A Method for Solving General Equilibrium Models with Incomplete Markets and Many Financial Assets[1]

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Abstract

This paper presents a new numerical method for solving stochastic general equilibrium models with dynamic portfolio choice over many financial assets. The method can be applied to models where there are heterogeneous agents, time-varying investment opportunity sets, and incomplete asset markets. We illustrate how the method is used by solving two versions of a two-country general equilibrium model with production and dynamic portfolio choice. We check the accuracy of our method by comparing the numerical solution to a complete markets version of the model against its known analytic properties. We then apply the method to an incomplete markets version where no analytic solution is available. In both models, with and without stationarity induced, and for different degrees of risk aversion the standard accuracy tests confirm the effectiveness of our method.

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Introduction

This paper presents a new numerical method for solving dynamic stochastic general equilibrium (DSGE) models with dynamic portfolio choice over many financial assets. The method can be applied to models where there are heterogeneous agents, time-varying investment opportunity sets, and incomplete asset markets. As such, our method can be used to solve models that analyze an array of important issues in international macroeconomics and finance. For example, questions concerning the role of revaluation effects in the process of external adjustment cannot be fully addressed without a model that incorporates the dynamic portfolio choices of home and foreign agents across multiple financial assets. Similarly, any theoretical assessment of the implications of greater international financial integration requires a model in which improved access to an array of financial markets has real effects; through capital deepening and/or improved risk sharing (because markets are incomplete). Indeed, there is an emerging consensus among researchers that the class of DSGE models in current use needs to be extended to include dynamic portfolio choice and incomplete markets (see, for example, Obstfeld (2004), and Gourinchas (2006)). This paper shows how an accurate approximation to the equilibrium in such models can be derived.

We illustrate the use of our solution method by solving two versions of a canonical two-country DSGE model. The full version of the model includes production, traded and nontraded goods, and an array of equity and bond markets. Households choose between multiple assets as part of their optimal consumption and saving decisions, but only have access to a subset of the world's financial markets. As a result, there is both dynamic portfolio choice and incomplete risk-sharing in the equilibrium. We also study the equilibrium in a simplified version of the model without nontraded goods. Here households still face a dynamic portfolio choice problem but the available array of financial assets is sufficient for complete risk-sharing. We use these two versions of our model to illustrate how well our solution method works in complete and incomplete market settings; in the stationary and non-stationary environments; with log-utility and with a higher degree of risk aversion. In particular, we present several tests to show that our approximations to all sets of equilibrium dynamics are very accurate.

The presence of portfolio choice and incomplete markets in a DSGE model gives rise to a number of problems that must be addressed by any solution method. First, and foremost, the method must address the complex interactions between the real and financial sides of the economy. On the one hand, portfolio decisions affect the degree of risk-sharing which in turn affects equilibrium real allocations. On the other, real allocations affect the behavior of returns via their implications for market-clearing prices, which in turn affect portfolio choices. Second, we need to track the distribution of households' financial wealth in order to account for the wealth effects that arise when risk-sharing is incomplete. This adds to the number of state variables needed to characterize the equilibrium dynamics of the economy and hence increases the complexity of finding the equilibrium. Third, it is well-known that transitory shocks can have very persistent effects on the distribution of financial wealth when markets are incomplete, leading to non-stationary wealth dynamics in the model. Such non-stationarity is typically removed using various approaches, as discussed in details in Schmitt-Grohe and Uribe (2003). Our solution method addresses all these problems and remains accurate in both stationary and non-stationary versions of the model.

The method we propose combines a perturbation technique commonly used in solving macro models with continuous-time approximations common in solving finance models of portfolio choice. In so doing, we contribute to the literature along several dimensions. First, relative to the finance literature, our method delivers optimal portfolios in a discrete-time general equilibrium setting in which returns are endogenously determined. It also enables us to characterize the dynamics of returns and the stochastic investment opportunity set as functions of macroeconomic state variables.[2] Second, relative to the macroeconomics literature, portfolio decisions are derived without assuming complete asset markets or constant returns to scale in production.[3]

Recent papers by Devereux and Sutherland (2008, 2007) and Tille and van Wincoop (2010) have proposed an alternative method for solving DSGE models with portfolio choice and incomplete markets.[4] Two key features differentiate their approach from the one we propose. First, their method requires at least third-order approximations to some of the model's equilibrium conditions in order to identify (first-order) variations in the portfolio holdings. By contrast, we are able to accurately characterize optimal portfolio holdings to second order from second-order approximations of the equilibrium conditions. This difference is important when it comes to solving models with a large state space (i.e. a large number of state variables). We have applied our method to models with 8 state variables and 10 decision variables (see Evans and Hnatkovska (2005), Hnatkovska (2010)). Second, we characterize the consumption and portfolio problem facing households using the approximations developed by John Campbell and his co-authors (for instance, Campbell, Chan, and Viceira (2003)) over the past decade. These approximations differ from those commonly used in solving DSGE models without portfolio choice, but they have proved very useful in characterizing intertemporal financial decision-making (see, for example, Campbell and Viceira (2002)). In particular, they provide simple closed-form expressions for portfolio holdings that are useful in identifying the role of different economic factors. In this sense, our approach can be viewed as an extension of the existing literature on dynamic portfolio choice to a general equilibrium setting.

The paper is structured as follows. Section 1 presents the model we use to illustrate our solution method. Section 2 describes the solution method in detail. Section 3 provides a step-by-step description of how the method is applied to our illustrative model. We present results on the accuracy of the solutions to both versions of our model in Section 4. Section 5 concludes.

[2] A number of approximate solution methods have been developed in partial equilibrium frameworks. Kogan and Uppal (2002) approximate portfolio and consumption allocations around the solution for a log-investor. Barberis (2000), Brennan, Schwartz, and Lagnado (1997) use discrete-state approximations. Brandt, Goyal, Santa-Clara, and Stroud (2005) solve for portfolio policies by applying dynamic programming to an approximated simulated model. Brandt and Santa-Clara (2004) expand the asset space to include asset portfolios and then solve for the optimal portfolio choice in the resulting static model.

[3] Solutions to portfolio problems with complete markets are developed in Heathcote and Perri (2004), Serrat (2001), Kollmann (2006), Baxter, Jermann, and King (1998), Uppal (1993), Engel and Matsumoto (2009). Pesenti and van Wincoop (2002) analyze equilibrium portfolios in a partial equilibrium setting with incomplete markets.

[4] Ghironi, Lee, and Rebucci (2009) also develop and analyze a model with portfolio choice and incomplete asset markets. To compute the steady state asset allocations they introduce financial transaction fees. In our frictionless model portfolio holdings are derived endogenously using the conditional distributions of asset returns.

1 The Model

This section describes the discrete-time DSGE model we employ to illustrate our solution method. Our starting point is a standard international asset pricing model with production, which we extend to incorporate dynamic portfolio choice over equities and an international bond. A frictionless production world economy in this model consists of two symmetric countries, called Home (H) and Foreign (F). Each country is populated by a continuum of identical households who consume and invest in different assets, and a continuum of firms that are split between the traded and nontraded goods' sectors. Firms are infinitely-lived, perfectly competitive, and issue equity claims to their dividend streams.

1.1 Firms

We shall refer to firms in the traded and nontraded sectors as "traded" and "nontraded". A representative traded firm in country H starts period t with a stock of firm-specific capital K_t. Period-t production is $Y_t = Z_t^{\mathrm{T}} K_t^{\theta}$ with $\theta > 0$, and Z_t^{T} denotes the current state of productivity. The output produced by traded firms in country F, $\hat{Y}_t$, is given by an identical production function using firm-specific foreign capital, $\hat{K}_t$, and productivity, $\hat{Z}_t^{\mathrm{T}}$. (Hereafter we use "ˆ" to denote foreign variables.) The goods produced by H and F traded firms are identical and can be costlessly transported between countries. Under these conditions, the law of one price prevails in the traded sector to eliminate arbitrage opportunities.

At the beginning of period t, each traded firm observes the productivity realization, produces output, and uses the proceeds to finance investment and to pay dividends to its shareholders. We assume that firms allocate output to maximize the value of the firm to its domestic shareholders every period. If the total number of outstanding shares is normalized to unity, the optimization problem facing a traded firm in country H can be summarized as

$$\max_{I_t} \mathbb{E}_t \sum_{i=0}^{\infty} M_{t+i,t}\ D_{t+i}^{\mathrm{T}}, \tag{1}$$

subject to

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t, & (2)\\ D_t^{\mathrm{T}} &= Z_t^{\mathrm{T}} K_t^{\theta} - I_t, & (3) \end{aligned}$$

where D_t^{T} is the dividend per share paid at t, I_t is real investment and $\delta > 0$ is the depreciation rate on physical capital. $\mathbb{E}_t$ denotes expectations conditioned on information at the start of period t. $M_{t+i,t}$ is the intertemporal marginal rate of substitution (IMRS) between consumption of tradables in period t and period $t+i$ of domestic households, and $M_{t,t} = 1$.[5] The representative traded firm in country F solves an analogous problem: It chooses investment, $\hat{I}_t$, to maximize the present discounted value of foreign dividends per share, $\hat{D}_t^{\mathrm{T}}$, using $\hat{M}_{t+i,t}$, the IMRS of F households.

[5] Although our specification in (1) is straightforward, we note that it can potentially induce home bias in households' traded equity holdings when markets are incomplete. If the array of assets available to households is insufficient for complete risk-sharing (as will be the case in one of the equilibria we study), the IMRS for H and F households will differ. Under these circumstances, households will prefer the dividend stream chosen by domestic traded firms. We abstract from these effects for clarity.

The output of nontraded firms in countries H and F is given by $Y_t^{\text{N}} = \eta Z_t^{\text{N}}$ and $\hat{Y}_t^{\text{N}} = \eta \hat{Z}_t^{\text{N}}$ respectively, where $\eta > 0$ is a constant. Nontraded firms have no investment decisions to make; they simply pass on sales revenue as dividends to their shareholders. Z_t^{N} and $\hat{Z}_t^{\text{N}}$ denote the period-t state of nontradable productivity in countries H and F, respectively.

Let $z_t \equiv [\ln Z_t^{\text{T}}, \ln \hat{Z}_t^{\text{T}}, \ln Z_t^{\text{N}}, \ln \hat{Z}_t^{\text{N}}]'$ denote the state of productivity in period t. We assume that the productivity vector, z_t, follows an AR(1) process:

$$z_t = a z_{t-1} + S_e^{1/2} e_t, \tag{4}$$

where a is a 4×4 matrix and e_t is a 4×1 vector of i.i.d. mean zero, unit variance shocks. $S_e^{1/2}$ is a 4×4 matrix of scaling parameters.

1.2 Households

Each country is populated by a continuum of households who have identical preferences over the consumption of traded and nontraded goods. The preferences of a representative household in country H are given by

$$\mathbb{E}_t \sum_{i=0}^{\infty} \theta_{t+i} \mathcal{U}(C_{t+i}^{\text{T}}, C_{t+i}^{\text{N}}), \tag{5}$$

where $\mathcal{U}(.)$ is a concave sub-utility function defined over the consumption of traded and nontraded goods, C_t^{T} and C_t^{N}. The period utility function is given by:

$$\mathcal{U}(C^{\text{T}}, C^{\text{N}}) = \frac{\left(\left[\mu_{\text{T}}^{1-\phi} (C^{\text{T}})^{\phi} + \mu_{\text{N}}^{1-\phi} (C^{\text{N}})^{\phi} \right]^{\frac{1}{\phi}} \right)^{1-\sigma} - 1}{1-\sigma},$$

with $\phi < 1$. μ_{T} and μ_{N} are the weights that the household assigns to traded and nontraded consumption, respectively. The elasticity of substitution between the two goods is $(1-\phi)^{-1} > 0$ and σ is the coefficient of relative risk aversion. Notice that preferences are not separable across the two goods. $\theta_{t+1} = \theta_t \beta(\tilde{C}_t^{\text{T}}, \tilde{C}_t^{\text{N}})$ is the endogenous discount factor that depends on average consumption of traded and nontraded goods in the domestic economy, denoted by $\tilde{C}_t^{\text{T}}, \tilde{C}_t^{\text{N}}$, with $\theta_0 = 1$. We assume $\partial\beta(\tilde{C}_t^{\text{T}}, \tilde{C}_t^{\text{N}})/\partial \tilde{C}_t^{\text{T}} < 0$, $\partial\beta(\tilde{C}_t^{\text{T}}, \tilde{C}_t^{\text{N}})/\partial \tilde{C}_t^{\text{N}} < 0$, so that as consumption increases, the discount factor decreases and household becomes more impatient. Following Kollmann (1996), Corsetti, Dedola, and Leduc (2008), Boileau and Normandin (2008), the discount function β is given by

$$\beta(\tilde{C}_t^{\text{T}}, \tilde{C}_t^{\text{N}}) = \left(1 + \left[\mu_{\text{T}}^{1-\phi} (C^{\text{T}})^{\phi} + \mu_{\text{N}}^{1-\phi} (C^{\text{N}})^{\phi} \right]^{\frac{1}{\phi}} \right)^{-\zeta},$$

with $\zeta \geq 0$. As in Boileau and Normandin (2008), Schmitt-Grohe and Uribe (2003), and Devereux and Sutherland (2008), the introduction of the endogenous discount factor ensures a stationary wealth process. Preferences for households in country F are identically defined in terms of foreign traded and nontraded consumption, $\hat{C}_t^{\text{T}}$ and $\hat{C}_t^{\text{N}}$.

Households can save by holding domestic equities (i.e., traded and nontraded), an international bond, and the equity issued by foreign traded firms. They cannot hold equity issued by foreign nontraded firms. Let $C_t \equiv C_t^{\mathrm{T}} + Q_t^{\mathrm{N}} C_t^{\mathrm{N}}$ denote total consumption expenditure, where Q_t^{N} is the relative price of H nontraded goods in terms of traded goods (our numeraire). The budget constraint of the representative H household can now be written as

$$W_{t+1} = R_{t+1}^{\mathrm{W}}(W_t - C_t), \tag{6}$$

where W_t is financial wealth and R_{t+1}^{W} is the (gross) return on wealth between period t and $t+1$. This return depends on how the household allocates wealth across the available array of financial assets, and on the realized returns on those assets. In particular,

$$R_{t+1}^{\mathrm{W}} \equiv R_t + \alpha_t^{\mathrm{H}}(R_{t+1}^{\mathrm{H}} - R_t) + \alpha_t^{\mathrm{F}}(R_{t+1}^{\mathrm{F}} - R_t) + \alpha_t^{\mathrm{N}}(R_{t+1}^{\mathrm{N}} - R_t), \tag{7}$$

where α_t^i and α_t^{N} respectively denote the shares of wealth allocated in period t by H households into equity issued by $i = \{\mathrm{H}, \mathrm{F}\}$ traded firms and H nontraded firms.[6] R_t is the risk-free return on bonds, R_{t+1}^{H} and R_{t+1}^{F} are the returns on equity issued by the H and F traded firms, and R_{t+1}^{N} is the return on equity issued by H nontraded firms. These returns are defined as

$$R_{t+1}^{\mathrm{H}} \equiv (P_{t+1}^{\mathrm{T}} + D_{t+1}^{\mathrm{T}})/P_t^{\mathrm{T}}, \qquad R_{t+1}^{\mathrm{F}} \equiv (\hat{P}_{t+1}^{\mathrm{T}} + \hat{D}_{t+1}^{\mathrm{T}})/\hat{P}_t^{\mathrm{T}}, \tag{8a}$$

$$R_{t+1}^{\mathrm{N}} \equiv \{(P_{t+1}^{\mathrm{N}} + D_{t+1}^{\mathrm{N}})/P_t^{\mathrm{N}}\}\{Q_{t+1}^{\mathrm{N}}/Q_t^{\mathrm{N}}\}, \tag{8b}$$

where P_t^{T} and P_t^{N} are period-t prices of equity issued by traded and nontraded firms in country H and D_t^{N} is the period-t flow of dividends from H nontraded firms. P_t^{N} and D_t^{N} are measured in terms of nontradables. The three portfolio shares $\{\alpha_t^{\mathrm{H}}, \alpha_t^{\mathrm{F}}, \alpha_t^{\mathrm{N}}\}$ are related to the corresponding portfolio holdings $\{A_t^{\mathrm{H}}, A_t^{\mathrm{F}}, A_t^{\mathrm{N}}\}$ by the identities: $P_t^{\mathrm{T}} A_t^{\mathrm{H}} \equiv \alpha_t^{\mathrm{H}}(W_t - C_t)$, $\hat{P}_t^{\mathrm{T}} A_t^{\mathrm{F}} \equiv \alpha_t^{\mathrm{F}}(W_t - C_t)$ and $Q_t^{\mathrm{N}} P_t^{\mathrm{N}} A_t^{\mathrm{N}} \equiv \alpha_t^{\mathrm{N}}(W_t - C_t)$.

The problem facing foreign household is defined symmetrically with hat ˆ over a variable denoting the foreign country.

1.3 Equilibrium

We now summarize the conditions that characterize the equilibrium in our model. The first-order conditions

[6] Note that in our model, households can not hold nontraded sector equities issued by foreign firms. This assumption is necessary to obtain less than perfect risk-sharing in equilibrium. There is support for limited tradability of claims to nontraded firms in the data. For instance based on data reported in Denis and Huizinga (2004) it can be shown that foreign ownership in firms belonging to the traded sector in Europe is two times larger than in firms belonging to the nontraded sector. Kang and Stulz (1997) show that foreign holdings of Japanese shares are heavily biased towards firms in manufacturing (traded) industries; while foreign ownership is underweighted in Electric, Power and Gas industries and Services (nontraded).

for the representative $\textsc{h}$ household's problem are

$$Q_t^{\mathrm{N}} = \frac{\partial \mathcal{U}/\partial C_t^{\mathrm{N}}}{\partial \mathcal{U}/\partial C_t^{\mathrm{T}}}, \tag{9a}$$
$$1 = \mathbb{E}_t \left[M_{t+1} R_t \right], \tag{9b}$$
$$1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{\mathrm{H}} \right], \tag{9c}$$
$$1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{\mathrm{F}} \right], \tag{9d}$$
$$1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{\mathrm{N}} \right], \tag{9e}$$

where $M_{t+1} \equiv M_{t+1,t} = \beta(\tilde{C}_t^{\mathrm{T}}, \tilde{C}_t^{\mathrm{N}})(\partial \mathcal{U}/\partial C_{t+1}^{\mathrm{T}})/(\partial \mathcal{U}/\partial C_t^{\mathrm{T}})$ is the IMRS between traded consumption in period t and period $t+1$. Our specification for utility implies that

$$\begin{aligned} M_{t+1} &= \beta(\tilde{C}_t^{\mathrm{T}}, \tilde{C}_t^{\mathrm{N}}) \left(C_{t+1}^{\mathrm{T}}/C_t^{\mathrm{T}} \right)^{\frac{(1-\sigma)(\phi-1)}{\phi}} \left(C_{t+1}/C_t \right)^{\frac{1-\sigma-\phi}{\phi}}, \\ &= \beta(\tilde{C}_t^{\mathrm{T}}, \tilde{C}_t^{\mathrm{N}}) \left(\Lambda_{t+1}^{\mathrm{T}}/\Lambda_t^{\mathrm{T}} \right)^{\frac{(1-\sigma)(\phi-1)}{\phi}} \left(\Lambda_{t+1}/\Lambda_t \right)^{\frac{1-\sigma-\phi}{\phi}} \left(W_{t+1}/W_t \right)^{-\sigma}, \end{aligned}$$

where $\Lambda_t \equiv C_t/W_t$ and $\Lambda_t^{\mathrm{T}} \equiv C_t^{\mathrm{T}}/W_t$. These ratios are restricted by the first-order conditions. In particular, (9b)-(9e) imply that $1 = \mathbb{E}_t[M_{t+1} R_{t+1}^{\mathrm{W}}]$, which after substituting for M_{t+1} gives

$$\Lambda_t^{\mathrm{T}\,\frac{(1-\sigma)(\phi-1)}{\phi}} \Lambda_t^{\frac{1-\sigma-\phi}{\phi}} (1-\Lambda_t)^{\sigma} = \beta(\tilde{C}_t^{\mathrm{T}}, \tilde{C}_t^{\mathrm{N}}) \mathbb{E}_t \left[\Lambda_{t+1}^{\mathrm{T}\,\frac{(1-\sigma)(\phi-1)}{\phi}} \Lambda_{t+1}^{\frac{1-\sigma-\phi}{\phi}} \left(R_{t+1}^{\mathrm{W}} \right)^{1-\sigma} \right]. \tag{10}$$

Similarly, combining (9a) with the fact that $C_t^{\mathrm{T}} + Q_t^{\mathrm{N}} C_t^{\mathrm{N}} = \Lambda_t W_t$ gives

$$\frac{C_t^{\mathrm{T}}}{W_t} = \Lambda_t \frac{\vartheta(Q_t^{\mathrm{N}})}{1+\vartheta(Q_t^{\mathrm{N}})} \qquad \text{and} \qquad \frac{Q_t^{\mathrm{N}} C_t^{\mathrm{N}}}{W_t} = \Lambda_t \frac{1}{1+\vartheta(Q_t^{\mathrm{N}})}, \tag{11}$$

where $\vartheta(Q_t^{\mathrm{N}}) \equiv (\mu_{\mathrm{T}}/\mu_{\mathrm{N}}) \left(Q_t^{\mathrm{N}} \right)^{\phi/(1-\phi)}$. Taken together, equations (10) and (11) pin down Λ_t and Λ_t^{T} as functions of the relative price, Q_t^{N}, and the return on wealth, R_{t+1}^{W}.

The first-order condition associated with the $\textsc{h}$ traded firm's optimization problem is

$$1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{\mathrm{K}} \right], \tag{12}$$

where $R_{t+1}^{\mathrm{K}} \equiv \theta Z_{t+1}^{\mathrm{T}} \left(K_{t+1} \right)^{\theta-1} + (1-\delta)$ is the return on capital. This condition determines the optimal investment of $\textsc{h}$ traded firms and thus implicitly identifies the level of traded dividends in period t, D_t^{T}, via equation (3). The first-order conditions for households and traded firms in country $\textsc{f}$ take an analogous form.

Solving for the equilibrium in this economy requires finding equity prices $\{P_t^{\mathrm{T}}, \hat{P}_t^{\mathrm{T}}, P_t^{\mathrm{N}}, \hat{P}_t^{\mathrm{N}}\}$, the risk-free return R_t, and goods prices $\{Q_t^{\mathrm{N}}, \hat{Q}_t^{\mathrm{N}}\}$, such that markets clear when households follow optimal consumption, savings and portfolio strategies, and firms make optimal investment decisions. Under the assumption that bonds are in zero net supply, market clearing in the bond market requires

$$0 = B_t + \hat{B}_t. \tag{13}$$

The traded goods market clears globally. In particular, since H and F traded firms produce a single good that can be costlessly transported between countries, the traded goods market clearing condition is

$$C_t^{\mathrm{T}} + \hat{C}_t^{\mathrm{T}} = Y_t^{\mathrm{T}} - I_t + \hat{Y}_t^{\mathrm{T}} - \hat{I}_t = D_t^{\mathrm{T}} + \hat{D}_t^{\mathrm{T}}. \tag{14}$$

Market clearing in the nontraded sector of each country requires that

$$C_t^{\mathrm{N}} = Y_t^{\mathrm{N}} = D_t^{\mathrm{N}} \qquad \text{and} \qquad \hat{C}_t^{\mathrm{N}} = \hat{Y}_t^{\mathrm{N}} = \hat{D}_t^{\mathrm{N}}. \tag{15}$$

Since the equity liabilities of all firms are normalized to unity, the market clearing conditions in the H and F traded equity markets are

$$1 = A_t^{\mathrm{H}} + \hat{A}_t^{\mathrm{H}} \qquad \text{and} \qquad 1 = A_t^{\mathrm{F}} + \hat{A}_t^{\mathrm{F}}. \tag{16}$$

Recall that nontraded equity can only be held by domestic households. Market clearing in these equity markets therefore requires that

$$1 = A_t^{\mathrm{N}} \qquad \text{and} \qquad 1 = \hat{A}_t^{\mathrm{N}}. \tag{17}$$

2 The Solution Method

In this section we discuss the solution to the nonlinear system of stochastic difference equations characterizing the equilibrium of our DSGE model. First, we outline why standard approximation methods (e.g., projections or perturbations) are inapplicable for solving DSGE models with incomplete markets and portfolio choice. We then provide an overview of our solution method and discuss how it relates to other methods in the literature.

2.1 Market Incompleteness and Portfolio Choice

The model in Section 1 is hard to solve because it combines dynamic portfolio choice with market incompleteness. In our model, markets are (dynamically) incomplete because households do not have access to the complete array of financial assets in the world economy. In particular, households cannot hold the equities issued by foreign nontraded firms. If we lifted this restriction, households would be able to completely share risks internationally (i.e., the H and F IMRS would be equal). In this special case, the problem of finding the equilibrium could be split into two sub-problems: First, we could use the risk-sharing conditions to find the real allocations as the solution to a social planning problem. Second, we could solve for the equilibrium prices and portfolio choices that support these allocations in a decentralized market setting. Examples of this approach include Obstfeld and Rogoff (1996) p. 302, Baxter, Jermann, and King (1998), Engel and Matsumoto (2009), and Kollmann (2006).

When markets are incomplete there are complex interactions between the real and financial sides of the economy; interactions that cannot be accommodated by existing solution methods if there are many financial assets. On the one hand, household portfolio decisions determine the degree of international risk-sharing, which in turn affects equilibrium real allocations. On the other, market-clearing prices affect the behavior of equilibrium returns, which in turn influence portfolio choices. We account for this interaction between the real and financial sides of the economy in our solution method by tracking the behavior of financial wealth across all households. More specifically, we track how shocks to the world economy affect the distribution of wealth given optimal portfolio choices (because risk-sharing is incomplete), and how changes in the distribution of wealth affect market-clearing prices. We also track how these distributional effects on prices affect returns and hence the portfolio choices of households.

In order to track the behavior of the world's wealth distribution, we must include the wealth of each household in the state vector; the vector of variables needed to described the complete state of the economy at a point in time. This leads to two technical problems. First, the numerical complexity in solving for an equilibrium in any model begins to increase with the number of variables in the state vector. The state vector for the simple model in Section 1 has 8 variables, but this is too many to apply a solution method based on a discretization of the state space (see, for example, chapter 12 of Judd, 1998). The second problem relates to the long-run distribution of wealth. In our model, and many others with incomplete markets (see, for example, Obstfeld and Rogoff (1995), Baxter and Crucini (1995), Correia, Neves, and Rebelo (1995)), such a long-run distribution is not well-defined as real shocks have permanent effects on the wealth of individual households. Our solution method aims to characterize the equilibrium behavior of the economy in a neighborhood around a particular initial wealth distribution. The advantage of this approach is that it does not require an assumption about how the international distribution of wealth is affected by such shocks in the long run. The disadvantage is that our characterization of the equilibrium dynamics will only be accurate while wealth remains close to the initial distribution. This does not appear to be an important limitation in practice. In Section 4 we show that our solution remains very accurate in simulations of 75 years of quarterly data. Furthermore, we show that our method remains fully operational when non-stationarity problem is removed. In fact, the accuracy of our method remain comparable in the stationary and non-stationary versions of our incomplete markets model.

The presence of portfolio choice also introduces technical problems. Perturbation solution methods use n'th-order Taylor approximations to the optimality and market-clearing conditions around the unique non-stochastic steady state of the economy. This approach is inapplicable to the household's portfolio choice problem because there is no unique steady state portfolio allocation: There is no risk in the non-stochastic steady state, so all assets have exactly the same (riskless) return. To address this problem, we use continuous-time approximations which do not require the existence of a unique portfolio allocation in the non-stochastic steady state, and then solve for it endogenously. Our method only requires us to pin down the initial wealth distribution.

The main methodological innovation in our solution method relates to the behavior of financial returns. Optimal portfolio choices in each period are determined by the conditional distribution of returns. In a partial equilibrium model the distribution of returns is exogenous, but in our general equilibrium setting we must derive the conditional distribution from the properties of the equilibrium asset prices and dividends. Our method does just this. We track how the conditional distribution of equilibrium returns changes with the state of the economy. This aspect of our method highlights an important implication of market incompleteness for portfolio choice: When risk-sharing is incomplete, the distributional effects of shocks on equilibrium asset prices can induce endogenous variations in the conditional distribution of returns even when the underlying shocks come from an i.i.d. distribution. Thus, our solution method allows us to examine how time-variation in portfolio choices and risk premia can arise endogenously when markets are incomplete.

2.2 Implementing the Method

We now provide the details of how the model in Section 1 is solved. The key novelty of our method is the log-approximation approach to the model equilibrium conditions. While these approximations are used widely for solving portfolio problems in partial equilibrium settings in the Finance literature, they are relatively underutilized in the Macroeconomic literature and in the general equilibrium applications. Next, we illustrate how the equilibrium conditions of our model are log-approximated and then discuss numerical approaches to solving the resulting system of linear equations.

Log-Approximations

Here we derive the log-approximations to the equations arising from the households' and firms' first-order conditions, budget constraints and market clearing conditions. These approximations are quite standard in both Macro and Finance aside from the point of approximation. Let x_t denote the state vector, where $x_t \equiv [z_t, k_t, \hat{k}_t, w_t, \hat{w}_t]'$, $k_t \equiv \ln(K_t/K)$, $\hat{k}_t \equiv \ln(\hat{K}_t/K)$, $w_t \equiv \ln(W_t/W_0)$ and $\hat{w}_t \equiv \ln(\hat{W}_t/\hat{W}_0)$ with K and $\hat{K}$ as the steady state capital stocks (steady state values have no t subscript). W_0 and $\hat{W}_0$ are the initial levels of H and F households' wealth. Hereafter, lowercase letters denote the log transformations for all other variables in deviations from their steady state or initial levels (e.g., $r_t \equiv \ln R_t - \ln R$, $p_t^{\mathrm{T}} \equiv \ln P_t^{\mathrm{T}} - \ln P^{\mathrm{T}}$, etc.). Appendix A.1 summarizes the approximation point of our economy and lists all equations used in the model's solution.

Following Campbell, Chan, and Viceira (2003) (CCV from hereafter) we approximate the equations characterizing the real side of the economy to first order, while those involving portfolios we approximate to second order. We also studied a version of our method that uses second-order approximations to all the equilibrium conditions, but found that the accuracy of the technique remains unaffected by this amendment (see Section 4.2). Therefore, for the sake of clarity, we present our method with first-order approximations of the real-side equations.[7] We focus below on the behavior of households and firms in country H; the behavior in country F is characterized in an analogous manner.

We begin, following CCV, with a first-order log-approximation to the budget constraint of the representative H household:

$$\begin{aligned} \Delta w_{t+1} &= \ln(1 - C_t/W_t) + \ln R_{t+1}^{\mathrm{W}}, \\ &= r_{t+1}^{\mathrm{W}} - \tfrac{\Lambda}{1-\Lambda}(c_t - w_t), \end{aligned} \tag{18}$$

[7] The ouline of the method with the second-order approximations of the real equations is available in the web appendix at http://www.econ.ubc.ca/vhnatkovska/research.htm

where $\Lambda \equiv C/W$ is the steady state consumption expenditure to wealth ratio. r^{W}_{t+1} is the log return on optimally invested wealth which CCV approximate as

$$r^{\mathrm{W}}_{t+1} = r_t + \boldsymbol{\alpha}'_t er_{t+1} + \tfrac{1}{2}\boldsymbol{\alpha}'_t \left(\mathrm{diag}\left(\mathbb{V}_t(er_{t+1})\right) - \mathbb{V}_t(er_{t+1})\boldsymbol{\alpha}_t\right), \tag{19}$$

where $\boldsymbol{\alpha}'_t \equiv [\ \alpha^{\mathrm{H}}_t \quad \alpha^{\mathrm{F}}_t \quad \alpha^{\mathrm{N}}_t\ ]$ is the vector of portfolio shares, $er'_{t+1} \equiv [\ r^{\mathrm{H}}_{t+1} - r_t \quad r^{\mathrm{F}}_{t+1} - r_t \quad r^{\mathrm{N}}_{t+1} - r_t\ ]$ is a vector of excess log equity returns, and $\mathbb{V}_t(.)$ is the variance conditioned on period-t information. It is important to emphasize how the approximation of the return on wealth in equation (19) differs from a standard second-order approximation. Note that in this approximation the vector of portfolio shares $\boldsymbol{\alpha}_t$ always appears multiplicatively, and not in deviations from its steady state value. Thus this approximation does not require us to know the steady state portfolios; instead, we will solve for the steady state values of $\boldsymbol{\alpha}_t$ as a part of our model solution. Furthermore, CCV show that the approximation error associated with (19) disappears in the limit where asset prices follow continuous–time diffusion processes.

Next, we turn to the first-order conditions in (9). Using standard log-normal approximations, we obtain

$$\begin{aligned} \mathbb{E}_t\left[r^{\varkappa}_{t+1}\right] - r_t + \tfrac{1}{2}\mathbb{V}_t\left(r^{\varkappa}_{t+1}\right) &= -\mathbb{CV}_t\left(m_{t+1}, r^{\varkappa}_{t+1}\right), & \text{(20a)}\\ r_t &= -\ln\beta_t - \mathbb{E}_t\left[m_{t+1}\right] - \tfrac{1}{2}\mathbb{V}_t(m_{t+1}), & \text{(20b)} \end{aligned}$$

where $r^{\varkappa}_{t+1}$ is the log return for equity $\varkappa = \{\mathrm{H},\ \mathrm{F},\ \mathrm{N}\}$, and $\mathbb{CV}_t\left(.,.\right)$ denotes the covariance conditioned on period-t information. $m_{t+1} \equiv \ln M_{t+1} - \ln M$ denotes the log IMRS which is given by

$$m_{t+1} = \ln\beta_t + \tfrac{(1-\sigma)(\phi-1)}{\phi}\Delta\lambda^{\mathrm{T}}_{t+1} + \tfrac{1-\sigma-\phi}{\phi}\Delta\lambda_{t+1} - \sigma\Delta w_{t+1}, \tag{21}$$

where $\lambda^{\mathrm{T}}_t = \ln(\Lambda^{\mathrm{T}}_t/\Lambda^{\mathrm{T}}) = \ln(C^{\mathrm{T}}_t/W_t\ ) - \ln(C^{\mathrm{T}}/W)$ and $\lambda_t = \ln\left(\Lambda_t/\Lambda\right) = \ln C_t/W_t - \ln(C/W)$. Substituting for log wealth from (18) and (19), equation (20a) can be rewritten in vector form as

$$\begin{aligned} \mathbb{E}_t\left[er_{t+1}\right] &= \sigma\mathbb{V}_t(er_{t+1})\boldsymbol{\alpha}_t - \tfrac{1}{2}\mathrm{diag}\left(\mathbb{V}_t(er_{t+1})\right)\\ &\quad -\tfrac{(1-\sigma)(\phi-1)}{\phi}\mathbb{CV}_t(\lambda^{\mathrm{T}}_{t+1}, er_{t+1}) - \tfrac{1-\sigma-\phi}{\phi}\mathbb{CV}_t\left(\lambda_{t+1}, er_{t+1}\right). \end{aligned} \tag{22}$$

This equation implicitly identifies the optimal choice of the H household's portfolio shares, $\boldsymbol{\alpha}_t$. Again, we note that this approximation does not require an assumption about the portfolio shares chosen in the steady state. We will determine those endogenously below.

Using equation (22) we can re-write the log return on wealth as

$$\begin{aligned} r^{\mathrm{W}}_{t+1} &= r_t + \left(\sigma - \tfrac{1}{2}\right)\boldsymbol{\alpha}'_t\mathbb{V}_t(er_{t+1})\boldsymbol{\alpha}_t - \tfrac{(1-\sigma)(\phi-1)}{\phi}\boldsymbol{\alpha}'_t\mathbb{CV}_t(\lambda^{\mathrm{T}}_{t+1}, er_{t+1})\\ &\quad -\tfrac{1-\sigma-\phi}{\phi}\boldsymbol{\alpha}'_t\mathbb{CV}_t\left(\lambda_{t+1}, er_{t+1}\right) + \boldsymbol{\alpha}'_t\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right). \end{aligned} \tag{23}$$

Substituting this expression into equation (18), we obtain a log-approximate version of the H household's budget constraint:

$$\begin{aligned}\Delta w_{t+1} \quad = \quad & -\tfrac{\Lambda}{1-\Lambda}\lambda_t + r_t + \left(\sigma - \tfrac{1}{2}\right)\boldsymbol{\alpha}_t'\mathbb{V}_t(er_{t+1})\boldsymbol{\alpha}_t - \tfrac{(1-\sigma)(\phi-1)}{\phi}\boldsymbol{\alpha}_t'\mathbb{CV}_t(\lambda_{t+1}^{\mathrm{T}}, er_{t+1}) \\ & -\tfrac{1-\sigma-\phi}{\phi}\boldsymbol{\alpha}_t'\mathbb{CV}_t\left(\lambda_{t+1}, er_{t+1}\right) + \boldsymbol{\alpha}_t'\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right). \end{aligned} \tag{24}$$

This equation shows that the growth in household's wealth between t and $t{+}1$ depends upon the consumption-wealth ratio in period t, the period-t risk free rate, r_t, portfolio shares, $\boldsymbol{\alpha}_t$, the variance-covariance matrix of excess returns, $\mathbb{V}_t(er_{t+1})$, and the covariances of consumption-wealth ratios (both traded and aggregate consumption) with the excess returns; as well as the unexpected return on assets held between t and $t+1$, $\boldsymbol{\alpha}_t'\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right)$. The first five terms comprise the expected growth rate of wealth under the optimal portfolio strategy. From equation (24) one can see a key novelty present in models with portfolio choice: the conditional distribution of household's wealth is state-dependent. In particular, the conditional second moments of w_{t+1} depend on $\boldsymbol{\alpha}_t'$ and conditional second moments of returns. As a result, the process for wealth in our model is conditionally heteroskedastic, as noted in Section 2.2.

The remaining equations characterizing the model's equilibrium are approximated in a standard way. Log-approximating the expressions in (11) around the initial value of W_t and Q_t^{N} gives the consumption-wealth ratios as

$$c_t^{\mathrm{T}} - w_t \quad = \quad \lambda_t + \tfrac{\phi}{(1+\vartheta)(1-\phi)}q_t^{\mathrm{N}}, \tag{25a}$$

$$c_t^{\mathrm{N}} - w_t \quad = \quad \lambda_t - \tfrac{1-\phi+\vartheta}{(1+\vartheta)(1-\phi)}q_t^{\mathrm{N}}, \tag{25b}$$

with ϑ denoting the initial value of $\vartheta(Q_t^{\mathrm{N}})$. Notice that equation (25a) defines λ_t^{T}. To derive the expression for λ_t, we log-approximate equation (10):

$$\begin{aligned}\tfrac{(1-\sigma)(\phi-1)}{\phi}\lambda_t^{\mathrm{T}} + \left(\tfrac{1-\sigma-\phi}{\phi} - \sigma\tfrac{\Lambda}{1-\Lambda}\right)\lambda_t \quad = \quad & \mathbb{E}_t\left[(1-\sigma)\,r_{t+1}^{\mathrm{W}} + \tfrac{(1-\sigma)(\phi-1)}{\phi}\lambda_{t+1}^{\mathrm{T}} + \tfrac{1-\sigma-\phi}{\phi}\lambda_{t+1}\right] \\ & + \tfrac{1}{2}\mathbb{V}_t\left[(1-\sigma)\,r_{t+1}^{\mathrm{W}} + \tfrac{(1-\sigma)(\phi-1)}{\phi}\lambda_{t+1}^{\mathrm{T}} + \tfrac{1-\sigma-\phi}{\phi}\lambda_{t+1}\right]. \end{aligned} \tag{26}$$

Optimal investment by H firms requires that

$$\mathbb{E}_t\left[r_{t+1}^{\mathrm{K}}\right] - r_t + \tfrac{1}{2}\mathbb{V}_t\left(r_{t+1}^{\mathrm{K}}\right) = -\mathbb{CV}_t\left(m_{t+1}, r_{t+1}^{\mathrm{K}}\right), \tag{27}$$

where r_{t+1}^{K} is the log return on capital approximated by

$$r_{t+1}^{\mathrm{K}} = \psi\left(z_{t+1}^{\mathrm{T}} - (1-\theta)k_{t+1}\right), \tag{28}$$

with $\psi \equiv 1 - \beta(1-\delta) < 1$. The dynamics of the H capital stock are approximated by

$$k_{t+1} = \tfrac{1}{\beta}k_t + \tfrac{\psi}{\beta\theta}z_t^{\mathrm{T}} - \tfrac{\varphi}{\theta\beta}d_t^{\mathrm{T}}, \tag{29}$$

where $\varphi = \psi - \delta\theta\beta > 0$.

We follow Campbell and Shiller (1988) in relating the log returns on equity to the log dividends and the log prices of equity:

$$r^{\mathrm{H}}_{t+1} = \rho^{\mathrm{H}} p^{\mathrm{T}}_{t+1} + (1-\rho^{\mathrm{H}}) d^{\mathrm{T}}_{t+1} - p^{\mathrm{T}}_t, \tag{30a}$$

$$r^{\mathrm{F}}_{t+1} = \rho^{\mathrm{F}} \hat{p}^{\mathrm{T}}_{t+1} + (1-\rho^{\mathrm{F}}) \hat{d}^{\mathrm{T}}_{t+1} - \hat{p}^{\mathrm{T}}_t, \tag{30b}$$

$$r^{\mathrm{N}}_{t+1} = q^{\mathrm{N}}_{t+1} + \rho^{\mathrm{N}} p^{\mathrm{N}}_{t+1} + (1-\rho^{\mathrm{N}}) d^{\mathrm{N}}_{t+1} - q^{\mathrm{N}}_t - p^{\mathrm{N}}_t, \tag{30c}$$

where $\rho^{\varkappa}$ is the reciprocal of one plus the dividend-to-price ratio. In the non-stochastic steady state, $\rho^{\varkappa} = \rho$ for $\varkappa = \{\mathrm{H}, \mathrm{F}, \mathrm{N}\}$.[8] Making this substitution, iterating forward, taking conditional expectations, and imposing $\lim_{j\to\infty} \mathbb{E}_t \rho^j p^{\mathrm{T}}_{t+j} = 0$, we can derive the H traded equity price as

$$p^{\mathrm{T}}_t = \sum_{i=0}^{\infty} \rho^i \left\{ (1-\rho)\mathbb{E}_t d^{\mathrm{T}}_{t+1+i} - \mathbb{E}_t r^{\mathrm{H}}_{t+1+i} \right\}. \tag{31}$$

Analogous expressions describe the log prices of F traded equity and nontraded equities.[9]

Finally, the market clearing conditions are approximated as follows. Market clearing in the goods' markets requires $C^{\mathrm{N}}_t = D^{\mathrm{N}}_t = \eta Z^{\mathrm{N}}_t$, $\hat{C}^{\mathrm{N}}_t = \hat{D}^{\mathrm{N}}_t = \eta \hat{Z}^{\mathrm{N}}_t$ and $D^{\mathrm{T}}_t + \hat{D}^{\mathrm{T}}_t = C^{\mathrm{T}}_t + \hat{C}^{\mathrm{T}}_t$. The first two conditions can be imposed without approximation as $c^{\mathrm{N}}_t = d^{\mathrm{N}}_t = z^{\mathrm{N}}_t$ and $\hat{c}^{\mathrm{N}}_t = \hat{d}^{\mathrm{N}}_t = \hat{z}^{\mathrm{N}}_t$. We rewrite the condition for traded goods as $\Lambda^{\mathrm{T}}_t \frac{W_t}{\hat{W}_t} + \hat{\Lambda}^{\mathrm{T}}_t = \frac{D^{\mathrm{T}}_t + \hat{D}^{\mathrm{T}}_t}{\hat{W}_t}$ and approximate it around the initial values for consumption-wealth ratios and steady state values for dividends:

$$\Lambda^{\mathrm{T}} \frac{W}{\hat{W}} (\lambda^{\mathrm{T}}_t + w_t - \hat{w}_t) + \hat{\Lambda}^{\mathrm{T}} \hat{\lambda}^{\mathrm{T}}_t = \frac{D^{\mathrm{T}}}{\hat{W}} (d^{\mathrm{T}}_t - \hat{w}_t) + \frac{\hat{D}^{\mathrm{T}}}{\hat{W}} (\hat{d}^{\mathrm{T}}_t - \hat{w}_t). \tag{32}$$

Market clearing in traded equity requires $A^{\mathrm{H}}_t + \hat{A}^{\mathrm{H}}_t = 1$ and $A^{\mathrm{F}}_t + \hat{A}^{\mathrm{F}}_t = 1$. Combining these conditions with the definitions for portfolio shares and the fact that the consumption-wealth ratio is equal to Λ_t for H households and $\hat{\Lambda}_t$ for F households, we obtain

$$\begin{aligned}
\exp(p^{\mathrm{T}}_t - w_t - \ln(1-\Lambda_t)) &= \alpha^{\mathrm{H}}_t + \hat{\alpha}^{\mathrm{H}}_t \exp(\hat{w}_t - w_t + \ln(1-\hat{\Lambda}_t) - \ln(1-\Lambda_t)), \\
\exp(\hat{p}^{\mathrm{T}}_t - \hat{w}_t - \ln(1-\hat{\Lambda}_t)) &= \hat{\alpha}^{\mathrm{F}}_t + \alpha^{\mathrm{F}}_t \exp(w_t - \hat{w}_t + \ln(1-\Lambda_t) - \ln(1-\hat{\Lambda}_t)).
\end{aligned}$$

We approximate the left-hand side of these expressions around the steady state values for $P^{\mathrm{T}}_t / W_t\,(1-\Lambda_t)$ and $\hat{P}^{\mathrm{T}}_t / \hat{W}_t (1-\hat{\Lambda}_t)$ and their right-hand side around the initial wealth ratio $\hat{W}_0/W_0$, which we take to equal one. A second-order approximation to both sides of the market clearing conditions gives

$$\alpha^{\mathrm{H}} \left[1 + \Re^{\mathrm{T}}_t + \tfrac{1}{2}(\Re^{\mathrm{T}}_t)^2 + \tfrac{\Lambda}{2(1-\Lambda)^2} \lambda_t^2 \right] = \alpha^{\mathrm{H}}_t + \hat{\alpha}^{\mathrm{H}}_t \left[1 - \Im_t + \tfrac{1}{2}\Im_t^2 + \tfrac{\Lambda}{2(1-\Lambda)^2} \left\{ \lambda_t^2 - \hat{\lambda}_t^2 \right\} \right], \tag{33a}$$

$$\alpha^{\mathrm{F}} \left[1 + \hat{\Re}^{\mathrm{T}}_t + \tfrac{1}{2}(\hat{\Re}^{\mathrm{T}}_t)^2 + \tfrac{\Lambda}{2(1-\Lambda)^2} \hat{\lambda}_t^2 \right] = \hat{\alpha}^{\mathrm{F}}_t + \alpha^{\mathrm{F}}_t \left[1 + \Im_t + \tfrac{1}{2}\Im_t^2 + \tfrac{\Lambda}{2(1-\Lambda)^2} \left\{ \hat{\lambda}_t^2 - \lambda_t^2 \right\} \right], \tag{33b}$$

[8] With the exogenous subjective discount factor we also have $\rho = \beta$; while when the discount factor is endogenous, $\rho = \beta(\tilde{C}^{\mathrm{T}}, \tilde{C}^{\mathrm{N}})$, which we calibrate to also equal β in the steady state.

[9] We confirm that the no-bubbles conditions are satisfied in our model.

where $\Re_t^{\mathrm{T}} \equiv p_t^{\mathrm{T}} - w_t + \frac{\Lambda}{1-\Lambda}\lambda_t$, $\hat{\Re}_t^{\mathrm{T}} \equiv \hat{p}_t^{\mathrm{T}} - \hat{w}_t + \frac{\Lambda}{1-\Lambda}\hat{\lambda}_t$, and $\Im_t \equiv w_t - \hat{w}_t + \frac{\Lambda}{1-\Lambda}(\hat{\lambda}_t - \lambda_t)$. α^{H} is the initial value of $\alpha_t^{\mathrm{H}} + \hat{\alpha}_t^{\mathrm{H}}$, and α^{F} is the initial value of $\hat{\alpha}_t^{\mathrm{F}} + \alpha_t^{\mathrm{F}}$. These values are pinned down by the steady state share of traded consumption in the total consumption expenditure. When the traded and nontraded sectors are of equal size, as in our model, $\alpha^{\mathrm{H}} = \alpha^{\mathrm{F}} = 1/2$. Market clearing in the nontraded equity (17) requires $\alpha_t^{\mathrm{N}} = \exp(q_t^{\mathrm{N}} + p_t^{\mathrm{N}} - w_t - \ln(1 - \Lambda_t))$ and $\hat{\alpha}_t^{\mathrm{N}} = \exp(\hat{q}_t^{\mathrm{N}} + \hat{p}_t^{\mathrm{N}} - \hat{w}_t - \ln(1 - \hat{\Lambda}_t))$. Using the same approach we obtain

$$\alpha_t^{\mathrm{N}}/\alpha^{\mathrm{N}} = 1 + \Re_t^{\mathrm{N}} + \tfrac{1}{2}(\Re_t^{\mathrm{N}})^2 + \tfrac{\Lambda}{2(1-\Lambda)^2}\lambda_t^2, \tag{34a}$$

$$\hat{\alpha}_t^{\mathrm{N}}/\hat{\alpha}^{\mathrm{N}} = 1 + \hat{\Re}_t^{\mathrm{N}} + \tfrac{1}{2}(\hat{\Re}_t^{\mathrm{N}})^2 + \tfrac{\Lambda}{2(1-\Lambda)^2}\hat{\lambda}_t^2, \tag{34b}$$

where $\Re_t^{\mathrm{N}} \equiv q_t^{\mathrm{N}} + p_t^{\mathrm{N}} - w_t + \frac{\Lambda}{1-\Lambda}\lambda_t$ and $\hat{\Re}_t^{\mathrm{N}} \equiv \hat{q}_t^{\mathrm{N}} + \hat{p}_t^{\mathrm{N}} - \hat{w}_t + \frac{\Lambda}{1-\Lambda}\hat{\lambda}_t$. α^{N} and $\hat{\alpha}^{\mathrm{N}}$ are the initial values of α_t^{N} and $\hat{\alpha}_t^{\mathrm{N}}$; $\alpha^{\mathrm{N}} = \hat{\alpha}^{\mathrm{N}} = 1/2$. All that now remains is the bond market clearing condition: $B_t + \hat{B}_t = 0$. Walras Law implies that this restriction is redundant given the other market clearing conditions and budget constraints.

An Overview

Let us provide an overview of our solution method and highlight its differences from the standard methods. The set of equations characterizing the equilibrium of a DSGE model with portfolio choice and incomplete markets can conveniently be written in a general form as

$$\begin{aligned} 0 &= \mathbb{E}_t f\left(Y_{t+1}, Y_t, X_{t+1}, X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), \\ X_{t+1} &= \mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), \end{aligned} \tag{35}$$

where $f(.)$ is a known function. X_t is a vector of state variables and Y_t is a vector of non-predetermined variables. In our model, X_t contains the state of productivity, the capital stocks and households' wealth, while Y_t includes consumption, dividends, asset allocations, prices and the risk-free rate. The function $\mathcal{H}(.,.)$, to be determined, governs how past states affect the current state. ε_t is a vector of i.i.d. mean zero, unit variance shocks. In our model, ε_t contains the four productivity shocks. $\mathcal{S}^{1/2}(X_t)$ is a state-dependent scaling matrix. The vector of innovations driving the equilibrium dynamics of the model is $U_{t+1} \equiv \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}$. This vector includes exogenous shocks, like the productivity shocks, and innovations to endogenous variables, such as households' wealth. These innovations have a conditional mean of zero and a conditional covariance equal to $\mathcal{S}(X_t)$, a function of the current state vector X_t:

$$\begin{aligned} \mathbb{E}(U_{t+1}|X_t) &= 0, \\ \mathbb{E}\left(U_{t+1}U_{t+1}'|X_t\right) &= \mathcal{S}^{1/2}(X_t)\,\mathcal{S}^{1/2}(X_t)' = \mathcal{S}(X_t). \end{aligned} \tag{36}$$

An important aspect of our formulation is that it explicitly allows for the possibility that innovations driving the equilibrium dynamics are conditionally heteroskedastic. It is important to note that we did not introduce conditional heteroskedasticity in the model since the productivity shocks, which are the only exogenous drivers in the model, follow a standard autoregressive process with i.i.d. innovations. Instead, conditional heteroskedasticity arises endogenously in the model due to market incompleteness and portfolio choice. In particular, as we showed in equation (24), the wealth of individual households is heteroskedastic and since it is included in the state vector X_t, the state vector itself becomes heteroskedastic. Our method simply recognizes this aspect of such a model and takes it into account when allowing the variance-covariance matrix of the state vector, $\mathcal{S}(X_t)$, to be state-dependent.[10] By contrast, standard perturbation methods assume that U_{t+1} follows an i.i.d. process, in which case $\mathcal{S}(X_t)$ would be a constant matrix.

Given our formulation in (35) and (36), a solution to the model is characterized by a decision rule for the non-predetermined variables

$$Y_t = \mathcal{G}(X_t, \mathcal{S}(X_t)), \tag{37}$$

a law of motion for state variables $\mathcal{H}(.)$, and a covariance function $\mathcal{S}(.)$ that satisfy the equilibrium conditions in (35):

$$0 = \mathbb{E}_t f\left(\mathcal{G}\left(\mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), \mathcal{S}\left(\mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right)\right)\right),\right.$$
$$\left.\mathcal{G}(X_t, \mathcal{S}(X_t)), \mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right).$$

Or, in a more compact notation,

$$0 = \mathcal{F}(X_t).$$

The first step in our method follows the perturbation procedure by approximating the policy functions as

$$\widehat{\mathcal{G}} = \sum\nolimits_i \psi_i \varphi_i(X_t), \qquad \widehat{\mathcal{H}} = \sum\nolimits_i \delta_i \varphi_i(X_t), \qquad \text{and}, \qquad \widehat{\mathcal{S}} = \sum\nolimits_i s_i \varphi_i(X_t),$$

for some unknown coefficient sequences $\{\psi_i\}$, $\{\delta_i\}$, and $\{s_i\}$. $\varphi_i(X_t)$ are ordinary polynomials in X_t. Next we approximate the function $f(.)$, as $\widehat{f}(.)$. The equations associated with the real side of the economy are approximated using Taylor series expansions, while those pertinent to the portfolio side are approximated using the continuous-time expansions of CCV. We denote the derivatives in these expansions as $\{\varsigma_i\}$.

Substituting $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$, and $\widehat{\mathcal{S}}$ into $\widehat{f}$ and taking expectations gives us an approximation for $\mathcal{F}$:

$$\widehat{\mathcal{F}}\left(X_t; \widehat{\mathcal{G}}, \widehat{\mathcal{H}}, \widehat{\mathcal{S}}, \varsigma, \psi, \delta, s\right) = \sum\nolimits_i \zeta_i \varphi_i(X_t),$$

where $\{\zeta_i\}$ are functions of $\{\varsigma_i\}$, $\{\psi_i\}$, $\{\delta_i\}$, and $\{s_i\}$. $\widehat{\mathcal{F}}$ is our residual function. To solve the model, we find the coefficient vectors ς, ψ, δ, and s that set the residual function equal to zero.[11]

[10] Variances and covariances of U_{t+1} entries corresponding to innovations to productivity shocks and capital are still homoskedastic, only the variances and covariances corresponding to wealth exhibit state-dependence.

[11] This step is reminiscent of the projection method introduced in Economics by Judd (1992). In its general formulation, the technique consists of choosing basis functions over the space of continuous functions and using them to approximate $\mathcal{G}(X_t, \sigma)$ and $\mathcal{H}(X_t, \sigma\varepsilon_{t+1})$. In most applications, families of orthogonal polynomials, like Chebyshev's polynomials, are used to form $\varphi_i(X_t, \sigma)$. Given the chosen order of approximation, the problem of solving the model translates into finding the coefficient vectors ψ and δ that minimize a residual function.

The Numerical Procedure

The next step in our solution procedure is deriving a general yet tractable set of equations that describe the equilibrium dynamics of the state variables. We use the method of undetermined coefficients (McCallum (1983), Christiano (2002)) to find these dynamics. Alternatively, the system of linear equations derived above could be solved with the algorithms in Klein (2000), Sims (2002), or Schmitt-Grohe and Uribe (2004).[12]

As is standard, we first conjecture that the $l \times 1$ vector of state variables x_t follows

$$x_{t+1} = \Phi_0 + (I - \Phi_1)x_t + \Phi_2 \tilde{x}_t + u_{t+1}, \tag{38}$$

where $\tilde{x}_t \equiv vec(x_t x_t')$, Φ_0 is the $l \times 1$ vector of constants, Φ_1 is the $l \times l$ matrix of autoregressive coefficients and Φ_2 is the $l \times l^2$ matrix of coefficients on the second-order terms. u_{t+1} is a vector of innovations with a zero conditional mean, and a conditional covariance that is a function of x_t :

$$\begin{aligned} \mathbb{E}\left(u_{t+1}|x_t\right) &= 0, \\ \mathbb{E}\left(u_{t+1}u_{t+1}'|x_t\right) &= \Omega(X_t) = \Omega_0 + \Omega_1 x_t x_t' \Omega_1'. \end{aligned} \tag{39}$$

This conjecture has two notable features: First, it introduces nonlinearity in the process for x_{t+1} by allowing its squares and cross-products in period t to enter the law of motion via the Φ_2 matrix. Second, the variance-covariance matrix of x_{t+1} depends on x_t. As we noted above, this conditional heteroskedasticity arises even though the productivity process is homoskedastic because x_t contains w_t and $\hat{w}_t$, and log wealth is endogenously heteroskedastic when asset markets are incomplete.

The period-t information set of our economy consists of x_t and $\tilde{x}_t$, which we conveniently combine in the extended state vector $X_t = [\begin{array}{ccc} 1 & x_t' & \tilde{x}_t' \end{array}]'$ with $\mathfrak{L} = 1 + l + l^2$ elements. Our solution method requires that we characterize the dynamics of X_t. In particular, we need to find an equation for the dynamics of $\tilde{x}_t$ consistent with (38) and (39). For this purpose, we first write the vectorized conditional variance of u_t as

$$vec\left(\Omega(X_t)\right) = \left[\begin{array}{ccc} \Sigma_0 & 0 & \Sigma_1 \end{array}\right] \left[\begin{array}{c} 1 \\ x_t \\ \tilde{x}_t \end{array}\right] = \Sigma X_t. \tag{40}$$

Next, we consider the continuous time analogue to (38) and derive the dynamics of $\tilde{x}_{t+1}$ via Ito's lemma. Appendix A.2 shows that the resulting process can be approximated in discrete time by

$$\tilde{x}_{t+1} = \tfrac{1}{2}D\Sigma_0 + \left((\Phi_0 \otimes I) + (I \otimes \Phi_0)\right)x_t + \left(I - (\Phi_1 \otimes I) - (I \otimes \Phi_1) + \tfrac{1}{2}D\Sigma_1\right)\tilde{x}_t + \tilde{u}_{t+1} \tag{41}$$

where
$$\tilde{u}_{t+1} = \left[(I \otimes x_t) + (x_t \otimes I)\right]u_{t+1},$$

[12] These algorithms, however, need to be amended to allow the variance-covariance function $S(X_t)$ to be state dependent. One possible way of doing so is by expanding the definition of the state vector to include $S(X_t)$ and then solve the model with the second-order approximations using this extended state vector.

$$D = \left[\mathbb{U} \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right], \qquad \text{and} \qquad \mathbb{U} = \sum_r \sum_s E_{rs} \otimes E'_{r,s}.$$

$E_{r,s}$ is the elementary matrix which has a unity at the $(r,s)^{th}$ position and zero elsewhere. Equation (41) approximates the dynamics of $\tilde{x}_{t+1}$ because it ignores the role played by cubic and higher-order terms involving the elements of x_t. In this sense, (41) represents a second–order approximation to the dynamics of the second–order terms in the state vector. Notice that the variance of u_{t+1} affects the dynamics of $\tilde{x}_{t+1}$ via the D matrix and that $\tilde{u}_{t+1}$ will generally be conditionally heteroskedastic.

We can now combine (38) and (41) into a single equation:

$$\begin{bmatrix} 1 \\ x_{t+1} \\ \tilde{x}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \Phi_0 & I - \Phi_1 & \Phi_2 \\ \frac{1}{2} D \Sigma_0 & (\Phi_0 \otimes I) + (I \otimes \Phi_0) & I - (\Phi_1 \otimes I) - (I \otimes \Phi_1) + \frac{1}{2} D \Sigma_1 \end{bmatrix} \begin{bmatrix} 1 \\ x_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} 0 \\ u_{t+1} \\ \tilde{u}_{t+1} \end{bmatrix},$$

or more compactly

$$X_{t+1} = \mathbb{A} X_t + U_{t+1}, \tag{42}$$

with $\mathbb{E}(U_{t+1}|X_t) = 0$ and $\mathbb{E}\left(U_{t+1} U'_{t+1} | X_t\right) \equiv \mathcal{S}(X_t)$. In Appendix A.3 we show that

$$\mathcal{S}(X_t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega(X_t) & \Gamma(X_t) \\ 0 & \Gamma(X_t)' & \Psi(X_t) \end{pmatrix}, \tag{43}$$

where

$$\begin{aligned} vec(\Gamma(X_t)) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \tilde{x}_t, \\ vec\left(\Gamma(X_t)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2 \tilde{x}_t, \\ vec(\Psi(X_t)) &= \Psi_0 + \Psi_1 x_t + \Psi_2 \tilde{x}_t. \end{aligned}$$

The Γ_i, Λ_i and Ψ_i matrices are functions of the parameters in (38) and (39); their precise form is shown in Appendix A.3.

Now that we are equipped with the process for the state vector, we can express all the endogenous variables in the model as linear combinations of X_t. After that, one can follow the method of undetermined coefficients directly as outlined in McCallum (1983), Christiano (2002) to solve the model.

2.3 Related Methods

Our solution method is most closely related to Campbell, Chan, and Viceira (2003). They developed an approximation for returns on household's wealth which preserves the multiplicative nature of portfolio weighting. Their expression for returns holds exactly in continuous time when asset prices follow diffusions and remains very accurate in discrete time for short time intervals. CCV apply this approximation method to study dynamic portfolio choice in a partial equilibrium setting where returns follow an exogenous process. Our solution method can be viewed as an extension of CCV to a DSGE setting where returns are endogenously determined.

Our approach also builds on the perturbation methods developed and applied in Judd and Guu (1993, 1997), Judd (1998), and further discussed in Collard and Juillard (2001), Jin and Judd (2002), Schmitt-Grohe and Uribe (2004) among others. These methods extend solution techniques relying on linearizations by allowing for second- and higher-order terms in the approximation of the policy functions. Applications of the perturbation technique to models with portfolio choice have been developed in Devereux and Sutherland (2008, 2007) and Tille and van Wincoop (2010). Both approaches are based on Taylor series approximations. Devereux and Sutherland (2008) use second-order approximations to the portfolio choice conditions and first-order approximations to the other optimality conditions in order to calculate the steady state portfolio allocations in a DSGE model. Our method also produces constant portfolio shares in the case where equilibrium returns are i.i.d. because $\mathcal{S}(X_t)$ is a constant matrix.

To study time-variation in portfolio choice, Devereux and Sutherland (2007) and Tille and van Wincoop (2010) use a method that incorporates third-order approximations of portfolio equations and second-order approximations to the rest of the model's equilibrium conditions. This approach delivers a first-order approximation for optimal portfolio holdings that vary with the state of the economy. By contrast, we are able to derive second-order approximations to portfolio holdings from a set of second-order approximations to the equilibrium conditions of the model *and* covariance function $\mathcal{S}(X_t)$. Approximating both *first* and *second* moments of the state vector to the second-order allows us to implicitly characterize the portfolio optimality conditions to the fourth order. This yields the second-order accurate dynamics of portfolio shares. If we approximated $\mathcal{S}(X_t)$ to the first-order, we would obtain first-order variations in portfolio shares, as in Devereux and Sutherland (2007) and Tille and van Wincoop (2010). Thus, we are able to study the dynamics of portfolio shares to a higher-order of accuracy while avoiding the numerical complexity of computing at least third-order approximations. This aspect of our method will be important in models with larger number of state variables and where agents choose between many assets. The model in Section 1 has 8 state variables and five assets, but was solved without much computational difficulty. We view this as an important practical advantage of our method that will make it particularly useful for solving international DSGE models. By their very nature, even a minimally specified two-country DSGE model will have many state variables and several assets. In Section 3.3 we also perform detailed accuracy comparisons of our method with that in Devereux and Sutherland (2007).

3 Results

In this section we evaluate the accuracy of our solution method. For this purpose we consider six versions of our model. We start by looking at a simplified version with complete markets. In this setup we study a log-utility case and a case with higher risk aversion. Results from the log-utility model are informative because they can be compared against known analytical properties of the equilibrium. The model with power utility allows us evaluate the impact of higher risk aversion for portfolio choice and method accuracy. We then look at the full model with incomplete markets with log and power utility. The results from the full model demonstrate the accuracy of our solution method in an application where no analytical characterization of the equilibrium is available.

Our model simplifies considerably if we let $(1-\phi)^{-1} \to \infty$, set $\sigma = 1$, require $\mu^{\mathrm{T}} = 1$ and $\mu^{\mathrm{N}} = 0$ in both countries, and assume that the variance of nontraded productivity shocks equal zero. These restrictions effectively eliminate the nontraded sectors in each country; the supply and demand for nontraded goods is zero, and so too is the price of nontraded equity. The equilibrium properties of the other variables will be identical to those in a world where households have log preferences defined over traded consumption and allocate their portfolios between H and F traded equities and the risk-free bond. In particular, the equilibrium will be characterized by complete risk-sharing if both H and F households start with the same initial level of wealth. Furthermore, since markets are complete, the non-stationarity problem does not arise in this simplified setup. So, we eliminate the endogenous discount factor in this version of our model by setting $\beta(\tilde{C}_t^{\mathrm{T}}, \tilde{C}_t^{\mathrm{N}}) = \beta$, $\forall t$, a constant.

Complete risk-sharing occurs in our simplified setting because all households have the same preferences and investment opportunity sets. We can see why this is so by returning to the conditions determining the households' portfolio choices. In particular, combining the log-approximated first-order conditions with the budget constraint in (22) under the assumption of log preferences gives

$$\boldsymbol{\alpha}_t = \Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}\,(\mathbb{V}_t(er_{t+1})) \qquad \text{and} \qquad \hat{\boldsymbol{\alpha}}_t = \Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}(\mathbb{V}_t(er_{t+1})), \tag{44}$$

where $\boldsymbol{\alpha}_t' \equiv [\ \alpha_t^{\mathrm{H}} \quad \alpha_t^{\mathrm{F}}\ ]$, $\hat{\boldsymbol{\alpha}}_t' \equiv [\ \hat{\alpha}_t^{\mathrm{H}} \quad \hat{\alpha}_t^{\mathrm{F}}\ ]$, $er_{t+1}' \equiv [\ r_{t+1}^{\mathrm{H}} - r_t \quad r_{t+1}^{\mathrm{F}} - r_t\ ]$, and $\Theta_t \equiv \mathbb{V}_t(er_{t+1})$. The key point to note here is that all households face the same set of returns and have the same information. So the right hand side of both expressions in (44) are identical in equilibrium. H and F households will therefore find it optimal to hold the same portfolio shares. This has a number of implications if the initial distribution of wealth is equal. First, households' wealth will be equalized across countries in all periods. Second, since households with log utility consume a constant fraction of wealth, consumption will also be equalized. This symmetry in consumption implies that $m_{t+1} = \hat{m}_{t+1}$, so risk sharing is complete. It also implies, together with the market clearing conditions, that bond holdings are zero and wealth is equally split between H and F equities (i.e., $A_t^{\mathrm{H}} = \hat{A}_t^{\mathrm{H}} = A_t^{\mathrm{F}} = \hat{A}_t^{\mathrm{F}} = 1/2$). We can use these equilibrium asset holdings as a benchmark for judging the accuracy of our solution technique.

When $\sigma \neq 1$, we can again use the log-approximated Euler equations to show that the households' optimal portfolio choices are given by

$$\begin{aligned}
\boldsymbol{\alpha}_t &= \tfrac{1}{\sigma}\Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}\,(\mathbb{V}_t(er_{t+1})) - \Theta_t^{-1}\mathbb{CV}_t(\lambda_{t+1}, er_{t+1}) \qquad \text{and} \\
\hat{\boldsymbol{\alpha}}_t &= \tfrac{1}{\sigma}\Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}\,(\mathbb{V}_t(er_{t+1})) - \Theta_t^{-1}\mathbb{CV}_t(\hat{\lambda}_{t+1}, er_{t+1}).
\end{aligned}$$

These equations express the optimal portfolios as the sum of two components. The first term on the right-hand-side is a familiar mean-variance component of asset demand. Except for the $\frac{1}{\sigma}$ term, it coincides with the solution under log-utility case. The second term is an intertemporal hedging demand. It depends on the covariance between aggregate consumption-wealth ratio and excess returns. In the full 2-sector model with power utility, the intertemporal hedging component of asset demand depends on the covariance of both traded and aggregate consumption-wealth ratios with excess returns, and is equal to

$$\tfrac{1}{\sigma}\Theta_t^{-1}\left(\tfrac{(1-\sigma)(\phi-1)}{\phi}\mathbb{CV}_t\left(\lambda_{t+1}^{\mathrm{T}}, er_{t+1}\right)+\tfrac{1-\sigma-\phi}{\phi}\mathbb{CV}_t(\lambda_{t+1}, er_{t+1})\right)$$

in country H and a symmetric expression in country F.

Next, we turn to the original version of our model, which is characterized by incomplete markets. To assess the accuracy of our method we consider two scenarios: (i) with an endogenous discount factor, and (ii) without an endogenous discount factor. The version with an endogenous discount factor is a stationary model, whose properties are well-understood. The version without an endogenous discount factor is characterized by non-stationary wealth dynamics. Our method can handle such situations without sacrificing accuracy. We use this version of the model to illustrate this feature of our approach.

Both the complete and incomplete markets versions of the model are computed by our solution method.[13] These calculations were performed assuming the technology parameter, θ, equal to 0.36 and a depreciation rate for capital, δ, of 0.02. When an exogenous discount factor is used we set β equal to 0.99, but when the discount factor is endogenous we calibrate the parameter ζ such that $\beta(\tilde{C}^{\mathrm{T}}, \tilde{C}^{\mathrm{N}})$ equals 0.99 in the steady state. In the complete markets version, the log of H and F traded productivity, $\ln Z_t$ and $\ln \hat{Z}_t$, are assumed to follow independent AR(1) processes with autocorrelation coefficients, a_{ii}, equal to 0.95 and innovation variance, S_e^{ii}, equal to 0.0001 for $i = \{\mathrm{H}, \mathrm{F}\}$. In the incomplete markets version we set the share parameters, μ^{T} and μ^{N}, equal to 0.5 and the elasticity of substitution, $(1-\phi)^{-1}$, equal to 0.74. The autocorrelation in traded and nontraded productivity are set to 0.78 and 0.99 respectively, and the innovations variances, S_e^{ii}, are assumed equal to 0.0001, for $i = \{\mathrm{T}, \hat{\mathrm{T}}, \mathrm{N}, \hat{\mathrm{N}}\}$. In the case of power utility, we set σ equal to 2. All of these parameter values are quite standard and were chosen so that each period in the model represents one quarter. Once the model is "solved", we simulate X_t over 300 quarters starting from an equal wealth distribution. The statistics we report are derived from 1200 simulations and so are based on 90,000 years of simulated quarterly data in the neighborhood of the initial wealth distribution.

3.1 Portfolios

We begin our assessment of the solution method by considering the equilibrium portfolio holdings. Panels A and B of Table 1 report statistics on the equilibrium asset holdings of H households computed from the simulations of the complete markets model. Theoretically speaking, with log utility we should see that $B_t = 0$ and $A_t^{\mathrm{H}} = A_t^{\mathrm{F}} = 1/2$. The simulation results in panel A conform closely to these predictions. The equity portfolio holdings show no variation and on average are exactly as theory predicts. Average bond holdings, measured as a share of wealth, are similarly close to zero, but show a little more variation. Overall, simulations based on our solution method appear to closely replicate the asset holdings theory predicts with complete risk sharing. Allowing for higher risk aversion (see panel B) does not change the properties of

[13] Implementation of the solution method for the complete markets version follows the steps described in Section 2.2, but excludes the restrictions involving the nontraded sectors.

equity holdings, but adds to the variability of bond holdings.

Panels C - F of Table 1 report statistics on the asset holdings of H households in the incomplete markets model, with panels C and D showing the results in the model with endogenous discount factor, while panels E and F are for the model with no endogenous discount factor. Households continue to diversify their portfolios between the equity issued by H and F firms producing tradable goods. The table shows that while these holdings are split equally on average, they are far from constant. Both the standard deviation and range of the tradable equity holdings are orders of magnitude larger than the simulated holdings from the complete markets model. Bond holdings also show significantly higher volatility when markets are incomplete. In this case, shocks to productivity in the nontradable sector affect H and F households differently and create incentives for international borrowing and lending. In equilibrium most of this activity takes place via trading in the bond market, so bond holdings display a good deal of volatility in our simulations of the incomplete markets model, with or without the endogenous discount factor.

Table 1: Portfolio Holdings

	A_t^{H}	A_t^{F}	A_t^{N}	B_t	A_t^{H}	A_t^{F}	A_t^{N}	B_t
	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
	A: Complete Markets, log-utility				B: Complete Markets, power-utility			
mean	0.5000	0.5000		-0.0011%	0.5000	0.5000		0.0015%
stdev	0.0000	0.0000		0.0170%	0.0000	0.0000		0.0847%
min	0.5000	0.5000		-0.0553%	0.5000	0.5000		-0.3539%
max	0.5000	0.5000		0.1219%	0.5000	0.5000		0.9647%
	C: Incomplete Markets, log-utility, with EDF				D: Incomplete Markets, power-utility, with EDF			
mean	0.5000	0.5000	1.0000	0.0003%	0.5000	0.5000	1.0000	-0.0105%
stdev	0.0007	0.0007	0.0000	0.1019%	0.0021	0.0021	0.0000	0.0915%
min	0.4964	0.4970	1.0000	-0.3758%	0.4897	0.4872	1.0000	-0.3744%
max	0.5030	0.5035	1.0000	0.4289%	0.5128	0.5103	1.0000	0.3681%
	E: Incomplete Markets, log-utility, no EDF				F: Incomplete Markets, power-utility, no EDF			
mean	0.4999	0.5001	1.0000	0.0101%	0.5000	0.5000	1.0000	-0.0101%
stdev	0.0015	0.0015	0.0000	0.1353%	0.0024	0.0024	0.0000	0.0700%
min	0.4895	0.4927	1.0000	-0.5136%	0.4874	0.4834	1.0000	-0.2862%
max	0.5073	0.5105	1.0000	0.5965%	0.5166	0.5126	1.0000	0.2984%

Note: A_t^{H}, A_t^{F}, and A_t^{N} correspond, respectively, to H household's holdings of equity issued by H, F traded firms, and H nontraded firms. B_t refers to H household's bond holdings as a share of H wealth.

3.2 Accuracy Tests

To assess the performance of our solution method we compute several tests of model accuracy. First, we report the size of Euler equation errors over our preferred simulation span of 300 quarters. Second, we examine the accuracy of our solution over various spans, ranging from 50 to 500 quarters. Third, we compare the accuracy of our method with the third-order approach proposed in Devereux and Sutherland (2007) and Tille and van Wincoop (2010).

Euler Equation Errors

Judd (1992) recommends using the size of the errors that households and firms make to assess the accuracy of an approximated solution. Given the definition of the stochastic discount factor M_{t+1} in our model the Euler equation errors for H households and firms are given by

$$\xi_t \equiv 1 - \frac{\left[\beta\left(\tilde{C}_t^{\mathrm{T}}, \tilde{C}_t^{\mathrm{N}}\right)\mathbb{E}_t C_{t+1}^{\frac{1-\sigma-\phi}{\phi}}\left(\frac{C_{t+1}^{\mathrm{T}}}{C_t^{\mathrm{T}}}\right)^{\frac{(1-\sigma)(\phi-1)}{\phi}} R_{t+1}^{\varkappa}\right]^{\frac{\phi}{1-\sigma-\phi}}}{C_t}, \tag{45}$$

where $R^{\varkappa} = \{R_{t+1}^{\mathrm{H}}, R_{t+1}^{\mathrm{F}}, R_{t+1}^{\mathrm{K}}, R_t\}$. Note that ξ_t provides a scale-free measure of the error and has an economic interpretation in terms of aggregate consumption. In the complete markets model the ξ_t vector contains four errors for each country: two for equity, one for capital, and one for bonds. In the incomplete markets model there are two more errors associated with the optional choice of nontraded equity holdings. We obtain the absolute values of the errors and report the mean, maximum and the upper percentiles of the distribution for the absolute errors in all six versions of the model in Table 2. All errors in the Table are of order 10^{-3}.

Table 2: Accuracy: Euler Equation Errors x10^{-3} (benchmark approximations)

	A_t^{H} (i)	A_t^{F} (ii)	A_t^{N} (iii)	K_t (iv)	B_t (v)	A_t^{H} (vi)	A_t^{F} (vii)	A_t^{N} (viii)	K_t (ix)	B_t (x)
	A: Complete Markets, log-utility					B: Complete Markets, power-utility				
mean	1.2483	1.2485		1.4259	1.5848	0.8823	0.8786		1.7404	1.8222
max	7.8886	7.7801		8.5921	9.3705	5.5097	5.0539		9.4665	9.9663
90^{th} percentile	2.5696	2.5715		2.9364	3.2658	1.8188	1.8146		3.5909	3.7567
95^{th} percentile	3.0647	3.0679		3.4980	3.8912	2.1723	2.1613		4.2710	4.4730
99^{th} percentile	4.0247	4.0176		4.6109	5.1161	2.8562	2.8447		5.5906	5.8502
	C: Incomplete Markets, log-utility, with EDF					D: Incomplete Markets, power-utility, with EDF				
mean	1.1427	1.1406	0.3715	1.3790	1.4634	0.5848	0.5854	1.2829	0.9634	1.0156
max	6.3436	6.3352	2.8230	7.2998	7.8704	3.7275	3.8266	8.1612	5.6180	5.8425
90^{th} percentile	2.3530	2.3541	0.7704	2.8426	3.0169	1.2067	1.2095	2.6436	1.9858	2.0938
95^{th} percentile	2.8078	2.8045	0.9237	3.3854	3.5956	1.4433	1.4422	3.1568	2.3764	2.5000
99^{th} percentile	3.6946	3.6991	1.2428	4.4627	4.7294	1.9114	1.9105	4.1394	3.1199	3.2821
	E: Incomplete Markets, log-utility, no EDF					F: Incomplete Markets, power-utility, no EDF				
mean	0.7796	0.7774	0.7057	1.1264	1.2132	0.7359	0.7369	1.4687	1.0432	1.0864
max	4.4551	4.6032	4.1723	6.0407	6.4713	4.6652	4.7642	9.0924	6.0736	6.3471
90^{th} percentile	1.6051	1.6056	1.4557	2.3208	2.5018	1.5188	1.5189	3.0286	2.1521	2.2442
95^{th} percentile	1.9162	1.9137	1.7403	2.7670	2.9829	1.8127	1.8145	3.6135	2.5737	2.6777
99^{th} percentile	2.5241	2.5288	2.2918	3.6414	3.9226	2.3935	2.3945	4.7347	3.3840	3.5217

Note: A_t^{H}, A_t^{F}, and A_t^{N} refer to the absolute errors from the Euler equations for H household's holdings of equity issued by H and F traded firms, and H nontraded firms; K_t and B_t correspond to the absolute errors from capital and bond Euler equations at H. All entries are of order 10^{-3}.

Columns (i)-(iii) and (vi)-(viii) show percentiles for the errors from H households' Euler equations for H, F and N equity; while columns (iv)-(v) and (ix)-(x) show the percentile from the H capital and bond Euler equations, respectively. There are several features of the Euler equation distributions to note. First, the Euler equation errors in all six models are small, with the largest error being less than 1 percent of aggregate consumption. On average, the errors are about a tenth of 1 percent of consumption, with the higher numbers obtained in the complete markets models. Second, the errors tend to be smaller in the models with power utility. This is mainly due to lower elasticity of intertemporal substitution and smoother consumption paths in these models. Finally, the presence of endogenous discount factor does not significantly affect the distribution of Euler equation errors under incomplete markets, as the errors are comparable across the versions with and without endogenous discount factor. Finally, we note that Euler equation errors obtained using our method are comparable to those reported in the accuracy checks for standard growth models without portfolio choice (e.g., Aruoba, Fernandez-Villaverde, and Rubio-Ramirez (2006) and Pichler (2005)).[14]

The approximations we use to characterize the solution to non-stationary versions of the model are only accurate in a neighborhood of the initial wealth distribution. If shocks to productivity push the wealth distribution outside this neighborhood in a few periods with high probability, we will not be able to accurately simulate long time series from the model's equilibrium. This is not a concern for the models in this paper. To illustrate this, we consider how the accuracy of the simulated equilibrium dynamics varies with the simulation span. For this purpose we examine the distributions of Euler equation errors for different simulation spans. To preserve space, we report the errors for the domestic equity holdings Euler equation in country H. The results for all other Euler equation are analogous. Table 3 reports the results for all six models simulated for various number of periods: 500, 400, 300 (our benchmark), 200, 100 and 50 quarters.

Euler equation errors in all six models are stable; including, importantly, the incomplete markets model with no endogenous discount factor (panels E and F). Furthermore, while the errors occasionally rise with the length of the simulation spans, the changes are moderate. Even at the horizon of 500 quarters, the Euler equation errors are small, on the order of less than one tenth of 1 percent of aggregate consumption. Overall, these results suggest that our method works well for both stationary and non-stationary versions of the model.

Wealth Dynamics and Simulation Spans

As an additional check of the accuracy of our model we examine the distribution of errors in the bond market clearing condition. Recall that the bond market clearing condition, $B_t + \hat{B}_t = 0$, was not used in our method, so the value of $B_t + \hat{B}_t$ implied by our solution provides a further accuracy check: If there is no approximation error in the equations we use for the other market clearing conditions and budged constraints, $B_t + \hat{B}_t$ should equal zero by Walras Law in our simulations of the model's solution.[15] We examine the accuracy of the

[14] In the benchmark approximations described above we relied on the first-order approximations of the equations characterizing the real side of the model and on the second-order approximations of the portfolio equations. We also solved all six versions of the model using second-order approximations of all equilibrium equations and computed the Euler equation errors from these extended approximations. We find that there is very little difference between the distributions of these errors and those reported in 2. The use of first- rather than second-order approximations to characterize the real side of the model does not appear to adversely affect the accuracy of our solution procedure. These results are available from the authors upon request.

[15] We thank Anna Pavlova for suggesting this accuracy evaluation.

Table 3: Accuracy: Euler Equation Errors x10^{-3}, various simulation spans

Span	mean (i)	max (ii)	90^{th} % (iii)	95^{th} % (iv)	99^{th} % (v)	mean (vi)	max (vii)	90^{th} % (viii)	95^{th} % (ix)	99^{th} % (x)
	A: Complete Markets, log-utility					B: Complete Markets, power-utility				
50	1.2478	6.2519	2.5808	3.0650	4.0271	0.8856	5.5097	1.8220	2.1773	2.8440
100	1.2513	6.3076	2.5736	3.0722	4.0467	0.8859	5.5097	1.8289	2.1781	2.8478
200	1.2483	7.1610	2.5681	3.0661	4.0310	0.8831	5.5097	1.8207	2.1732	2.8469
300	1.2483	7.8886	2.5696	3.0647	4.0247	0.8823	5.5097	1.8188	2.1723	2.8562
400	1.2475	7.8886	2.5709	3.0632	4.0327	0.8828	5.5097	1.8213	2.1735	2.8583
500	1.2475	7.8886	2.5714	3.0612	4.0267	0.8830	5.5097	1.8213	2.1736	2.8613
	C: Incomplete markets, log-utility, with EDF					D: Incomplete markets, power-utility, with EDF				
50	1.1490	6.3070	2.3579	2.8245	3.7208	0.5861	3.7275	1.2078	1.4386	1.9044
100	1.1447	6.3070	2.3580	2.8088	3.6884	0.5866	3.7275	1.2062	1.4381	1.9123
200	1.1454	6.3436	2.3575	2.8106	3.6989	0.5866	3.7275	1.2095	1.4439	1.9172
300	1.1427	6.3436	2.3530	2.8078	3.6946	0.5848	3.7275	1.2067	1.4433	1.9114
400	1.1414	7.0921	2.3514	2.8042	3.6928	0.5835	3.7275	1.2051	1.4430	1.9090
500	1.1427	7.0921	2.3569	2.8133	3.6980	0.5833	3.7275	1.2055	1.4430	1.9095
	E: Incomplete markets, log-utility, no EDF					F: Incomplete markets, power-utility, no EDF				
50	0.7841	4.4551	1.6105	1.9217	2.5457	0.7384	4.6652	1.5155	1.8100	2.3935
100	0.7813	4.4551	1.6119	1.9123	2.5232	0.7383	4.6652	1.5176	1.8063	2.3937
200	0.7820	4.4551	1.6099	1.9171	2.5271	0.7378	4.6652	1.5224	1.8132	2.3991
300	0.7796	4.4551	1.6051	1.9162	2.5241	0.7359	4.6652	1.5188	1.8127	2.3935
400	0.7786	4.8553	1.6039	1.9141	2.5227	0.7342	4.6652	1.5164	1.8098	2.3934
500	0.7790	4.8553	1.6079	1.9167	2.5232	0.7343	4.6652	1.5173	1.8115	2.3921

Note: 1, 5, 95, 99, *mean* and max refer to the corresponding percentiles, mean and maximum of the error distribution in the bond market clearing condition. All entries are of order 10^{-3}.

simulated equilibrium dynamics by computing the empirical distribution of $(B_t + \hat{B}_t)/(2R_t(W_t - C_t))$ within a simulation of a given span, and then comparing the distributions across different spans. The scaling allows us to interpret the bond market errors as shares of H household's wealth.

Table 4 reports the mean, maximum (of absolute values) and percentiles of the bond market error distribution in the six versions of our model from simulations spanning 50 to 500 quarters. All errors are of order 10^{-3}. Panel A shows statistics for the error distributions computed from the complete markets version of the log-utility model. Here, the dispersion of the error distribution increases with the span of our simulations, but the upper and lower percentiles of the distributions remain a very small percentage of wealth. The bond market errors in this version of the model are economically insignificant. These results are not surprising. The initial wealth distribution used in the simulations is equal to the long run distribution in the complete markets version of the log-utility model. Consequently, realizations of equilibrium wealth should never be too far from their initial values even when the span of the simulations is very long. Panel C reports the bond market clearing errors in the incomplete markets model in which stationarity is induced through endogenous discount factor. In this model errors are larger, on average, and tend to increase with the length of simulations. Their size, however, remains economically insignificant, reaching at maximum 1 percent of wealth at the longest simulation span of 500 quarters. When we consider the non-stationary

Table 4: Accuracy: Bond Market Clearing, x10^{-3}

Span	1% (i)	5% (ii)	mean (iii)	95% (iv)	99% (v)	max (vi)	1% (vii)	5% (viii)	mean (ix)	95% (x)	99% (xi)	max (xii)
	A: Complete Markets, log-utility						B: Complete Markets, power-utility					
50	-0.0948	-0.0585	0.0000	0.0703	0.1746	0.5508	-0.1078	-0.0704	-0.0001	0.0880	0.2112	0.7672
100	-0.1497	-0.1057	-0.0006	0.1503	0.2799	0.8670	-0.2395	-0.1617	0.0033	0.2401	0.5018	1.2546
200	-0.2618	-0.1907	-0.0055	0.2396	0.4863	1.2196	-0.7432	-0.4774	0.0067	0.6268	1.3544	3.3767
300	-0.3449	-0.2507	-0.0106	0.3170	0.5411	1.2196	-1.9154	-1.1445	0.0188	1.4040	3.2894	9.6551
400	-0.4277	-0.3036	-0.0148	0.3734	0.6059	1.2196	-4.8997	-2.7076	0.0457	3.1465	7.9508	26.7188
500	-0.4641	-0.3313	-0.0168	0.3944	0.6445	1.3708	-7.8948	-4.1792	0.0687	4.7366	12.4470	43.3476
	C: Incomplete markets, log-utility, with EDF						D: Incomplete markets, power-utility, with EDF					
50	-0.0218	-0.0171	-0.0068	0.0013	0.0110	0.0529	-0.0376	-0.0313	-0.0149	-0.0021	-0.0002	0.0389
100	-0.0435	-0.0314	-0.0075	0.0218	0.0590	0.1476	-0.0830	-0.0657	-0.0258	-0.0012	0.0310	0.1274
200	-0.1156	-0.0744	0.0002	0.1160	0.2580	0.8068	-0.2550	-0.1903	-0.0548	0.0315	0.1452	0.6393
300	-0.2794	-0.1596	0.0219	0.3189	0.7158	2.5031	-0.6888	-0.4863	-0.1156	0.0838	0.3449	1.7166
400	-0.6867	-0.3351	0.0718	0.7633	1.8296	6.8373	-1.8230	-1.2150	-0.2538	0.1651	0.8012	4.3353
500	-1.0855	-0.4982	0.1174	1.1736	2.9023	11.4175	-2.9657	-1.9237	-0.3805	0.2225	1.2211	7.1576
	E: Incomplete markets, log-utility, no EDF						F: Incomplete markets, power-utility, no EDF					
50	-0.0108	-0.0073	0.0000	0.0111	0.0226	0.0674	-0.0369	-0.0308	-0.0145	-0.0020	0.0000	0.0387
100	-0.0158	-0.0094	0.0077	0.0468	0.0886	0.1926	-0.0815	-0.0643	-0.0252	-0.0012	0.0312	0.1248
200	-0.0261	-0.0106	0.0453	0.2044	0.3702	1.0351	-0.2493	-0.1853	-0.0524	0.0351	0.1522	0.6685
300	-0.0357	-0.0102	0.1312	0.5657	1.0351	3.1759	-0.6716	-0.4703	-0.1085	0.0963	0.3762	1.8499
400	-0.0445	-0.0092	0.3233	1.4184	2.6681	8.5485	-1.7732	-1.1702	-0.2365	0.1959	0.8753	4.5838
500	-0.0508	-0.0087	0.4974	2.2271	4.2652	14.2283	-2.8840	-1.8519	-0.3544	0.2686	1.3330	7.4516

Note: 1, 5, 95, 99 and *mean* refer to the corresponding percentiles and mean of the error distribution in the bond market clearing condition. All entries are of order 10^{-3}.

incomplete markets version of the log-utility model, the size of the errors rises in absolute level and with the simulation span. The change in the error distribution is pronounced in the upper percentiles as the span increases beyond 300 quarters. For perspective, recall from Table 1 that the estimated bond holdings of country H households under log utility range from -0.51% to 0.59% of wealth over simulations spanning 300 quarters. The support of the corresponding bond error distribution is an order of magnitude smaller. Beyond 300 quarters, the support of the distributions approaches the range of variation in the estimated bond holdings. At least some of the bond errors in these simulations are economically significant.

The results for the power-utility case are summarized in Panels B, D, and F. Here the dispersion of the error distribution increases with the span of our simulations, but remains significantly below the dispersion of household's bond holdings reported in Table 1 for up to 300 quarters. As before, the errors are somewhat larger in the incomplete markets model with no endogenous discount factor as compared to the stationary incomplete markets model with endogenous discount factor.

The results in Table 4 have two important implications for the applicability and accuracy of our solution method. First, we can simulate very long accurate equilibrium time series from models if we can use the known long-run wealth distribution as a point of approximation in our solution method. Second, our method is capable of generating accurate equilibrium time series over empirically relevant time spans in the neighborhood of an assumed initial wealth distribution. For the model studied here, the results in Panels E and F indicate that the accuracy of the simulated series begins to deteriorate in an economically significant way in spans greater than 300 quarters or 75 years. For this reason all the accuracy statistics reported in Tables 1 - 4 were based on simulations with a span of 300 quarters.

3.3 Comparison with Devereux-Sutherland method

It is informative to show how our method compares to that developed in Devereux and Sutherland (2007) (DS hereafter) and Tille and van Wincoop (2010) in terms of accuracy. For this purpose all six models presented in this paper were re-solved using the third-order approach of Devereux and Sutherland (2007) and Tille and van Wincoop (2010).[16] In this section we present the results. We begin by reporting the portfolio holdings obtained using Devereux and Sutherland (2007) in Table 5.

Table 5: Portfolio Holdings from Devereux-Sutherland method

	A_t^{H}	A_t^{F}	A_t^{N}	B_t	A_t^{H}	A_t^{F}	A_t^{N}	B_t
	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
	A: Complete Markets, log-utility				B: Complete Markets, power-utility			
mean	0.5001	0.4999		0.0000%	0.5002	0.4997		0.0100%
stdev	0.0035	0.0034		0.2300%	0.0035	0.0035		0.2300%
min	0.4848	0.4847		-0.9500%	0.4855	0.4838		-0.8900%
max	0.5138	0.5147		0.9600%	0.5169	0.5150		0.9500%
	C: Incomplete Markets, log-utility, with EDF				D: Incomplete Markets, power-utility, with EDF			
mean	0.5000	0.5000	1.0000	0.0000%	0.4998	0.5001	1.0000	0.0000%
stdev	0.0039	0.0039	0.0000	0.2000%	0.0107	0.0104	0.0000	0.9100%
min	0.4811	0.4838	1.0000	-0.8200%	0.4490	0.4523	1.0000	-4.5000%
max	0.5213	0.5193	1.0000	1.0100%	0.5460	0.5464	1.0000	3.8900%
	E: Incomplete Markets, log-utility, no EDF				F: Incomplete Markets, power-utility, no EDF			
mean	0.5003	0.5000	1.0000	0.0200%	0.5003	0.5000	1.0000	-0.0200%
stdev	0.0040	0.0041	0.0000	0.2800%	0.0132	0.0132	0.0000	1.6800%
min	0.4801	0.4809	1.0000	-1.2100%	0.4307	0.4365	1.0000	-9.140%
max	0.5205	0.5205	1.0000	1.2000%	0.5573	0.5793	1.0000	8.100%

Note: The portfolio holdings in this Table are obtained by applying Devereux-Sutherland solution method. A_t^{H}, A_t^{F}, and A_t^{N} correspond, respectively, to H household's holdings of equity issued by H, F traded firms, and H nontraded firms. B_t refers to H household's bond holdings as a share of H wealth.

Comparing portfolio holdings in Table 5 with the portfolio holdings obtained from our method and reported in Table 1 reveals several important differences. First, recall that Panel A reports portfolio holdings from the model with the known analytical solution for portfolios $B_t = 0$ and $A_t^{\text{H}} = A_t^{\text{F}} = 1/2$. Our solution method delivers exactly that solution (Panel A of Table 1), while DS method produces equity and bond holdings that are correct on average, but are significantly more volatile. In fact, the volatility of equity holdings is infinitely higher in the DS method, while the volatility of bond holdings is 14 times higher in

[16] This section borrows from Kazimov (2010), who also presents detailed accuracy comparisons of the methods.

their method relative to our approach. Second, the higher volatility of portfolio holdings in the DS method extends to all other models considered in this paper and is particularly pronounced for bond holdings.

Next, we compare the accuracy of our method relative to DS using Euler equation errors. For brevity we report a summary measure of the errors which is computed from the portfolio Euler equation. Interested reader is referred to Kazimov (2010) who reports the Euler equation errors from the individual Euler equations and for all six models presented in this paper after solving them with the DS method.

Table 6 presents the mean, max, and the upper percentiles of the distribution of the absolute errors from the portfolio Euler equation obtained from equation (45) with $R^{\varkappa} = R^{\mathrm{w}}_{t+1}$.

Table 6: Portfolio Euler equation errors x10^{-3}

	Complete markets			
	A: Log utility		B: Power utility	
	EH	DS	EH	DS
mean	0.0010	2.1256	0.6239	2.1306
max	0.0010	13.6812	3.3746	11.8019
90th percentile	0.0010	4.3925	1.2864	4.3922
95th percentile	0.0010	5.2219	1.5309	5.2447
99th percentile	0.0010	6.8736	2.0054	6.9020
	Incomplete markets, with EDF			
	C: Log utility		D: Power utility	
	EH	DS	EH	DS
mean	0.4101	1.6629	0.9121	1.2042
max	2.3609	9.5096	5.9565	6.7974
90th percentile	0.8490	3.4324	1.8811	2.4758
95th percentile	1.0182	4.0818	2.2493	2.9481
99th percentile	1.3591	5.3646	2.9557	3.8613
	Incomplete markets, no EDF			
	E: Log utility		F: Power utility	
	EH	DS	EH	DS
mean	0.0349	1.6383	1.0898	1.3176
max	0.6013	9.3046	6.8903	7.1804
90th percentile	0.0881	3.3695	2.2445	2.7175
95th percentile	0.1220	4.0138	2.6816	3.2213
99th percentile	0.2070	5.3348	3.5172	4.2534

Note: This Table reports the errors from portfolio Euler equation in our method (denoted by EH) and in the Devereux-Sutherland method (denoted by DS). All errors are of the order 10^{-3}.

Notice that the portfolio Euler equation errors obtained using our solution method are comparable with the errors in the individual Euler equations, and on many occasions are even smaller. But more importantly, portfolio Euler equation errors obtained using our method are singificantly smaller than the portfolio Euler equation errors obtained using DS solution method. This conclusion applies to all six models considered in the paper. These comparisons highlight the accuracy of our method relative to the existing approaches.

4 Conclusion

We have presented a numerical method for solving general equilibrium models with many financial assets, heterogeneous agents and incomplete markets. Our method builds on the log-approximations of Campbell, Chan, and Viceira (2003) and the second-order perturbation technique developed by Judd (1998) and others. To illustrate its use, we applied our solution method to complete and incomplete markets versions of a two-country general equilibrium model with production, and to the versions of the model with log- and power-utility. The numerical solution to the complete markets version closely conforms to the predictions of theory and is highly accurate based on a number of standard tests. This gives us confidence in the accuracy of our technique. The power of our method is illustrated by solving the incomplete markets version of the model. The array of assets in this model is insufficient to permit complete risk-sharing among households, so the equilibrium allocations cannot be found by standard analytical techniques. Our accuracy tests show that simulations of our solution to this version of the model are very accurate over spans of 75 years of quarterly data. Our solution method can be applied to more richly specified models than the one examined here. For example, the method can be applied to solve models with more complex preferences, capital adjustment costs, or portfolio constraints. As a result, we believe that our method will be useful in the future analysis of many models in international macroeconomics and finance.

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A Appendix:

A.1 Model equations and the approximation point

The system of equations characterizing the equilibrium of our model consists of

1. Process for productivity

$$z_t = az_{t-1} + S_e^{1/2} e_t$$

2. H and F budget constraints

$$\begin{aligned} W_{t+1} &= R^{\mathrm{W}}_{t+1}\left(W_t - C^{\mathrm{T}}_t - Q^{\mathrm{N}}_t C^{\mathrm{N}}_t\right) \\ R^{\mathrm{W}}_{t+1} &= R_t + \alpha^{\mathrm{H}}_t (R^{\mathrm{H}}_{t+1} - R_t) + \alpha^{\mathrm{F}}_t (R^{\mathrm{F}}_{t+1} - R_t) + \alpha^{\mathrm{N}}_t (R^{\mathrm{N}}_{t+1} - R_t) \end{aligned}$$

and

$$\begin{aligned} \hat{W}_{t+1} &= \hat{R}^{\mathrm{W}}_{t+1}(\hat{W}_t - \hat{C}^{\mathrm{T}}_t - \hat{Q}^{\mathrm{N}}_t \hat{C}^{\mathrm{N}}_t) \\ \hat{R}^{\mathrm{W}}_{t+1} &= R_t + \hat{\alpha}^{\mathrm{H}}_t (R^{\mathrm{H}}_{t+1} - R_t) + \hat{\alpha}^{\mathrm{F}}_t (R^{\mathrm{F}}_{t+1} - R_t) + \hat{\alpha}^{\mathrm{N}}_t (\hat{R}^{\mathrm{N}}_{t+1} - R_t). \end{aligned}$$

3. H and F bond and equity Euler equations

$$\begin{aligned} 1 = \mathbb{E}_t\left[M_{t+1}R_t\right], &\qquad 1 = \mathbb{E}_t\left[\hat{M}_{t+1}R_t\right], \\ 1 = \mathbb{E}_t\left[M_{t+1}R^{\mathrm{H}}_{t+1}\right], &\qquad 1 = \mathbb{E}_t\left[\hat{M}_{t+1}R^{\mathrm{H}}_{t+1}\right], \\ 1 = \mathbb{E}_t\left[M_{t+1}R^{\mathrm{F}}_{t+1}\right], &\qquad 1 = \mathbb{E}_t\left[\hat{M}_{t+1}R^{\mathrm{F}}_{t+1}\right], \\ 1 = \mathbb{E}_t\left[M_{t+1}R^{\mathrm{N}}_{t+1}\right], &\qquad 1 = \mathbb{E}_t\left[\hat{M}_{t+1}\hat{R}^{\mathrm{N}}_{t+1}\right], \end{aligned}$$

where

$$M_{t+1} = \beta\left(C^{\mathrm{T}}_{t+1}/C^{\mathrm{T}}_t\right)^{\frac{(1-\sigma)(\phi-1)}{\phi}}\left((C^{\mathrm{T}}_{t+1} + Q^{\mathrm{N}}_{t+1}C^{\mathrm{N}}_{t+1})/(C^{\mathrm{T}}_t + Q^{\mathrm{N}}_t C^{\mathrm{N}}_t)\right)^{\frac{1-\sigma-\phi}{\phi}}$$

and

$$\hat{M}_{t+1} = \beta\left(\hat{C}^{\mathrm{T}}_{t+1}/\hat{C}^{\mathrm{T}}_t\right)^{\frac{(1-\sigma)(\phi-1)}{\phi}}\left((\hat{C}^{\mathrm{T}}_{t+1} + \hat{Q}^{\mathrm{N}}_{t+1}\hat{C}^{\mathrm{N}}_{t+1})/(\hat{C}^{\mathrm{T}}_t + \hat{Q}^{\mathrm{N}}_t \hat{C}^{\mathrm{N}}_t)\right)^{\frac{1-\sigma-\phi}{\phi}}.$$

Under log-utility M_{t+1} simplifies to $\beta\left(C^{\mathrm{T}}_t + Q^{\mathrm{N}}_t C^{\mathrm{N}}_t\right)/\left(C^{\mathrm{T}}_{t+1} + Q^{\mathrm{N}}_{t+1}C^{\mathrm{N}}_{t+1}\right) = \beta W_t/W_{t+1}$ and $\hat{M}_{t+1} = \beta\hat{W}_t/\hat{W}_{t+1}$.

4. H and F optimality conditions determining relative goods prices

$$Q^{\mathrm{N}}_t = \left(\frac{\mu_{\mathrm{N}}}{\mu_{\mathrm{T}}}\right)^{1-\phi}\left(\frac{C^{\mathrm{N}}_t}{C^{\mathrm{T}}_t}\right)^{\phi-1}, \qquad \text{and} \qquad \hat{Q}^{\mathrm{N}}_t = \left(\frac{\mu_{\mathrm{N}}}{\mu_{\mathrm{T}}}\right)^{1-\phi}\left(\frac{\hat{C}^{\mathrm{N}}_t}{\hat{C}^{\mathrm{T}}_t}\right)^{\phi-1}.$$

5. Capital Euler equation at H and F

$$
\begin{aligned}
1 &= \mathbb{E}_t\left[M_{t+1}R^{\text{K}}_{t+1}\right], \qquad \text{with } R^{\text{K}}_{t+1} \equiv \theta Z^{\text{T}}_{t+1}\left(K_{t+1}\right)^{\theta-1} + (1-\delta) \\
1 &= \mathbb{E}_t\left[\hat{M}_{t+1}\hat{R}^{\text{K}}_{t+1}\right], \qquad \text{with } \hat{R}^{\text{K}}_{t+1} \equiv \theta \hat{Z}^{\text{T}}_{t+1}(\hat{K}_{t+1})^{\theta-1} + (1-\delta)
\end{aligned}
$$

6. From the portfolio Euler equation in each country, $1 = \mathbb{E}_t[M_{t+1}R^{\text{W}}_{t+1}]$ and $1 = \mathbb{E}_t[\hat{M}_{t+1}\hat{R}^{\text{W}}_{t+1}]$, after substituting for M_{t+1} and $\hat{M}_{t+1}$ we get the expressions for consumption-wealth ratios

$$
\begin{aligned}
\Lambda_t^{\text{T}\,\frac{(1-\sigma)(\phi-1)}{\phi}}\Lambda_t^{\frac{1-\sigma-\phi}{\phi}}(1-\Lambda_t)^{\sigma} &= \beta\mathbb{E}_t\left[\Lambda_{t+1}^{\text{T}\,\frac{(1-\sigma)(\phi-1)}{\phi}}\Lambda_{t+1}^{\frac{1-\sigma-\phi}{\phi}}\left(R^{\text{W}}_{t+1}\right)^{1-\sigma}\right] \\
\hat{\Lambda}_t^{\text{T}\,\frac{(1-\sigma)(\phi-1)}{\phi}}\hat{\Lambda}_t^{\frac{1-\sigma-\phi}{\phi}}\left(1-\hat{\Lambda}_t\right)^{\sigma} &= \beta\mathbb{E}_t\left[\hat{\Lambda}_{t+1}^{\text{T}\,\frac{(1-\sigma)(\phi-1)}{\phi}}\hat{\Lambda}_{t+1}^{\frac{1-\sigma-\phi}{\phi}}\left(\hat{R}^{\text{W}}_{t+1}\right)^{1-\sigma}\right].
\end{aligned}
$$

7. Market clearing conditions

   (a) traded goods

$$C^{\text{T}}_t + \hat{C}^{\text{T}}_t = D^{\text{T}}_t + \hat{D}^{\text{T}}_t$$

   (b) nontraded goods

$$C^{\text{N}}_t = Y^{\text{N}}_t = D^{\text{N}}_t \qquad \text{and} \qquad \hat{C}^{\text{N}}_t = \hat{Y}^{\text{N}}_t = \hat{D}^{\text{N}}_t.$$

   (c) bond

$$0 = B_t + \hat{B}_t.$$

   (d) traded equity

$$1 = A^{\text{H}}_t + \hat{A}^{\text{H}}_t \qquad \text{and} \qquad 1 = A^{\text{F}}_t + \hat{A}^{\text{F}}_t,$$

   which can equivalently be written as

$$
\begin{aligned}
P^{\text{T}}_t &= \alpha^{\text{H}}_t\left(W_t - C_t\right) + \hat{\alpha}^{\text{H}}_t\left(\hat{W}_t - \hat{C}_t\right) \\
\hat{P}^{\text{T}}_t &= \alpha^{\text{F}}_t\left(W_t - C_t\right) + \hat{\alpha}^{\text{F}}_t\left(\hat{W}_t - \hat{C}_t\right)
\end{aligned}
$$

   (e) nontraded equity

$$1 = A^{\text{N}}_t \qquad 1 = \hat{A}^{\text{N}}_t,$$

   which is equivalent to

$$\alpha^{\text{N}}_t = Q^{\text{N}}_t P^{\text{N}}_t / \left(W_t - C_t\right) \qquad \hat{\alpha}^{\text{N}}_t = \hat{Q}^{\text{N}}_t \hat{P}^{\text{N}}_t / \left(\hat{W}_t - \hat{C}_t\right).$$

The approximation point is given by the steady state levels of the following variables: $R = R^{\text{K}} = \hat{R}^{\text{K}} = R^{\text{H}} = R^{\text{F}} = R^{\text{N}} = \hat{R}^{\text{N}} = R^{\text{W}} = \hat{R}^{\text{W}} = \frac{1}{\beta}$. $K = \hat{K} = (\beta\theta)^{1/(1-\theta)}(1-\beta+\beta\delta)^{1/(\theta-1)}$, $D^{\text{T}} = \hat{D}^{\text{T}} = K^{\theta} - \delta K$, $P^{\text{T}} = \hat{P}^{\text{T}} = \beta D^{\text{T}}/(1-\beta)$. $D^{\text{N}} = \hat{D}^{\text{N}} = \eta$, so that $C^{\text{N}} = \hat{C}^{\text{N}} = \eta$ and $P^{\text{N}} = \hat{P}^{\text{N}} = \beta\eta/(1-\beta)$. As we discussed in Section 4.2, wealth at H and F is approximated around an initial level, W_0 and $\hat{W}_0$. When $W_0 = \hat{W}_0$, then $C_0^{\mathrm{T}} = \hat{C}_0^{\mathrm{T}} = D^{\mathrm{T}}$. Then portfolios are approximated around $\alpha^{\mathrm{N}} = \hat{\alpha}^{\mathrm{N}} = \mu_{\mathrm{N}}^{1-\phi}\left(C_0^{\mathrm{N}}/C_0\right)^{\phi}$ and $\alpha^{\mathrm{H}} = \alpha^{\mathrm{F}} = \mu_{\mathrm{T}}^{1-\phi}\left(C_0^{\mathrm{T}}/C_0\right)^{\phi}$, where α^{H} and α^{F} denote the initial values of $(\alpha_t^{\mathrm{H}} + \hat{\alpha}_t^{\mathrm{H}})$ and $(\hat{\alpha}_t^{\mathrm{F}} + \alpha_t^{\mathrm{F}})$, respectively, as before.

A.2 Derivation of Equation (41)

We start with quadratic and cross-product terms, $\tilde{x}_t$ and approximate their laws of motion using Ito's lemma. In continuous time, the discrete process for x_{t+1} in (38) becomes

$$dx_t = \left[\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t\right] dt + \Omega(\tilde{x}_t)^{1/2} dW_t$$

Then by Ito's lemma:

$$\begin{aligned} dvec(x_t x_t') &= \left[(I \otimes x_t) + (x_t \otimes I)\right]\left(\left[\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t\right] dt + \Omega(\tilde{x}_t)^{1/2} dW_t\right) \\ &\quad + \tfrac{1}{2}\left[(I \otimes U)\left(\frac{\partial x}{\partial x'} \otimes I\right) + \left(\frac{\partial x}{\partial x'} \otimes I\right)\right] d\left[x, x\right]_t \\ &= \left[(I \otimes x_t) + (x_t \otimes I)\right]\left(\left[\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t\right] dt + \Omega(\tilde{x}_t)^{1/2} dW_t\right) \\ &\quad + \tfrac{1}{2}\left[\mathbb{U}\left(\frac{\partial x}{\partial x'} \otimes I\right) + \left(\frac{\partial x}{\partial x'} \otimes I\right)\right] vec\left\{\Omega(\tilde{x}_t)\right\} dt \\ &= \left[(I \otimes x_t) + (x_t \otimes I)\right]\left(\left[\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t\right] dt + \Omega(\tilde{x}_t)^{1/2} dW_t\right) + \tfrac{1}{2} D vec\left\{\Omega(\tilde{x}_t)\right\} dt, \end{aligned} \quad \text{(A1)}$$

where $$D = \left[\mathbb{U}\left(\frac{\partial x}{\partial x'} \otimes I\right) + \left(\frac{\partial x}{\partial x'} \otimes I\right)\right], \qquad \mathbb{U} = \sum_r \sum_s E_{rs} \otimes E'_{r,s},$$

and $E_{r,s}$ is the elementary matrix which has a unity at the $(r,s)^{th}$ position and zero elsewhere. The law of motion for the quadratic states in (A1) can be rewritten in discrete time as

$$\begin{aligned} \tilde{x}_{t+1} &\cong \tilde{x}_t + \left[(I \otimes x_t) + (x_t \otimes I)\right]\left[\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t\right] + \tfrac{1}{2} D vec\left(\Omega(\tilde{x}_t)\right) \\ &\quad + \left[(I \otimes x_t) + (x_t \otimes I)\right]\varepsilon_{t+1}, \\ &\cong \tfrac{1}{2} D\Sigma_0 + \left[(\Phi_0 \otimes I) + (I \otimes \Phi_0)\right] x_t + \left[I - (\Phi_1 \otimes I) - (I \otimes \Phi_1) + \tfrac{1}{2} D\Sigma_1\right]\tilde{x}_t + \tilde{\varepsilon}_{t+1}, \end{aligned}$$

where $\tilde{\varepsilon}_{t+1} \equiv \left[(I \otimes x_t) + (x_t \otimes I)\right]\varepsilon_{t+1}$. The last equality is obtained by using an expression for $vec\left(\Omega(X_t)\right)$ in (40), where $\Sigma_0 = vec(\Omega_0)$ and $\Sigma_1 = \Omega_1 \otimes \Omega_1$, and by combining together the corresponding coefficients on a constant, linear and second-order terms.

A.3 Derivation of Equation (43)

Recall that $U_{t+1} = [\begin{array}{ccc} 0 & \varepsilon_{t+1} & \tilde{\varepsilon}_{t+1} \end{array}]'$, so $\mathbb{E}\left(U_{t+1}|X_t\right) = 0$ and

$$\mathbb{E}\left(U_{t+1}U'_{t+1}|X_t\right) \equiv \mathcal{S}\left(X_t\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega\left(X_t\right) & \Gamma\left(X_t\right) \\ 0 & \Gamma\left(X_t\right)' & \Psi(X_t) \end{pmatrix}$$

To evaluate the covariance matrix, we assume that $vec(x_{t+1}\tilde{x}'_{t+1}) \cong 0$ and define:

$$\begin{aligned}
\Gamma\left(X_t\right) &\equiv \mathbb{E}_t\varepsilon_{t+1}\tilde{\varepsilon}'_{t+1}, \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - \mathbb{E}_t x_{t+1}\mathbb{E}_t\tilde{x}'_{t+1}, \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - (\Phi_0 + (I-\Phi_1)x_t + \Phi_2\tilde{x}_t) \\
&\quad \times \left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]'\right), \\
&\cong -\Phi_0\left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]'\right) \\
&\quad -(I-\Phi_1)x_t\left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]'\right) - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0 D', \\
&= -\tfrac{1}{2}\Phi_0\Sigma'_0 D' - \Phi_0 x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' - \tfrac{1}{2}(I-\Phi_1)x_t\Sigma'_0 D' \\
&\quad -\Phi_0\tilde{x}'_t\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]' - (I-\Phi_1)x_t x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0 D'.
\end{aligned}$$

Hence

$$\begin{aligned}
vec\left(\Gamma\left(X_t\right)\right) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2\tilde{x}_t, \\
\Gamma_0 &= -\frac{1}{2}\left(D\Sigma_0\otimes\Phi_0\right)vec(I), \\
\Gamma_1 &= -\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\otimes\Phi_0 + \tfrac{1}{2}\left(D\Sigma_0\otimes(I-\Phi_1)\right), \\
\Gamma_2 &= -\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\otimes\Phi_0 - \tfrac{1}{2}\left(D\Sigma_0\otimes\Phi_2\right) \\
&\quad -\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\otimes(I-\Phi_1).
\end{aligned}$$

Note also from above that

$$\begin{aligned}
\Gamma\left(X_t\right)' &= -\tfrac{1}{2}D\Sigma_0\Phi'_0 - \left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]x_t\Phi'_0 - \Sigma_0 x'_t(I-\Phi_1)' \\
&\quad -\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\tilde{x}_t\Phi'_0 - \left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]x_t x'_t(I-\Phi_1)' - \tfrac{1}{2}D\Sigma_0\tilde{x}'_t\Phi'_2.
\end{aligned}$$

So

$$\begin{aligned}
vec\left(\Gamma\left(X_t\right)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2\tilde{x}_t, \\
\Lambda_0 &= -\tfrac{1}{2}\left(\Phi_0\otimes D\Sigma_0\right)vec(I), \\
\Lambda_1 &= -\left(\Phi_0\otimes\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\right) + \tfrac{1}{2}\left((I-\Phi_1)\otimes D\Sigma_0\right), \\
\Lambda_2 &= -\left(\Phi_0\otimes\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\right) - \tfrac{1}{2}\left(\Phi_2\otimes D\Sigma_0\right) \\
&\quad -\left((I-\Phi_1)\otimes\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\right).
\end{aligned}$$

Next, consider the variance of $\tilde{\varepsilon}_{t+1}$:

$$
\begin{aligned}
\Psi\left(X_t\right) &\equiv \mathbb{E}_t\tilde{\varepsilon}_{t+1}\tilde{\varepsilon}_{t+1}' = \mathbb{E}_t\tilde{x}_{t+1}\tilde{x}_{t+1}' - \mathbb{E}_t\tilde{x}_{t+1}\mathbb{E}_t\tilde{x}_{t+1}', \\
&= \mathbb{E}_t\tilde{x}_{t+1}\tilde{x}_{t+1}' - \left(\tfrac{1}{2}D\Sigma_0 + \left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t + \left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\tilde{x}_t\right) \\
&\quad \times\left(\tfrac{1}{2}\Sigma_0'D' + x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' + \tilde{x}_t'\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]'\right), \\
&\cong -\tfrac{1}{2}D\Sigma_0\left(\tfrac{1}{2}\Sigma_0'D' + x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' + \tilde{x}_t'\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]'\right) \\
&\quad -\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t\left(\tfrac{1}{2}\Sigma_0'D' + x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]'\right) \\
&\quad -\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}\mathbb{D}\Sigma_1\right]\tilde{x}_t\tfrac{1}{2}\Sigma_0'D', \\
&= -\tfrac{1}{4}D\Sigma_0\Sigma_0'D' - \tfrac{1}{2}D\Sigma_0x_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' - \tfrac{1}{2}\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_t\Sigma_0'D' \\
&\quad -\tfrac{1}{2}D\Sigma_0\tilde{x}_t'\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]' - \left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]x_tx_t'\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]' \\
&\quad -\tfrac{1}{2}\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\tilde{x}_t\Sigma_0'D'.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
vec\left(\Psi\left(X_t\right)\right) &= \Psi_0 + \Psi_1x_t + \Psi_2\tilde{x}_t, \\
\Psi_0 &= -\tfrac{1}{4}\left(D\Sigma_0\otimes D\Sigma_0\right)vec(I), \\
\Psi_1 &= -\tfrac{1}{2}\left(\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\otimes D\Sigma_0\right) - \tfrac{1}{2}\left(D\Sigma_0\otimes\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\right), \\
\Psi_2 &= -\tfrac{1}{2}\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\otimes D\Sigma_0 - \tfrac{1}{2}\left(D\Sigma_0\otimes\left[I-((\Phi_1\otimes I)+(I\otimes\Phi_1))+\tfrac{1}{2}D\Sigma_1\right]\right) \\
&\quad -\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right]\otimes\left[(\Phi_0\otimes I)+(I\otimes\Phi_0)\right].
\end{aligned}
$$