ r_{t+1}^{\mathrm{H}} - r_t \quad r_{t+1}^{\mathrm{F}} - r_t \quad r_{t+1}^{\mathrm{N}} - r_t\ ]$ and $\widehat{er}_{t+1}' \equiv [\ r_{t+1}^{\mathrm{H}} - r_t \quad r_{t+1}^{\mathrm{F}} - r_t \quad \hat{r}_{t+1}^{\mathrm{N}} - r_t\ ]$ are the vectors of excess returns. [This equation is analogous to (39) in the one-sector model.] From the Euler equations we also know that the risk-premium on each asset is given by the covariance of the return on that asset with wealth:

$$\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}(\Theta_t) = \mathbb{CV}_t(w_{t+1}, er_{t+1}) \qquad \text{and} \qquad \mathbb{E}_t \widehat{er}_{t+1} + \tfrac{1}{2}\mathrm{diag}(\widehat{\Theta}_t)) = \mathbb{CV}_t(\hat{w}_{t+1}, \widehat{er}_{t+1}). \tag{45}$$

Hence, the portfolio rules in (44) can be rewritten as

$$\boldsymbol{\alpha}_t = \Theta_t^{-1}\mathbb{CV}_t(w_{t+1}, er_{t+1}) \qquad \text{and} \qquad \hat{\boldsymbol{\alpha}}_t = \widehat{\Theta}_t^{-1}\mathbb{CV}_t(\hat{w}_{t+1}, \widehat{er}_{t+1}). \tag{46}$$

When $\mathcal{S}(X_t)$ is constant, so too are the second moments of the state vector X_t and linear functions of X_t. This means that $\mathbb{CV}_t(w_{t+1}, er_{t+1})$ and $\mathbb{CV}_t(\hat{w}_{t+1}, \widehat{er}_{t+1})$ are constant with the result that both the risk premia in (45) and the portfolio shares in (46) are time-invariant.

Restricting $\mathcal{S}(X_t)$ to a constant matrix also has implications for portfolio holdings. To illustrate, consider the flow of H tradable equity held by H households. From the definition of A_t^{H} in (43) we can write

$$\begin{aligned} P_t^{\text{H}}\Delta A_t^{\text{H}} &= \alpha_t^{\text{H}}W_t^{\text{C}} - \alpha_{t-1}^{\text{H}}W_{t-1}^{\text{C}}\frac{P_t^{\text{H}}}{P_{t-1}^{\text{H}}}, \\ &= \Delta\alpha_t^{\text{H}}W_t^{\text{C}} + \left[\alpha_{t-1}^{\text{H}}\Delta W_t^{\text{C}} - \left(\frac{P_t^{\text{H}}}{P_{t-1}^{\text{H}}} - 1\right)W_{t-1}^{\text{C}}\alpha_{t-1}^{\text{H}}\right]. \end{aligned} \tag{47}$$

The first term on the right captures the variations in portfolio holdings due to shifts in the risk-premia, $\Delta\alpha_t^{\text{H}}$. The second term identifies portfolio rebalancing factors arising from changes in wealth or asset valuations. Clearly, if $\boldsymbol{\alpha}_t$ is constant because $\mathcal{S}(X_t)$ is restricted to be a constant matrix, only the rebalancing factors drive portfolio holdings. We can assess the quantitative significance of this restriction by computing the contribution of the terms in (47) to the variance of $P_t^{\text{H}}\Delta A_t^{\text{H}}$ from simulations of the model (i.e., using our solution method where $\mathcal{S}(X_t)$ is unrestricted). This calculation reveals that both terms contribute to the variations in portfolio holdings, but in different directions [see Evans and Hnatkovska (2005) for details]. In other words, shocks that increase the first term, lower the second, but their combined effect increases $P_t^{\text{H}}\Delta A_t^{\text{H}}$. Thus, in this particular case, variations in the risk-premia are the dominating factor driving portfolio holdings, a feature that could not be captured if we restricted $\mathcal{S}(X_t)$ to a constant matrix.

5.4 Accuracy

In this section we conduct the accuracy tests outlined in section 4 for the two-sector model. Table 6 reports the 90th, 95th and 99th percentiles of the maxima, averages, and standard deviations of the absolute values of the third-order terms, $|vec(x_t\tilde{x}_t')|$. As the table shows, the numbers are comparable with those computed from the one-sector model.

Table 6. Accuracy: 3rd Order Terms (Two-Sector Model)

	90% (i)	95% (ii)	99% (iii)
max	2.30E-03	3.24E-03	8.31E-03
mean	4.87E-05	6.16E-05	2.32E-04
stdev	2.26E-03	2.74E-03	1.03E-02

Note: max, mean and stdev refer to the corresponding summary statistic calculated for each element in the absolute vector of third-order terms, $|vec(x_t\tilde{x}_t')|$. 90%, 95%, and 99% stand for the respective percentiles of the distributions of these summary statistics across the cross-section of $|vec(x_t\tilde{x}_t')|$.

Table 7 reports the upper percentiles of the absolute errors from each of the five Euler equations in the home country. Columns (i) and (ii) refer to the H household's investment into H and F traded equity, column

(iii) summarizes errors for the optimal N equity investments, while (iv) and (v) refer to the capital and bond Euler equations, respectively. Interestingly, the percentiles of all the Euler equation errors in the two-sector model are below those found in the one-sector model.

Table 7. Accuracy: EE Errors (Two-Sector Model)

	A_t^{H} (i)	A_t^{F} (ii)	A_t^{N} (iii)	K_t (iv)	B_t (v)
90^{th} percentile	0.0016	0.0016	0.0015	0.0023	0.0025
95^{th} percentile	0.0019	0.0019	0.0017	0.0028	0.0030
99^{th} percentile	0.0025	0.0025	0.0023	0.0036	0.0039

Note: A_t^{H}, A_t^{F}, and A_t^{N} refer to the absolute errors from the Euler equations for H household's holdings of equity issued by H and F traded firms, and H nontraded firms; K_t and B_t correspond to the absolute errors from capital and bond Euler equations at H.

Table 8 reports the empirical percentiles of the den Haan and Marcet test statistics. Here we used a constant, productivity shocks, as well as the levels, first and second lags of the capital stock and differenced wealth in both countries as instruments, $\omega(X_t)$. As in the one-sector model we first implement the test for each Euler equation individually and report the results in columns (ii)-(vi). We also test the joint significance of the residuals in the Euler equations pertinent to each country. However, due to collinearity among the residuals in those equations, we implement the test for a subset of three equations: one tradable equity, nontradable equity and capital. The results are summarized in column (i).

Table 8. Accuracy: den Haan and Marcet χ^2 Test (Two-Sector Model)

	$A_t^{\text{H}}, A_t^{\text{F}}, K_t$ (i)	A_t^{H} (ii)	A_t^{F} (iii)	A_t^{N} (iv)	K_t (v)	B_t (vi)
lower 5%	0.01	0.05	0.03	0.05	0.04	0.06
upper 5%	0.18	0.07	0.08	0.06	0.03	0.03

Note: A_t^{H}, A_t^{F}, and A_t^{N} correspond to the percentiles of the χ^2 test statistics calculated based on the errors from H country Euler equation for H and F tradable equity, and N equity; K_t and B_t refer to the percentiles of the χ^2 test statistics for capital and bond Euler equations at H.

Overall, the results in Table 8 confirm the accuracy of our solution technique. For each individual Euler equation we find a close correspondence between the empirical distribution of the J_T test statistic and its true χ^2_{qg} distribution. The one exception is in column (i) where we consider the three Euler equations jointly. Here the empirical distribution appears to be too far to the right. We attribute this result to the near collinearity of the weighting matrix. The average condition number across the 100 simulations used to compute the empirical distribution was 170,284. By comparison, the average condition number for the weighting matrix in the individual equations was approximately 24,000.

6 Conclusion

We have presented a numerical method for solving general equilibrium models with many assets, heterogeneous agents and incomplete markets. Our method builds on the log-approximations of Campbell, Chan and Viceira (2003) and the second-order perturbation and projection techniques developed by Judd (1992) and others. To illustrate its use, we have applied our solution method to one– and two-sector versions of a two country general equilibrium model with production. The numerical solution to the one-sector model closely conforms to the predictions of theory and are highly accurate based on a number of standard tests. This gives us confidence in the accuracy of our technique. The power of our method is illustrated by solving the two-sector version of the model. The array of assets in this model is insufficient to permit complete risk sharing among households, so the equilibrium allocations cannot be found by standard analytic techniques. To the best of our knowledge, our method provides the only way to analyze general equilibrium models with portfolio choice and incomplete markets.

In principle, our solution method can be applied to more complicated models than the one- and two-sector models described above. For example, the method can be applied to solve models with more complex preferences, capital adjustment costs, or portfolio constraints. The only requirement is that the equilibrium conditions can be expressed in a log-approximate form. We believe that the solution method presented here will be useful in the future analysis of such models.

References

[1] Aruoba, S. B., Jesús Fernández-Villaverde, and Juan F. Rubio-Ramirez (2005). "Comparing Solution Methods for Dynamic Equilibrium Economies", Journal of Economic Dynamics and Control, in press.

[2] Barberis, Nicholas (2000). "Investing for the Long Run when Returns are Predictable", Journal of Finance 55(1), 225-264.

[3] Baxter, Marianne, Urban J. Jermann, and Robert G. King (1998). "Nontraded Goods, Nontraded Factors and International Non-diversification", Journal of International Economics 44, 211-229.

[4] Brandt, Michael W., Amit Goyal, and Pedro Santa-Clara (2001). "Dynamic Portfolio Choice: A Simulation Approach", Anderson Graduate School of Management Finance Paper 4-01.

[5] Brandt, Michael W. and Pedro Santa-Clara (2004). "Dynamic Portfolio Selection by Augmenting the Asset Space", NBER Working Papers 10372.

[6] Brennan, Michael J., Eduardo S. Schwartz, Ronald Lagnado (1997). "Strategic Asset Allocation", Journal of Economic Dynamics and Control 21, 1377-1403.

[7] Campbell, John Y. and Robert J. Shiller (1989). "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors", The Review of Financial Studies 1(3), 195-228.

[8] Campbell, John Y. (1994). "Inspecting the Mechanism: An Analytical Approach to the Stochastic Growth Model," Journal of Monetary Economics 33(3), 463-506.

[9] Campbell, John Y., Yeung Lewis Chan, and Luis M. Viceira (2003). "A Multivariate Model of Strategic Asset Allocation", Journal of Financial Economics 67(1), 41-80.

[10] Collard, Fabrice and Michel Juillard (2001). "Accuracy of Stochastic Perturbation Methods: The Case of Asset Pricing Models", Journal of Economic Dynamics and Control 25, 979-999.

[11] Danthine, Jean-Pierre and John B. Donaldson (1994). "Asset Pricing Implications of Real Market Frictions", Columbia - Graduate School of Business, Paper 95-04.

[12] Den Haan, Wouter J. and Marcet, Albert (1994). "Accuracy in Simulations", Review of Economic Studies 61(1), 3-17.

[13] Engel, Charles and Akito Matsumoto (2004). "Home Bias Under New Open Economy Macroeconomics", IMF Working Paper 05/165.

[14] Evans, Martin D. D. and Viktoria V. Hnatkovska (2005). "International Capital Flows, Returns and World Financial Integration", Georgetown University Working Paper and NBER Working Paper 11701.

[15] Heathcote, Jonathan and Fabrizio Perri (2004). "The International Diversification Puzzle is not as Bad as You Think", Georgetown University Working Paper.

[16] Hnatkovska, Viktoria (2005). "Home Bias and High Turnover: Dynamic Portfolio Choice with Incomplete Markets", mimeo, Georgetown University.

[17] Jin, He-Hui and Kenneth L. Judd (2002). "Perturbation Methods for General Dynamic Stochastic Models", manuscript, Stanford University.

[18] Judd, Kenneth L. (1992). "Projection Methods for Solving Aggregate Growth Models", Journal of Economics Theory 58, 410-452.

[19] ______ (1998). "Numerical Methods in Economics", MIT Press, Cambridge, Massachusetts.

[20] Judd, Kenneth L. and Sy-Ming Guu (1993). "Perturbation Solution Methods for Economic Growth Models", in Economic and Financial Modeling with Mathematica, Hal Varian, ed., Springer-Verlag: New York.

[21] ______ (1997). "Asymptotic Methods for Aggregate Growth Models", Journal of Economic Dynamics and Control 21(6), 1025-1042.

[22] ______ (2000). "The Economic Effects of New Assets: An Asymptotic Approach", Hoover Institution, Stanford University Working Paper.

[23] Kogan, Leonid and Raman Uppal (2000). "Risk Aversion and Optimal Portfolio Policies in Partial and General Equilibrium Economies", NBER Working Paper 8609.

[24] Kollmann, Robert (2005). "A Dynamic Equilibrium Model of International Portfolio Holdings: Comment", University of Paris XII Working Paper.

[25] Obstfeld, Maurice and Kenneth Rogoff (1996). "Foundations of International Macroeconomics", The MIT Press, Cambridge, Massachusetts.

[26] Pesenti, Paolo and Eric van Wincoop (1996). "Do Nontraded Goods Explain the Home Bias Puzzle?" NBER Working Paper 5784.

[27] Pichler, Paul (2005). "Evaluating Approximate Equilibria of Dynamic Economic Models", University of Vienna, Department of Economics Paper 0510.

[28] Serrat, Angel (2001). "A Dynamic Equilibrium Model of International Portfolio Holdings", Econometrica 69(6), 1467-1489.

[29] Schmitt-Grohe, Stephanie and Martin Uribe (2004). "Solving Dynamic General Equilibrium Models Using a Second-order Approximation to the Policy Function", Journal of Economic Dynamics and Control 28, 755-775.

[30] Uppal, Raman (1993). "A General Equilibrium Model of International Portfolio Choice", The Journal of Finance 48(2), 529-553.

A Appendix:

A.1 Derivation of (34)

We start with quadratic and cross-product terms, $\tilde{x}_t$ and approximate their laws of motion using Ito's lemma. In continuous time, the discrete process for x_{t+1} in (32) becomes

$$dx_t = [\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t$$

Then by Ito's lemma:

$$\begin{aligned} dvec(x_t x_t') &= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) \\ &\quad + \tfrac{1}{2} \left[(I \otimes U) \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right] d\, [x, x]_t \\ &= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) \\ &\quad + \tfrac{1}{2} \left[\mathbb{U} \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right] vec \left\{ \Omega(\tilde{x}_t) \right\} dt \\ &= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) + \tfrac{1}{2} D vec \left\{ \Omega(\tilde{x}_t) \right\} dt, \end{aligned} \quad \text{(A1)}$$

where
$$D = \left[\mathbb{U} \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right], \qquad \mathbb{U} = \sum_r \sum_s E_{rs} \otimes E'_{r,s},$$

and $E_{r,s}$ is the elementary matrix which has a unity at the $(r,s)^{th}$ position and zero elsewhere. The law of motion for the quadratic states in (A1) can be rewritten in discrete time as

$$\begin{aligned} \tilde{x}_{t+1} &\cong \tilde{x}_t + [(I \otimes x_t) + (x_t \otimes I)] [\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t] + \tfrac{1}{2} D vec\, (\Omega(\tilde{x}_t)) \\ &\quad + [(I \otimes x_t) + (x_t \otimes I)]\, \varepsilon_{t+1}, \\ &\cong \tfrac{1}{2} D \Sigma_0 + [(\Phi_0 \otimes I) + (I \otimes \Phi_0)]\, x_t + \left[I - (\Phi_1 \otimes I) - (I \otimes \Phi_1) + \tfrac{1}{2} D \Sigma_1 \right] \tilde{x}_t + \tilde{\varepsilon}_{t+1}, \end{aligned}$$

where $\tilde{\varepsilon}_{t+1} \equiv [(I \otimes x_t) + (x_t \otimes I)]\, \varepsilon_{t+1}$. The last equality is obtained by using an expression for $vec\, (\Omega(X_t))$ in (33), where $\Sigma_0 = vec(\Omega_0)$ and $\Sigma_1 = \Omega_1 \otimes \Omega_1$, and by combining together the corresponding coefficients on a constant, linear and second-order terms.

A.2 Derivation of (36)

Recall that $U_{t+1} = [\begin{array}{ccc} 0 & \varepsilon_{t+1} & \tilde{\varepsilon}_{t+1} \end{array}]'$, so $\mathbb{E}\, (U_{t+1} | X_t) = 0$ and

$$\mathbb{E} \left(U_{t+1} U'_{t+1} | X_t \right) \equiv \mathcal{S}\, (X_t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega\, (X_t) & \Gamma\, (X_t) \\ 0 & \Gamma\, (X_t)' & \Psi(X_t) \end{pmatrix}$$

To evaluate the covariance matrix, we assume that $vec(x_{t+1}\tilde{x}'_{t+1}) \cong 0$ and define:

$$
\begin{aligned}
\Gamma(X_t) &\equiv \mathbb{E}_t \varepsilon_{t+1}\tilde{\varepsilon}'_{t+1}, \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - \mathbb{E}_t x_{t+1}\mathbb{E}_t\tilde{x}'_{t+1}, \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - (\Phi_0 + (I-\Phi_1)x_t + \Phi_2\tilde{x}_t) \\
&\quad \times \left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1\otimes I)+(I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]'\right), \\
&\cong -\Phi_0\left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1\otimes I)+(I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]'\right) \\
&\quad -(I-\Phi_1)x_t\left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]'\right) - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0 D', \\
&= -\tfrac{1}{2}\Phi_0\Sigma'_0 D' - \Phi_0 x'_t\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' - \frac{1}{2}(I-\Phi_1)x_t\Sigma'_0 D' \\
&\quad -\Phi_0\tilde{x}'_t\left[I - ((\Phi_1\otimes I)+(I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]' - (I-\Phi_1)x_t x'_t\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0 D'.
\end{aligned}
$$

Hence

$$
\begin{aligned}
vec(\Gamma(X_t)) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2\tilde{x}_t, \\
\Gamma_0 &= -\frac{1}{2}(D\Sigma_0\otimes\Phi_0)\,vec(I), \\
\Gamma_1 &= -\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\otimes\Phi_0 + \tfrac{1}{2}(D\Sigma_0\otimes(I-\Phi_1)), \\
\Gamma_2 &= -\left[I - ((\Phi_1\otimes I)+(I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\otimes\Phi_0 - \tfrac{1}{2}(D\Sigma_0\otimes\Phi_2) \\
&\quad -\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\otimes(I-\Phi_1).
\end{aligned}
$$

Note also from above that

$$
\begin{aligned}
\Gamma(X_t)' &= -\tfrac{1}{2}D\Sigma_0\Phi'_0 - \left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t\Phi'_0 - \Sigma_0 x'_t(I-\Phi_1)' \\
&\quad -\left[I - ((\Phi_1\otimes I)+(I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\tilde{x}_t\Phi'_0 - \left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t x'_t(I-\Phi_1)' - \tfrac{1}{2}D\Sigma_0\tilde{x}'_t\Phi'_2.
\end{aligned}
$$

So

$$
\begin{aligned}
vec\left(\Gamma(X_t)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2\tilde{x}_t, \\
\Lambda_0 &= -\tfrac{1}{2}(\Phi_0\otimes D\Sigma_0)\,vec(I), \\
\Lambda_1 &= -\left(\Phi_0\otimes\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\right) + \tfrac{1}{2}((I-\Phi_1)\otimes D\Sigma_0), \\
\Lambda_2 &= -\left(\Phi_0\otimes\left[I - ((\Phi_1\otimes I)+(I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\right) - \tfrac{1}{2}(\Phi_2\otimes D\Sigma_0) \\
&\quad -\left((I-\Phi_1)\otimes\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\right).
\end{aligned}
$$

Next, consider the variance of $\tilde{\varepsilon}_{t+1}$:

$$
\begin{aligned}
\Psi\left(X_t\right) &\equiv \mathbb{E}_t\tilde{\varepsilon}_{t+1}\tilde{\varepsilon}_{t+1}' = \mathbb{E}_t\tilde{x}_{t+1}\tilde{x}_{t+1}' - \mathbb{E}_t\tilde{x}_{t+1}\mathbb{E}_t\tilde{x}_{t+1}', \\
&= \mathbb{E}_t\tilde{x}_{t+1}\tilde{x}_{t+1}' - \left(\tfrac{1}{2}D\Sigma_0 + \left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t + \left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\tilde{x}_t\right) \\
&\quad \times\left(\tfrac{1}{2}\Sigma_0'D' + x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' + \tilde{x}_t'\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]'\right), \\
&\cong -\tfrac{1}{2}D\Sigma_0\left(\tfrac{1}{2}\Sigma_0'D' + x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' + \tilde{x}_t'\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]'\right) \\
&\quad -\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t\left(\tfrac{1}{2}\Sigma_0'D' + x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]'\right) \\
&\quad -\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}\mathbb{D}\Sigma_1\right]\tilde{x}_t\tfrac{1}{2}\Sigma_0'D', \\
&= -\tfrac{1}{4}D\Sigma_0\Sigma_0'D' - \tfrac{1}{2}D\Sigma_0x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' - \tfrac{1}{2}\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t\Sigma_0'D' \\
&\quad -\tfrac{1}{2}D\Sigma_0\tilde{x}_t'\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]' - \left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_tx_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' \\
&\quad -\tfrac{1}{2}\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\tilde{x}_t\Sigma_0'D'.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
vec\left(\Psi\left(X_t\right)\right) &= \Psi_0 + \Psi_1x_t + \Psi_2\tilde{x}_t, \\
\Psi_0 &= -\frac{1}{4}\left(D\Sigma_0\otimes D\Sigma_0\right)vec(I), \\
\Psi_1 &= -\tfrac{1}{2}\left(\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\otimes D\Sigma_0\right) - \tfrac{1}{2}\left(D\Sigma_0\otimes\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\right), \\
\Psi_2 &= -\frac{1}{2}\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\otimes D\Sigma_0 - \tfrac{1}{2}\left(D\Sigma_0\otimes\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\right) \\
&\quad -\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\otimes\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right].
\end{aligned}
$$

A.3 Derivation of Results R1 and R2

Let $m_t = \pi_m X_t$ and $n_t = \pi_n X_t$ for two variables m_t and n_t.We want to find the conditional covariance between the two:

$$
\begin{aligned}
\mathbb{CV}_t\left(m_{t+1}, n_{t+1}\right) &= \begin{bmatrix} \pi_m^0 & \pi_m^1 & \pi_m^2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Omega(X_t) & \Gamma\left(X_t\right) \\ 0 & \Gamma\left(X_t\right)' & \Psi(X_t) \end{bmatrix}\begin{bmatrix} \pi_n^{0\prime} \\ \pi_n^{1\prime} \\ \pi_n^{2\prime} \end{bmatrix}, \\
&= \pi_m^1\Omega(X_t)\pi_n^{1\prime} + \pi_m^2\Gamma\left(X_t\right)'\pi_n^{1\prime} + \pi_m^1\Gamma\left(X_t\right)\pi_n^{2\prime} + \pi_m^2\Psi(X_t)\pi_n^{2\prime}, \\
&= \left(\pi_n^1\otimes\pi_m^1\right)vec\left(\Omega(X_t)\right) + \left(\pi_n^1\otimes\pi_m^2\right)vec\left(\Gamma\left(X_t\right)'\right) \\
&\quad + \left(\pi_n^2\otimes\pi_m^1\right)vec\left(\Gamma\left(X_t\right)\right) + \left(\pi_n^2\otimes\pi_m^2\right)vec\left(\Psi\left(X_t\right)\right), \\
&= \left(\pi_n^1\otimes\pi_m^1\right)\Sigma_0 + \left(\pi_n^1\otimes\pi_m^2\right)\Lambda_0 + \left(\pi_n^2\otimes\pi_m^1\right)\Gamma_0 + \left(\pi_n^2\otimes\pi_m^2\right)\Psi_0 \\
&\quad + \left(\left(\pi_n^1\otimes\pi_m^2\right)\Lambda_1 + \left(\pi_n^2\otimes\pi_m^1\right)\Gamma_1 + \left(\pi_n^2\otimes\pi_m^2\right)\Psi_1\right)x_t \\
&\quad + \left(\left(\pi_n^1\otimes\pi_m^1\right)\Sigma_1 + \left(\pi_n^1\otimes\pi_m^2\right)\Lambda_2 + \left(\pi_n^2\otimes\pi_m^1\right)\Gamma_2 + \left(\pi_n^2\otimes\pi_m^2\right)\Psi_2\right)\tilde{x}_t.
\end{aligned}
$$

So, to summarize,

$$
\begin{aligned}
\mathbb{CV}_t\left(m_{t+1}, n_{t+1}\right) &= \mathcal{A}\left(\pi_m, \pi_n\right) X_t, \\
\mathcal{A}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{ccc} \mathcal{A}^0_{m,n} & \mathcal{A}^1_{m,n} & \mathcal{A}^2_{m,n} \end{array}\right], \\
\mathcal{A}^0_{m,n} &= \left(\pi^1_n \otimes \pi^1_m\right) \Sigma_0 + \left(\pi^1_n \otimes \pi^2_m\right) \Lambda_0 + \left(\pi^2_n \otimes \pi^1_m\right) \Gamma_0 + \left(\pi^2_n \otimes \pi^2_m\right) \Psi_0, \\
\mathcal{A}^1_{m,n} &= \left(\pi^1_n \otimes \pi^2_m\right) \Lambda_1 + \left(\pi^2_n \otimes \pi^1_m\right) \Gamma_1 + \left(\pi^2_n \otimes \pi^2_m\right) \Psi_1, \\
\mathcal{A}^2_{m,n} &= \left(\pi^1_n \otimes \pi^1_m\right) \Sigma_1 + \left(\pi^1_n \otimes \pi^2_m\right) \Lambda_2 + \left(\pi^2_n \otimes \pi^1_m\right) \Gamma_2 + \left(\pi^2_n \otimes \pi^2_m\right) \Psi_2.
\end{aligned}
$$

To obtain the products of vectors involving the state vector X_t, we note that

$$
\begin{aligned}
\pi_m X_t X_t' \pi_n' &= \left[\begin{array}{ccc} \pi^0_m & \pi^1_m & \pi^2_m \end{array}\right] \left[\begin{array}{ccc} 1 & x_t' & \tilde{x}_t' \\ x_t & x_t x_t' & 0 \\ \tilde{x}_t & 0 & 0 \end{array}\right] \left[\begin{array}{c} \pi^{0\prime}_n \\ \pi^{1\prime}_n \\ \pi^{2\prime}_n \end{array}\right], \\
&= \left(\pi^0_m + \pi^1_m x_t + \pi^2_m \tilde{x}_t\right) \pi^{0\prime}_n + \left(\pi^0_m x_t' + \pi^1_m x_t x_t'\right) \pi^{1\prime}_n + \pi^0_m \tilde{x}_t' \pi^{2\prime}_n, \\
&= \left(\pi^0_n \otimes \pi^0_m\right) + \left(\pi^0_n \otimes \pi^1_m\right) x_t + \left(\pi^0_n \otimes \pi^2_m\right) \tilde{x}_t + \left(\pi^1_n \otimes \pi^0_m\right) x_t \\
&\quad + \left(\pi^1_n \otimes \pi^1_m\right) \tilde{x}_t + \left(\pi^2_n \otimes \pi^0_m\right) \tilde{x}_t.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\pi_m X_t X_t' \pi_n' &= \mathcal{B}\left(\pi_m, \pi_n\right) X_t, \\
\mathcal{B}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{ccc} \mathcal{B}^0_{m,n} & \mathcal{B}^1_{m,n} & \mathcal{B}^2_{m,n} \end{array}\right], \\
\mathcal{B}^0_{m,n} &= \left(\pi^0_n \otimes \pi^0_m\right) vec(I) = vec(\pi^0_n * \pi^0_m), \\
\mathcal{B}^1_{m,n} &= \left(\pi^0_n \otimes \pi^1_m\right) + \left(\pi^1_n \otimes \pi^0_m\right), \\
\mathcal{B}^2_{m,n} &= \left(\pi^0_n \otimes \pi^2_m\right) + \left(\pi^1_n \otimes \pi^1_m\right) + \left(\pi^2_n \otimes \pi^0_m\right).
\end{aligned}
$$