

Speculation and Financial Wealth Distribution under Belief Heterogeneity^{*†}

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February 19, 2014

Abstract

Under limited commitment that prevents agents from pledging their future non-financial wealth, agents with incorrect beliefs always survive by holding on their non-financial wealth. Friedman (1953)'s market selection hypothesis suggests that their financial wealth trends towards zero in the long run. However, in this paper, I construct an example in dynamic general equilibrium in which over-optimistic agents not only survive but also prosper by holding an increasingly larger share of a real asset and driving up the price of the asset: they *prosper by speculation*. The endogenous price dynamics implied by different beliefs is an essential ingredient to the mechanism.

*I am grateful to Daron Acemoglu and Ivan Werning for their infinite support and guidance during my time at MIT. I wish to thank Guido Lorenzoni and Robert Townsend for their advice since the beginning of this project, Ricardo Caballero and the MIT Macro seminar, Macro Lunch, Theory Lunch, International breakfast participants for helpful comments and discussions. I also thank Markus Brunnermeier, Jinhui Bai, Tim Cogley, Behzad Diba, John Geanakoplos, Mark Huggett, Felix Kubler, Dirk Krueger, Per Krusell, Viktor Tsyrennikov, and other participants at conferences and seminars at UCLA, Georgetown, Princeton, UCL, LSE, University of Wisconsin-Madison, Yale, Cowles Foundation, SED meeting in Montreal, Stanford Institute for Theoretical Economics, NYU-Columbia NBER mathematical economics meeting for comments and discussions on the later versions of the paper.

†The previous version of this paper was circulated as Cao (2010) - "Collateral shortages, asset price, and investment volatility with heterogeneous beliefs" - which can be found at <http://www9.georgetown.edu/faculty/dc448/papers.html>

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1 Introduction

The events leading to the financial crisis of 2007-2008 have highlighted the importance of belief heterogeneity and how financial markets create opportunities for agents with different beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds profited from the securities by short-selling them.

One reason why economic theory has paid relatively little attention to the heterogeneity of beliefs and how it interacts with financial markets is the *market selection hypothesis*. The hypothesis, originally formulated by Friedman (1953), claims that in the long run, there should be limited differences in beliefs because agents with incorrect beliefs will be taken advantage of and eventually driven out of the markets by those with the correct beliefs. Therefore, agents with incorrect beliefs will have no influence on economic activity in the long run. This hypothesis has been formalized and extended in recent work by Sandroni (2000) and Blume and Easley (2006). However these papers assume that financial markets are complete, an assumption that plays a central role in allowing agents to pledge all their wealth, including financial and non-financial wealth.¹

In this paper, I present a dynamic general equilibrium framework in which agents differ in their beliefs but markets are endogenously incomplete because of collateral constraints. Collateral constraints limit the extent to which agents can pledge their future non-financial wealth and ensure that agents with incorrect beliefs never lose so much as to be driven out of the market. Consequently, all agents, regardless of their beliefs, survive in the long run and continue to trade on the basis of their heterogeneous beliefs.

In this environment, it is natural to ask the question what happens to the financial wealth of the agents with incorrect beliefs. The market selection hypothesis suggests that these agents will lose most of their financial wealth in the long run, leaving them only their non-financial wealth. The answer to the question is not simple. The long run distribution of financial wealth between agents depends on the exact structure of incomplete financial markets. For example, when agents are allowed to trade in only one real asset, over-optimistic agents (agents with incorrect beliefs) prosper by holding an increasingly larger share of the real asset and by driving up the price of the asset: they *prosper by speculation*. In the same example, when these agents can use the real asset as collateral to borrow, they end up with

¹In this paper, it is important to differentiate financial wealth, i.e., the value of financial asset holdings, from non-financial wealth, i.e., non-financial endowment such as wages. The literature on survival focuses on total wealth distribution, i.e., the total financial and non-financial wealth, while the focus of this paper is on financial wealth distribution. See Lustig and Van Nieuwerburgh (2005) for a similar distinction between housing wealth and human wealth.

low financial wealth, that is, their share in the real asset net of their borrowing, in the long run. The simultaneous determination in equilibrium of financial wealth distribution and asset price dynamics is essential for these results and sets the current paper apart from other papers in the survival literature.

The study of financial wealth distribution thus requires the solution of the endogenous asset price dynamics. The infinite-horizon, dynamic general equilibrium approach adopted in this paper provides a transparent mapping between the financial wealth distribution and economic variables such as asset prices and portfolio choices. It also allows for a characterization of the effects of financial regulation on asset price volatility.²

More specifically, I study an economy in dynamic general equilibrium with both aggregate shocks and idiosyncratic shocks and heterogeneous, infinitely-lived agents. The shocks follow a Markov process. Consumers differ in their beliefs on the transition matrix of the Markov process. For simplicity, these belief differences are never updated because there is no learning; in other words agents in this economy agree to disagree.³ There is a unique final consumption good, one real asset and potentially one bond. The real asset, modelled as a *Lucas tree* as in Lucas (1978), is in fixed supply. I assume that agents cannot short sell the real asset. Endogenously incomplete (financial) markets are introduced by assuming that borrowing, i.e. selling bond, has to be collateralized by the real asset. I refer to equilibria of the economy with these assets as *collateral constrained equilibria*^{4,5}.

Households (consumers) can differ in many aspects, such as risk-aversion and endowments. Most importantly, they differ in their beliefs concerning the transition matrix governing the transitions across the exogenous states of the economy. Given the consumers' subjective expectations, they choose their consumption and real asset and bond holdings to

²In the earlier version of this paper - Cao (2010) - I show that the dynamic stochastic general equilibrium model with endogenously incomplete markets presented here also includes well-known models as special cases, including recent models, such as those in Fostel and Geanakoplos (2008) and Geanakoplos (2009), as well as more classic models including those in Kiyotaki and Moore (1997) and Krusell and Smith (1998). For instance, a direct generalization of the current model allows for capital accumulation with adjustment costs in the same model in Krusell and Smith (1998) and shows the existence of a recursive equilibrium. The generality is useful in making this framework eventually applicable to a range of questions on the interaction between financial markets, heterogeneity, aggregate capital accumulation and aggregate activity.

³Alternatively, one could assume that even though agents differ with respect to their initial beliefs, they partially update them. In this case, similar results would apply provided that the learning process is sufficiently slow (as will be the case when individuals start with relatively firm priors).

⁴In Appendix D, I show that, under some restriction, this setup is equivalent to the one in which there are many bonds with different levels of collateral as in Geanakoplos and Zame (2002).

⁵The liquidity constrained equilibrium in Kehoe and Levine (2001) corresponds to a special case of collateral constrained equilibrium when the margin on collateralized borrowing is set to 1. The numerical solution in this paper completely characterizes the equilibrium which Kehoe and Levine (2001) conjecture that the dynamics can be very complicated. In the paper, the authors also show that the dynamics in liquidity constrained equilibrium is more complicated than the dynamics in debt constrained equilibrium.

maximize their intertemporal expected utility.

Before analyzing the dynamic of financial wealth distribution, I show that in any collateral constrained equilibrium, every agent survives in the standard definition a la Blume and Easley (2006) and Sandroni (2000) because of the constraints. When agents lose their bets, they can simply walk away from their debt at the only cost of losing collateral and keep their current and future (non-financial) endowments. They can return and trade again in the financial markets in the same period.⁶ They cannot walk away from their debt under complete markets because they can commit to delivering all their future endowments. Using this simple observation, I establish the existence of collateral constrained equilibria with a stationary structure - *Markov equilibria* - in which equilibrium prices and quantities depend only on the distribution of *normalized financial wealth distribution*. In addition, I also develop an algorithm to compute these equilibria.

The numerical solutions of these equilibria allows me to answer the questions on the dynamics of financial wealth distribution, as well as asset prices. I use the algorithm to solve for collateral constrained equilibria and present these dynamics in a reasonably calibrated example. In this example, I assume that there are two types of agents: pessimists who have correct beliefs and optimists who are over-optimistic about the return of the real asset.

First of all, to answer the question asked at the beginning of the introduction on the financial wealth of agents with incorrect beliefs, I start the numerical analysis by studying a collateral constrained economy in which agents are allowed to trade only in the real asset subject to the no-short-selling constraint. This economy corresponds to the liquidity constrained economy in Kehoe and Levine (2001), but allows for heterogeneous beliefs.⁷ In this economy, not only do agents with incorrect beliefs (the optimists) survive but also prosper by postponing consumption to invest in the real asset. The speculative activities of the optimists - combined with their increasing financial wealth - constantly pushes up the price of the real asset which is essential for their increasing financial wealth. I call this phenomenon *prosper by speculation*.^{8,9} However when I allow the agents to borrow using the real asset as collateral. In this case, the optimists loose their financial wealth and end up with low financial wealth in the long run. Thus the structure of the financial markets really matters for the (long run) financial wealth of agents with incorrect beliefs through the price of the real asset.

⁶The collateral constraints are a special case of limited commitment because there will be no need for collateral if agents can fully commit to their promises.

⁷This is also Harrison and Kreps (1978) with risk-aversion.

⁸Appendix B shows that the existence of non-financial wealth of the optimists is crucial for this result.

⁹Interestingly, the increasing price dynamics is such that the pessimists do not always want to short-sell the asset. They start trying to short-sell the asset only when the price of the real asset is too high, at which point their short-selling constraint is strictly binding.

One implication of this result is that - also in the numerical example above - simple and extreme forms of financial regulations such as shutting down collateralized borrowing or uniformly restricting leverage surprisingly increase the long run financial wealth of the optimists and long run asset price volatility. The intuition for greater volatility under such regulations is similar to the intuition for why long run asset price volatility is higher under collateral constrained economies than under complete markets economies. Financial regulations act as further constraints protecting the agents with incorrect beliefs. Thus, in the long run these agents hold most of the assets that they believe, incorrectly, to have high rates of return. The shocks to the rates of return on these assets then create large movements in the marginal utilities of the agents and, hence, generate large volatility of the prices of the assets.

The endogenous price of the real asset which plays an important role in the distribution of financial wealth between agents with different beliefs also exhibits interesting dynamics. For example, in the last economy with collateralized borrowing, the dynamic general equilibrium captures the "*debt-deflation*" channel as in Mendoza (2010), which models a small open economy. The economy in this example also follows two different dynamics in different times, "*normal business cycles*" and "*debt-deflation cycles*," depending on whether the collateral constraints are binding for any of the agents. In a debt-deflation cycle, some collateral constraint binds. When a bad shock hits the economy, the constrained agents are forced to liquidate their real asset holdings. This *fire sale* of the real asset reduces the price of the asset and tightens the constraints further, starting a vicious circle of falling asset prices. This example shows that the debt-deflation channel still operates when we are in a closed-economy with an endogenous interest rate, as opposed to exogenous interest rates as in Mendoza (2010). Moreover, due to this mechanism, asset price volatility also tends to be higher at low levels of asset prices near the debt-deflation region. This pattern has been documented in several empirical studies, including Heathcote and Perri (2011). Similar nonlinear dynamics are also emphasized in a recent paper by Brunnermeier and Sannikov (2013).

The rest of the paper proceeds as follows. The next section reviews the related literature. In Section 3, I present the general model of an endowment economy and an analysis of the survival of agents with incorrect beliefs and asset price volatility under collateral constraints. In this section, I also define collateral constrained equilibrium as well as a recursive form of the equilibrium called Markov equilibrium. Section 4 focuses on a numerical example with two agents to present the equilibrium dynamics of financial wealth distribution and asset prices. Section 5 concludes with potential applications of the framework in this paper. Appendix A shows the existence of Markov equilibrium and derives a numerical algorithm to compute the equilibrium. Appendix B shows the importance of non-financial wealth for the

prosper by speculation mechanism and Appendix C shows the robustness of the numerical results under a different set of parameters. Finally, Appendix D shows the equivalence between collateral constrained equilibrium in this paper and the collateral equilibrium in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003).

2 Related literature

This paper is related to the growing literature studying collateral constraints in dynamic general equilibrium, started by early papers including Kiyotaki and Moore (1997) and Geanakoplos and Zame (2002).¹⁰ The dynamic analysis of collateral constrained equilibria is related to Kubler and Schmedders (2003). They pioneer the introduction of financial markets with collateral constraints into a dynamic general equilibrium model with aggregate shocks and heterogeneous agents. The technical contribution of this paper relative to Kubler and Schmedders (2003) is to introduce heterogeneous beliefs using the rational expectations equilibrium concept in Radner (1972): even though agents assign different probabilities to both aggregate and idiosyncratic shocks, they agree on the equilibrium outcomes, including prices and quantities, once a shock is realized. This rational expectations concept differs from the standard rational expectation concept, such as the one used in Lucas and Prescott (1971), in which subjective probabilities should coincide with the true conditional probabilities given all available information.

Related to the literature on the survival of agents with incorrect beliefs such as Blume and Easley (2006) and Sandroni (2000) under complete markets, and Coury and Sciubba (2005), Beker and Chattopadhyay (2009), and Cogley, Sargent, and Tsyrennikov (2011) under incomplete markets, my paper focuses on the dynamics of the financial wealth distribution among agents - taking non-financial wealth as exogenously given - while the existing literature investigates total wealth.

While the focus of my paper is on financial wealth, the survival (in terms of consumption) result is easily obtained because of the collateral constraints. As mentioned in footnote 6, collateral constraints are a special case of limited commitment. However, this special case of limited commitment is different from an alternative limited commitment in the literature in which agents are assumed to be banned from trading in financial markets after their defaults such as in Kehoe and Levine (1993) and Alvarez and Jermann (2001). In this paper, agents can always return to the financial markets and trade using their non-financial endowment after defaulting and losing all their financial wealth. Given this outside option, the financial constraints are more stringent than they are in other papers. Beker and Espino (2010) and

¹⁰In Appendix D I show a mapping between the equilibria in the two papers.

Tsyrennikov (2012) have a similar survival mechanism based on the limited commitment framework in Alvarez and Jermann (2001).¹¹ Kogan et al. (2006) and Borovicka (2010) explore yet another survival mechanism based on the preferences of agents but use complete markets. The survival of irrational traders is also studied in De Long et al. (1990) and De Long et al. (1991) but they do not have a fully dynamic framework to study the long run survival of the traders.

In a recent paper, Cogley, Sargent, and Tsyrennikov (2011) study an incomplete markets economy with belief heterogeneity in which agents can only trade in state-incontingent bonds (but they can pledge their future non-financial wealth). They show numerically that, due to precautionary saving, agents with incorrect beliefs not only survive but also prosper and drive agents with correct beliefs out of the market in the long run. They call this phenomenon "survival by precautionary saving."¹² The prosper by speculation mechanism mentioned in my introduction offers another way in which agents with incorrect beliefs can survive (and prosper). Because the real asset is long-lived in my paper, its price dynamics is important for my mechanism, in contrast to Cogley, Sargent, and Tsyrennikov (2011) in which financial assets are short-lived (either bonds or Arrow securities). Indeed, I show that differences in beliefs alone do not suffice for prospering in my paper, we also need the existence of non-financial wealth of the agents.

My paper is also related to the literature on the effect of heterogeneous beliefs on asset prices studied in Xiong and Yan (2009) and Cogley and Sargent (2008). These authors, however, consider only complete markets. Also assuming complete markets, Kubler and Schmedders (2011) show the importance of beliefs heterogeneity and wealth distribution on asset prices in a model with overlapping-generations. Simsek (2012) studies the effects of belief heterogeneity on asset prices, but in a static setting. He assumes exogenous wealth distributions to investigate whether heterogeneous beliefs affect asset prices. In contrast, I study the effects of the endogenous wealth distribution on asset prices.

In the classic Harrison and Kreps (1978) paper, the authors show that beliefs heterogeneity can lead to asset price bubbles, but they assume linear utility function. My paper includes this set-up as a special case and allows for risk-aversion. I show that only when asset prices deviate too far away from their fundamental values, rational agents try to short-sell the real asset but are constrained by the short-selling constraint.

The channel through which asset prices deviate from their fundamental values is different from the limited arbitrage mechanism in Shleifer and Vishny (1997). In their paper, the

¹¹Beker and Espino (2013) apply the framework to U.S. data to explain the magnitude of short-term momentum and long-term reversal in the excess returns of U.S. stock market.

¹²Blume and Easley (2006) - Section 5 - offers a similar example with incomplete markets and short-lived assets in which agents with incorrect beliefs dominates by saving more than agents with correct beliefs do.

deviation arises because agents with correct beliefs hit their financial constraints before being able to arbitrage away the price anomalies. In this paper, agents with incorrect beliefs hit their financial constraint more often and are protected by the constraint. Moreover, in the equilibria computed in Section 4, agents with the correct belief (the pessimists) often do not hit their borrowing constraint.

On the normative question of how financial regulation affects asset price volatility, in a recent paper, Brumm et al. (2011) show that at high levels of margin requirement (more than 80%) increasing the requirement, i.e. restricting leverage, decreases asset price volatility. This result is different from the normative result in my paper. In their paper, agents trade due to their difference in risk-aversion (under Epstein-Zin recursive preferences), while in my paper, agents trade due to their difference in beliefs. Under belief heterogeneity, there is a new mechanism that does not exist in Brumm et al. (2011): a higher margin requirement actually protects the agents with incorrect beliefs. Thus they are financially wealthier in the long run and their trading activity drives up asset price volatility.

In Cao (2010), I introduce capital accumulation to the current framework. The model presented there is a generalization of Krusell and Smith (1998) with financial markets and adjustment costs.^{13,14} In particular, the existence theorem 1 in Appendix A shows that a recursive equilibrium in Krusell and Smith (1998) exists. Krusell and Smith (1998) derives numerically such an equilibrium, but they do not formally show its existence. Cao (2010) is also related to Kiyotaki and Moore (1997). I provide a microfoundation for the borrowing constraint in Kiyotaki and Moore (1997) using the endogeneity of the set of actively traded financial assets.

In a recent breakthrough paper, Brunnermeier and Sannikov (2013) present an economy in continuous time with collateral constraint and large shocks. Instead of linearizing around the steady state, the authors are able to solve for the global equilibrium of the economy in which asset prices and aggregate economic activities depend on the financial wealth distribution. The long run stationary distribution of the economy has a U-shape form. Most of the time the economy stays in the linear region in which the collateral constraint is not binding. Occasionally, large shocks push the economy toward a highly nonlinear region in

¹³In Cao (2010), I also allow for many types of assets (for examples trees, land, housing and machines) that differ in their dividend process and their collateral value. For example, some of the assets can be used as collateral to borrow and others cannot. Moreover, that version of the model can also allow for assets in flexible supply and production in order to study the effects of belief heterogeneity on the aggregate capital accumulation and aggregate economic activity. The model with fixed supply asset in the current paper is a special case of assets in flexible supply with adjustment costs approaching infinity, to highlight asset pricing implications.

¹⁴See Feng, Miao, Peralta-Alva, and Santos (2009) for an existence proof in an environment without financial markets.

which the collateral constraint is nearly binding. My paper is a discrete time counterpart of Brunnermeier and Sannikov (2013) in the sense that I also solve for the global nonlinear equilibrium of the economy. In contrast to Brunnermeier and Sannikov (2013), under belief heterogeneity, the stationary distribution of the economy concentrates on the nonlinear region in which the collateral constraint is binding or nearly binding.

3 General model

In this general model, there are heterogeneous agents who differ in their beliefs about the future streams of dividends and the individual endowments. In order to study the effects of belief heterogeneity on asset prices, I allow for only one real asset in fixed supply. After presenting the model and defining collateral constrained equilibrium in Subsection 3.1, I show some general properties of the equilibrium in Subsection 3.2 and a stationary form of collateral constrained equilibrium in Subsection 3.3.

3.1 The endowment economy

Consider an endowment, a single consumption (final) good economy in infinite horizon with infinitely-lived agents (consumers). Time runs from $t = 0$ to ∞ . There are H types of consumers

$$h \in \mathcal{H} = \{1, 2, \dots, H\}$$

in the economy with a continuum of measure 1 of identical consumers in each type. These consumers might differ in many dimensions including their per period utility function $U_h(c)$ (i.e., risk-aversion), and their endowment of final good e_h . The consumers might also differ in their initial endowment of a real asset, Lucas' tree,¹⁵ that pays off real dividend in terms of the consumption good. However, the most important dimension of heterogeneity is the heterogeneity in belief over the evolution of the exogenous state of the economy. There are S possible exogenous states (or equivalently exogenous shocks)

$$s \in \mathcal{S} = \{1, 2, \dots, S\}.$$

The states capture both idiosyncratic uncertainty (uncertain individual endowments) , and aggregate uncertainty (uncertain aggregate dividends).¹⁶

¹⁵See Lucas (1978).

¹⁶A state s can be a vector $s = (A, \epsilon_1, \dots, \epsilon_H)$ where A consists of aggregate shocks and ϵ_h are idiosyncratic shocks.

The evolution of the economy is captured by the realizations of the shocks over time: $s^t = (s_0, s_1, \dots, s_t)$ is the history of realizations of shocks up to time t . I assume that the shocks follow a Markov process with the transition probabilities $\pi(s, s')$. In order to rule out transient states, I make the following assumption:

Assumption 1 \mathcal{S} is ergodic.

In contrast to the standard rational expectation literature, I assume that the agents do not have the perfect estimate of the transition matrix π . Each of them has their own estimate of the matrix, π^h .¹⁷ However, these estimates are not very far from the truth: there exist u and U strictly positive such that

$$u < \frac{\pi^h(s, s')}{\pi(s, s')} < U \quad \forall s, s' \in S \text{ and } h \in \mathcal{H}, \quad (1)$$

where $\pi(s, s') = 0$ if and only if $\pi^h(s, s') = 0$ in which case let $\frac{\pi^h(s, s')}{\pi(s, s')} = 1$. This formulation allows for time-varying belief heterogeneity as in He and Xiong (2011). In particular, agents might share the same beliefs in good states, $\pi^h(s, \cdot) = \pi^{h'}(s, \cdot)$, but their beliefs can start diverging in bad states, $\pi^h(s, \cdot) \neq \pi^{h'}(s, \cdot)$.¹⁸ Notice that (1) implies that every agent believes that \mathcal{S} is ergodic.

Real Asset: There is one real asset that pays off state-dependent dividend $d(s)$ in the final good. The asset can be both purchased and used as collateral to borrow. The ex-dividend price of each unit of the asset in history s^t is denoted by $q(s^t)$. I assume that agents cannot short-sell the real asset.¹⁹ The total supply 1 of the asset is given at the beginning of the economy, under the form of asset endowments to the consumers.

Collateralized Bond: In addition to the real asset, the agents in this economy can also borrow subject collateral constraints. The agents borrow by selling one-period-ahead bonds but bonds have to be collateralized by the real asset. In history s^t , a bond that pays off one unit of consumption good next period is sold at price $p(s^t)$.

When agents borrow by selling bonds, they have to simultaneously purchase the real asset to use it as collateral for the bonds. This transaction is equivalent to buying the real asset using leverage. The difference between how much they pay for the real asset and how much can borrow against it is called margin in the literature.

¹⁷Learning can be easily incorporated into this framework by allowing additional state variables which are the current beliefs of agents in the economy. As in Blume and Easley (2006) and Sandroni (2000), agents who learn slower will disappear under complete markets. However they all survive under collateral constraints. The dynamics of asset prices described here corresponds to the short-run behavior of asset prices in the economy with learning.

¹⁸Simsek (2012) shows, in a static model, that only the divergence in beliefs about bad states matters for asset prices.

¹⁹I can relax this assumption by allowing for limited short-selling.

Consumers: Consumers are the most important actors in this economy. They can be hedge fund managers or banks' traders in financial markets. In each state s^t , each consumer is endowed with a potentially state dependent (non-financial) endowment $e_t^h = e^h(s_t)$ units of the consumption good. I suppose that there is a strictly positive lower bound on these endowments. This lower bound guarantees a lower bound on consumption if a consumer decides to default on all her debt.²⁰

Assumption 2 $\min_{h,s} e_h(s) > \underline{e} > 0$.

Consumers maximize their intertemporal expected utility with the per period utility functions $U_h(\cdot) : \mathbf{R}^+ \rightarrow \mathbf{R}$ that satisfy

Assumption 3 $U_h(\cdot)$ is concave and strictly increasing.²¹

I also assume that consumers share the same discount factor β .²² Consumer h takes the sequences of prices $\{q_t, p_t\}$ as given and solves

$$\max_{\{c_t^h, \theta_{t+1}^h, \phi_{t+1}^h\}} \mathbf{E}_0^h \left[\sum_{t=0}^{\infty} \beta^t U_h(c_t^h) \right] \quad (2)$$

subject to the budget constraint

$$c_t^h + q_t \theta_{t+1}^h + p_t \phi_{t+1}^h \leq e_t^h + \phi_t^h + (q_t + d_t) \theta_t^h \quad (3)$$

and the collateral constraint

$$\phi_{t+1}^h + (1 - m) \theta_{t+1}^h \min_{s^{t+1}|s^t} (q_{t+1} + d_{t+1}) \geq 0, \quad (4)$$

where $0 \leq m \leq 1$ is the margin that determines how much an agent can borrow per unit of the real asset holding.²³

²⁰I also introduce the disutility of labor in the general existence proof in Cao (2010) in order to study employment in this environment. The existence of equilibria for finite horizon allows for labor choice decision. When we have strictly positive labor endowments, l^h , we can relax Assumption 2 on final-good endowments, e^h .

²¹Notice that I do not require U_h to be strictly concave. This assumption allows for linear utility functions in Geanakoplos (2009) and Harrison and Kreps (1978).

²²The general formulation and solution method in Cao (2010) allow for heterogeneity in the discount rates. In this paper I assume homogeneous discount factor to focus on beliefs heterogeneity.

²³We can also consider the alternative collateral constraint

$$\phi_{t+1}^h + (1 - m) \theta_{t+1}^h \min_{s^{t+1}|s^t} q(s^{t+1}) \geq 0.$$

This is the constraint used in Kiyotaki and Moore (1997) The quantitative implications of such constraint are very similar to the ones in this paper.

The agents are also subject to the no-short sale constraint on the real asset

$$\theta_{t+1}^h \geq 0. \quad (5)$$

One implicit condition from the assumption on utility functions is that consumption is positive, i.e., $c_t^h \geq 0$.

The most important feature of the objective function is the superscript h in the expectation operator $\mathbf{E}^h[\cdot]$, which represents the subjective beliefs when an agent calculates her future expected utility. Entering period t , agent h holds θ_t^h old units of real asset and ϕ_t^h units of collateralized bonds. She can trade old units of real asset at price q_t and buy new units of real asset θ_{t+1}^h for time $t + 1$ at the same price. She can also buy and sell bonds ϕ_{t+1}^h at price p_t . If she sells bonds she is subject to collateral constraint (4). The collateral constraint (4) has the usual property of financial constraints that higher (future) asset prices should enable more borrowing. When $m = 1$, agents are not allowed to borrow, so they can only trade in the real asset.

In this environment, I define an equilibrium as follows

Definition 1 *A collateral constrained equilibrium for an economy with initial asset holdings*

$$\{\theta_0^h\}_{h \in \{1, 2, \dots, H\}}$$

and initial shock s_0 is a collection of consumption, real asset, and bond holdings and prices in each history s^t ,

$$(\{c_t^h(s^t), \theta_{t+1}^h(s^t), \phi_{t+1}^h(s^t)\}_{h \in \mathcal{H}}, q_t(s^t), p_t(s^t))$$

satisfying the following conditions:

i) The markets for final good, real asset, and bond in each period clear:

$$\begin{aligned} \sum_{h \in \mathcal{H}} \theta_{t+1}^h(s^t) &= 1 \\ \sum_{h \in \mathcal{H}} \phi_{t+1}^h(s^t) &= 0 \\ \sum_{h \in \mathcal{H}} c_t^h(s^t) &= \sum_{h \in \mathcal{H}} e^h(s_t) + d(s_t) \end{aligned}$$

ii) For each consumer h , $\{c_t^h(s^t), \theta_{t+1}^h(s^t), \phi_{t+1}^h(s^t)\}$ solves the individual maximization problem subject to the budget constraint (3), and the collateral constraint (4).

The collateral constrained equilibrium is based on the exogenous collateral constraint (4) which is often assumed in the literature. In Appendix D, I present a micro-foundation for this collateral constraint based on the limited commitment that requires agents to hold collateral when they borrow.

Proposition 1 *Assume that each aggregate state s^t has only two possible future successor states s^{t+1} , each collateral constrained equilibrium has an equivalent collateral equilibrium a la Geanakoplos and Zame (2002) and Kubler and Schmedders (2003), i.e., the two equilibria have the same prices (of bonds and real asset) and consumption allocation (but the equilibria might differ in the portfolio holdings of the agents in the economy).*

Appendix D formalizes and proves this proposition.

3.2 Survival in collateral constrained equilibrium

In this subsection, I show that all agents (including the ones with incorrect beliefs) survive in a collateral constrained equilibrium.

Given the endowment economy, we can easily show that the total supply of final good in each period is bounded by a constant \bar{e} . Indeed, in each period, the total supply of final good is bounded by

$$\bar{e} = \max_{s \in \mathcal{S}} \left(\sum_{h \in \mathcal{H}} e^h(s) + d(s) \right). \quad (6)$$

The first term on the right hand side is the total endowment of all consumers. The second term is the dividend from the real asset. In collateral constrained equilibrium, the market clearing condition for the final good implies that the total consumption of all consumers is bounded from above by \bar{e} .

We can show that in any collateral constrained equilibrium, the consumption of each consumer is bounded from below by a strictly positive constant \underline{c} . Two assumptions are important for this result. First, the no-default-penalty assumption allows consumers, at any moment in time, to walk away from their past debt and only lose their collateral. After defaulting, they can always keep their non-financial wealth - inequality (8) below. Second, increasingly large speculation by postponing current consumption is not an optimal plan in equilibrium, because total consumption is bounded by \bar{e} , in inequality (9).²⁴ This assumption prevents agents from constantly postponing their consumption to speculate in the real asset and represents the main difference with the survival channel in Alvarez and Jermann (2000)

²⁴Even though an atomistic consumer may have unbounded consumption, in equilibrium, prices will adjust such that a consumption plan in which consumption sometime exceeds \bar{e} will not be optimal.

which is used by Beker and Espino (2010) and Tsyrennikov (2012) for heterogeneous beliefs. Formally, we arrive at

Proposition 2 *Suppose that there exists a \underline{c} such that*

$$U_h(\underline{c}) < \underbrace{\frac{1}{1-\beta}U_h(\underline{e})}_{\text{endowment}} - \underbrace{\frac{\beta}{1-\beta}U_h(\bar{e})}_{\text{speculation}}, \quad \forall h \in \mathcal{H}, \quad (7)$$

where \bar{e} is defined in (6). Then in a collateral constrained equilibrium, the consumption of each consumer in each history always exceeds \underline{c} .

Proof. This result is shown in an environment with homogenous beliefs [Lemma 3.1 in Duffie et al. (1994) and Kubler and Schmedders (2003)]. It can be done in the same way under heterogenous beliefs. I replicate the proof in order to provide the economic intuition in this environment.

By the market clearing condition in the market for the final good, the consumption of each consumer in each future period is bounded by future aggregate endowment. In each period, a feasible strategy of consumer h is to default on all of her past debt at the only cost of losing all her collateral. However, she can still at least consume her endowment from the current period onwards. Therefore

$$U_h(c_{h,t}) + \mathbf{E}_t^h \left[\sum_{r=1}^{\infty} \beta^r U_h(c_{h,t+r}) \right] \geq \frac{1}{1-\beta} U_h(\underline{e}). \quad (8)$$

Notice that in equilibrium, $\sum_h c_{h,t+r} \leq \bar{e}$ and so $c_{h,t+r} \leq \bar{e}$. Hence

$$U_h(c_h) + \frac{\beta}{1-\beta} U_h(\bar{e}) \geq \frac{1}{1-\beta} U_h(\underline{e}). \quad (9)$$

This implies

$$U_h(c_h) \geq \frac{1}{1-\beta} U_h(\underline{e}) - \frac{\beta}{1-\beta} U_h(\bar{e}) > U_h(\underline{c}).$$

Thus, $c_h \geq \underline{c}$. ■

Condition (7) is satisfied immediately if $\lim_{c \rightarrow 0} U_h(c) = -\infty$, for example, with log utility or with CRRA utility with the CRRA coefficient exceeding 1.

One immediate corollary of Proposition 2 is that every consumer survives in equilibrium. Sandroni (2000) shows that in complete markets equilibrium, almost surely the consumption of agents with incorrect beliefs converges to 0 at infinity. Therefore, collateral constrained equilibrium differs from complete markets equilibrium when consumers strictly differ in their beliefs.

The survival mechanism in collateral constrained equilibrium is similar to the one in Alvarez and Jermann (2000), Beker and Espino (2010), and Tsyrennikov (2012). In particular, the first term on the right hand side of (7) captures the fact that the agents always have the option to default and go to autarky. In which case, they only consume their endowment which exceeds \underline{e} in each period and which is the lower bound for consumption in Alvarez and Jermann (2000). However, the two survival mechanisms also differ because, in this paper, agents can always default on their promises and lose all their real asset holdings. Yet they can always return to financial markets to trade right after defaulting. The second term in the right hand side of (7) shows that this possibility might actually hurt the agents if they have incorrect beliefs. The prospect of higher reward for speculation, i.e. high \bar{e} , will induce these agents to constantly postpone consumption to speculate. As a result, their consumption level might fall well below \underline{e} . Indeed, the lower bound of consumption \underline{c} is decreasing in \bar{e} . The more there is of the total final good, the more profitable speculative activities are and the more incentives consumers have to defer current consumption to engage in these activities.

This survival mechanism is also different from the limited arbitrage mechanism in Shleifer and Vishny (1997), in which asset prices differ from their fundamental values because agents with correct beliefs hit their financial constraints (or short-selling constraints) before they can arbitrage away the difference between assets' fundamental value and their market price. In this paper, agents with incorrect beliefs hit their financial constraint more often than the agents with correct beliefs do and are protected by the constraint. In the equilibria computed in Section 4, agents with the correct belief (the pessimists) sometime do not hit their borrowing constraint (or short-selling constraints).

The next subsection is devoted to showing the existence of these equilibria with a stationary structure and Appendix A presents an algorithm to compute the equilibria.

3.3 Markov equilibrium

Proposition 2 is established under the presumption that collateral constrained equilibria exist. In Appendix A, I show that under weak conditions on endowments and utility functions, a collateral constrained equilibrium exists and has the following stationary structure.

Following Kubler and Schmedders (2003), I define the *normalized financial wealth* of each agent as

$$\omega_t^h = \frac{(q_t + d_t) \theta_t^h + \phi_t^h}{q_t + d_t}. \quad (10)$$

Let $\omega(s^t) = (\omega^1(s^t), \dots, \omega^H(s^t))$ denote the *normalized financial wealth distribution*. Then in equilibrium $\omega(s^t)$ always lies in the $(H-1)$ -dimensional simplex Ω , i.e., $\omega^h \geq 0$ and $\sum_{h=1}^H \omega^h =$

1. ω^h 's are non-negative because of the collateral constraint (4) that requires the value of each agent's asset holding to exceed the liabilities from their past borrowings. The sum of ω^h equals 1 because of the real asset market clearing and bond market clearing conditions. Using the definition of normalized financial wealth, I can define a Markov equilibrium as follows.

Definition 2 *A Markov equilibrium is a collateral constrained equilibrium in which the prices of real asset and bond and the allocation of consumption, real asset and bond in each history depend only on the exogenous shock s_t and the endogenous normalized financial wealth distribution $\omega(s^t)$.*

In Appendix A, I show the conditions under which a Markov equilibrium exists, and I develop a numerical method to compute Markov equilibria. Markov equilibria inherit all the properties of collateral constrained equilibria. In particular, in a Markov equilibrium, every consumer survives (Proposition 2). Regarding asset prices, the construction of Markov equilibria shows that asset prices can be history-dependent in the long run through the evolution of the normalized financial wealth distribution, defined in (10).

This result is in contrast with the one in Sandroni (2000) in which - under complete markets - asset prices depend on the wealth distribution that converges in the long run. So, in the long run, asset prices only depend on the current exogenous state s_t . In a Markov equilibrium, the normalized financial wealth distribution constructed in (10) constantly moves over time, even in the long run. For example, if an agent h with incorrect belief loses all her real asset holding due to leverage, next period, she can always use her endowment to speculate in the real asset again. In this case, ω^h will jump from 0 to a strictly positive number. Asset prices therefore depend on the past realizations of the exogenous shocks, which determine the evolution of the normalized financial wealth distribution ω . Consequently, we have the following result.

Remark 1 *When the aggregate endowment is constant across states $s \in \mathcal{S}$, and shocks are I.I.D., long run asset price volatility is higher in Markov equilibria than it is in complete markets equilibria.*

As shown in Sandroni (2000), in the long run, under complete markets, the economy converges to an economy with homogenous beliefs because agents with incorrect beliefs will eventually be driven out of the markets and the real asset price $q(s^t)$ converges to a price independent of time and state. Hence, due to I.I.D. shocks and constant aggregate endowment, under complete markets, asset price volatility converges to zero in the long run. In Markov equilibrium, asset price volatility remains above zero as the exogenous shocks

constantly change the normalized financial wealth distribution that, in turn, changes real asset price.²⁵

There are two components of asset price volatility. The first and standard component comes from the volatility in the dividend process and the aggregate endowment. The second component comes from the financial wealth distribution when agents strictly differ in their beliefs. In general, it depends on the correlation of the two components, that we might have asset price volatility higher or lower under collateral constraints versus under complete markets. However, the second component disappears under complete markets because only agents with the correct belief survive in the long run. In contrast, under collateral constraints, this component persists. As a result, when we shut down the first component, asset price is more volatile under collateral constraints than it is under complete markets in the long run. In general, whether the same comparison holds depends on the long-run correlation between the first and the second volatility components.

4 The dynamics of financial wealth distribution and asset prices

In this section, I focus on a special case of the general framework analyzed in section 3. I restrict myself to an economy with only two types of agents. In such an economy, the normalized financial wealth distribution can be summarized by only one number, which is the fraction of the normalized financial wealth held by one of the two types of agents. Under this simplification, I can compute collateral constrained equilibria. In the following Subsections 4.2 through 4.4, I compute Markov equilibria in a simple calibrated economy to show the complex joint dynamics of financial wealth distribution and asset prices, including the prosper by speculation mechanism that I described in the introduction.

4.1 Two-agent economy

Consider a special case of the general model presented in Section 3. There are two exogenous states $\mathcal{S} = \{G, B\}$ and one real asset of which the dividend depends on the exogenous state:

$$d(G) > d(B).$$

²⁵This result holds except in knife-edge cases in which, even in collateral constrained equilibrium, asset price is independent of the normalized financial wealth and the exogenous shocks, or when normalized financial wealth does not move over time. These cases never appear in numerical solutions.

The exogenous state follows an I.I.D. process, with the probability of high dividend, π , unknown to agents in this economy.²⁶

There are two types of consumers (measure one in each type), the optimists, O, and the pessimists, P, who differ in their beliefs. They have different estimates of the probability of high dividend π^h , $h \in \{O, P\}$. I suppose that $\pi^O > \pi^P$, i.e., optimists always think that good states are more likely than the pessimists believe. Again, due to the collateral constraint, in equilibrium, ω_t^h must always be positive and

$$\omega_t^O + \omega_t^P = 1.$$

The pay-off relevant state space

$$\{(\omega_t^O, s_t) : \omega_t^O \in [0, 1] \text{ and } s_t \in \{G, B\}\}$$

is compact.²⁷ Appendix A shows the existence of collateral constrained equilibria under the form of Markov equilibria in which prices and allocations depend solely on that state defined above. Appendix provides an algorithm to compute such equilibria. As explained in Subsection 3.3, the equilibrium asset prices depend not only on the exogenous state but also on the normalized financial wealth ω_t^O .

I choose plausible parameters to illustrate the dynamics of financial wealth distribution and asset prices. I assume that the optimists correspond to the investment banking sector that is bullish about the profitability of the mortgage-backed securities market (the real asset) and the pessimists correspond to the rest of the economy.

The parameters are chosen such that the income size of the investment banking sector is about 3% (e^O) of the U.S. economy, and the size of the mortgage-backed security market in the U.S. is about 20% of the U.S. annual GDP. In particular, let

$$\begin{aligned} \beta &= 0.95 \\ d(G) &= 1 > d(B) = 0.2 \\ U(c) &= \log(c), \end{aligned}$$

and the beliefs are

$$\pi^O = 0.9 > \pi^P = 0.5.$$

²⁶In the online appendix (Appendix E), I allow for richer structures of shocks as well as of consumers' endowment.

²⁷Given that the optimists prefer holding the real asset, i.e., $\theta_t^O > 0$, ω_t^O corresponds to the fraction of the asset owned by the optimists.

To study the issue of survival and its effect on asset prices, I assume that the pessimists have the correct belief, i.e., $\pi = \pi^P = 0.5$. Thus the optimists are over-optimistic.

I fix the endowments of the pessimists and the optimists at

$$\begin{aligned} e^P &= \begin{bmatrix} 100 & 100.8 \end{bmatrix} \\ e^O &= \begin{bmatrix} 3 & 3 \end{bmatrix}. \end{aligned}$$

The endowment of the pessimists is chosen as slightly counter-cyclical so that the aggregate endowment is kept constant at 104.²⁸

In the next Subsections, I study the properties of equilibria under different market structures (different margin requirement m 's) using the numerical method developed in Appendix A.²⁹

4.2 Liquidity constrained equilibrium

As a benchmark, I assume that $m = 1$. In this case, a collateral constrained economy is equivalent to an economy in which agents can only trade in the real asset

$$c_t^h + q_t \theta_{t+1}^h \leq e_t^h + (q_t + d_t) \theta_t^h \quad (11)$$

and $\theta_{t+1}^h \geq 0$. This economy corresponds to the liquidity constrained economy studied in Kehoe and Levine (2001) with belief heterogeneity.

The left panel in Figure 1 shows the relationship between the price of the asset and the current fraction of the real asset θ_t^O held by the optimists, when the current exogenous state $s_t = G$ (bold green line) and $s_t = B$ (blue line) respectively. The two black bands show the prices of the real asset evaluated using the belief of the optimists (dashed-dotted upper band) and the belief of the pessimist (dotted lower band).³⁰ The scale is on the left axis. Similarly, the right panel shows the future fraction of the real asset held by the optimists, θ_{t+1}^O . We can easily see that when θ_t^O is far from 1, θ_{t+1}^O lies strictly between 0 and 1. Thus, both sets of agents are marginal buyers of the real asset.³¹ Given that the consumption of

²⁸In Appendix C, I show that the results in this Subsection are robust to changes in the degree of belief heterogeneity and the variability of dividends.

²⁹For each equilibrium, the algorithm takes about 15 minutes to converge in an IBM X201 laptop. This running time can be shortened further by parallel computing.

³⁰ $P^h = \frac{\beta}{1-\beta} (\pi^h(G) d(G) + (1 - \pi^h(G)) d(B))$

³¹In contrast to the limits to arbitrage channel in Shleifer and Vishny (1997), the agents with correct beliefs, i.e. the pessimists, are not constrained by the short-selling constraint all the time. The dynamics of asset price is such that the pessimists are happy to hold the asset as well. Only when θ_t^O is sufficiently high, or equivalently when the asset is sufficiently over-valued, the pessimists start their attempt to short-sell the

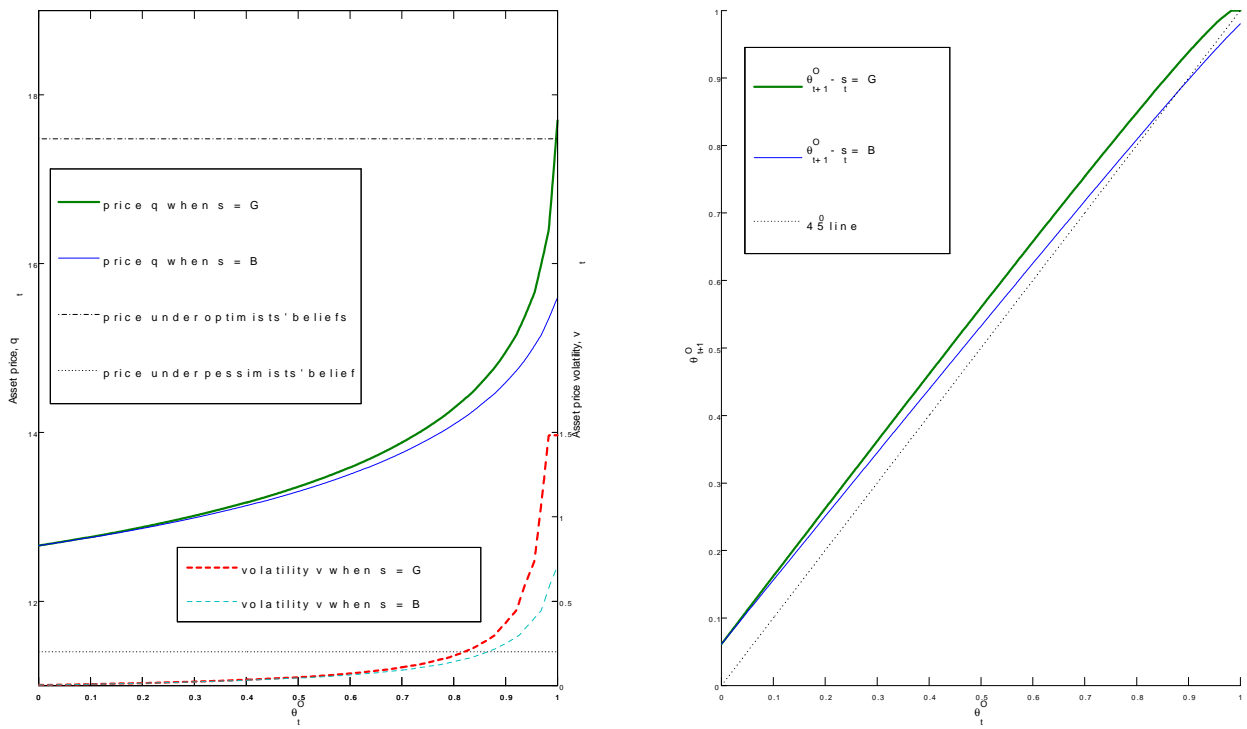


Figure 1: Asset prices and asset holdings without leverage

the pessimists changes relatively little due to their high level of non-tradable endowment, the price of the asset is close to the pessimists' estimate (lower band). This is true if the non-short selling constraint of the pessimists is not going to bind in the near future, i.e., θ_t^O , sufficiently far away from 1. When θ_t^O is close to 1, θ_{t+1}^O is equal to 1, thus $\theta_{t+1}^P = 0$, i.e., the optimists are the only marginal buyers of the real asset, thus its price is close to the optimists' valuation (upper band). Notice also that when $s_t = G$ and θ_t^O is close to 1, the price of the asset exceeds the valuation of any of the agents in the economy. This phenomenon is similar to the one in Harrison and Kreps (1978), in which agents expect to be able to sell their assets to other agents in the economy. Interestingly, it is only when θ_t^O is close to 1 and the price of the real asset is too high that the pessimists start trying to short-sell the real asset. However, they are constrained by the short-selling constraint.

I define the volatility of asset price as one period ahead volatility³²

$$v_t(s^t) = std_{s^{t+1}|s^t}(q_{t+1}(s^{t+1})). \quad (12)$$

The dashed lines in Figure 1 show asset price volatility as functions of θ_t^O (bold red dashed line for $s_t = G$ and blue-green dashed line for $s_t = B$, the scale is on the right axis). Asset price volatility is the highest when $s_t = G$ and θ_t^O is close to 1. The budget constraint (11) and the policy function θ_{t+1}^O in the right panel of Figure 1 show that the consumption of the optimists c_t^O is close to $e_t^O + d_t\theta_t^O$. As θ_t^O is close to 1, the optimists' consumption changes more with the changes in d_t , which translates into higher asset price volatility because the optimists are always the marginal buyer of the real asset. The same logic explains why asset price volatility is increasing in θ_t^O .

Another implication of the policy functions in the right panel is that in the stationary distribution, the optimists will end up holding most of the real asset (see Figure 9). The optimists therefore are not driven out of the markets as suggested in Sandroni (2000) but in fact prosper. They survive by forgoing current consumption, even if the real asset pays off low dividend, to hold on to an ever-increasing share of the asset. This survival by "speculation" channel is complementary to the survival by precautionary saving channel suggested by Cogley, Sargent, and Tsyrennikov (2011). However, due to the fact that the real asset is long-lived in this framework but not in Cogley, Sargent, and Tsyrennikov (2011), in Appendix B, I show that in addition to belief heterogeneity, the optimists need non-financial wealth to prosper.

asset and hit the no-short-selling constraint.

³²This definition corresponds to the instantaneous volatility in continuous time asset pricing models. See Xiong and Yan (2009) for the most recent use of the application in the context of belief heterogeneity and complete markets.

The stationary distribution combined with the fact that volatility is increasing in θ_t^O , shows that the price of the real asset tend to be very volatile in the long run, as the optimists hold most of the real asset. We will return to this result when we study the effect of financial regulation on volatility in Subsection 4.4.

4.3 Collateral Constrained Equilibrium

Now let $m = 0$. The agents can borrow up to the minimum future value of their real asset holdings. This economy is thus a mixture of liquidity constrained and debt constrained economies studied in Kehoe and Levine (2001).

I rewrite the budget constraint of the optimists using normalized financial wealth, ω_t^O ,

$$c_t^O + q_t \theta_{t+1}^O + p_t \phi_{t+1}^O \leq e_t^O + (q_t + d_t) \omega_t^O.$$

Therefore, their total wealth $e_t^O + (q_t + d_t) \omega_t^O$ affects their demand for the asset. If non-financial endowment e_t^O of the optimists is small relative to the price of the asset, their demand for the asset is more elastic with respect to their financial wealth $(q_t + d_t) \omega_t^O$.

Figure 2 plots the price of the real asset as a function of the optimists' normalized financial wealth ω_t^O for the good state $s_t = G$ (bold green line) and for the bad state $s_t = B$ (blue line). The scale is on the left axis and the two black bands are the valuations of the optimists and the pessimists as in the left panel of Figure 1. Interest rate r is also endogenously determined in this economy. Most of the time it hovers around the common discount factors of the two agents, i.e. $r(s^t) \approx \frac{1}{\beta} - 1$ because, as the pessimists have relatively large endowments, they are the marginal buyers of the collateralized bonds.

To understand the shape of the price functions in Figure 2, it is helpful to look at the equilibrium portfolio choice of the agents in Figure 3. The bold blue lines on the left and right panels are the real asset holdings of the optimists, θ_{t+1}^O , as functions of the normalized financial wealth ω_t^O when the state s_t is G or B respectively. Similarly, the green lines correspond to the bond holdings of the optimists, ϕ_{t+1}^O and the red dashed-dotted lines correspond to the borrowing constraint $-\theta_{t+1}^O \min_{s^{t+1}|s^t} (q(s^{t+1}) + d(s_{t+1}))$. Negative ϕ_{t+1}^O means the optimists borrow from the pessimists to invest into the real asset and the green and red lines overlap when the borrowing constraint is binding. Leverage induces the optimists to hold the total supply of the real asset. The pessimists are no longer the marginal buyers of the real asset as in Subsection 4.2, so the price of the real asset is very close to the valuation of the optimists. Figure 3 also shows that the collateral constraint (4) is binding when $\omega_t^O < 0.1$. Due to this binding constraint, the optimists are restricted in their ability to smooth consumption, so their consumption becomes relatively low. Low consumption

drives up marginal utility and consequently lower the price of the real asset. Thus the slope of the price function in normalized financial wealth is higher when $\omega_t^O < 0.1$. As prices and portfolio choices are all endogenous in equilibrium, we can re-interpret the high slope under the light of the "*debt-deflation*" channel: In the region in which the borrowing constraint binds - or is going to bind in the near future - when a bad shock hits the economy, that is $s_{t+1} = B$, the optimists are forced to liquidate their real asset holdings. This *fire sale* of the real asset reduces its price and tightens the constraints further, thus setting off a vicious cycle of falling asset price and binding collateral constraint. These dynamics of asset price under borrowing constraint are called the *debt-deflation* channel in a small-open economy in Mendoza (2010). This example shows that the channel still operates when we are in a closed-economy with endogenous interest rate, $r(s^t)$ as opposed to exogenous interest rates in Mendoza (2010) or in Kocherlakota (2000).

On the right side of Figure 2, when ω_t^O is close to 1, the *debt-deflation* channel is not present as the collateral constraint is not binding or going to be binding in the near future. High asset price elasticity with respect to the normalized financial wealth of the optimists is due to their high exposure to the asset. As the optimists own most of the real asset without significant borrowing from the pessimists (Figure 3), the dividend from the real asset directly impacts the consumption of the optimists. Higher dividend implies higher consumption, lower marginal utility and higher asset price. The behavior of asset price when ω_t^O is close to 1 is thus similar to the case without leverage in Subsection 4.2.

Figure 2 also plots asset price volatility, v_t in (12) as functions of the normalized financial wealth of the optimists, the bold red dashed line for $s_t = G$ and the green-blue dashed line for $s_t = B$. The scale is on the right axe. These functions show that at low levels of financial wealth of the optimists, asset price is mostly inversely related to asset price volatility. This negative relationship between price level and price volatility has been observed in several empirical studies, for example, Heathcote and Perri (2011). High asset price volatility at low levels of ω_t^O is due to the *debt-deflation* channel described above.³³ High asset price volatility at high levels of ω_t^O is due to large changes in the marginal utility of the optimists as in Subsection 4.2. Comparing Figure 2 and the left panel of Figure 1, for higher ω_t^O , asset price volatility is also lower in this economy than it is in the no leverage economy in Subsection 4.2 because the ability to borrow allows the optimists to mitigate the drop in consumption when the bad shock with low dividend realizes.

In order to study the dynamics of asset price, we need to combine asset price as a function of the normalized financial wealth shown in Figure 2 with the evolution of the exogenous state

³³Asset price volatility flattens below certain level of ω_t^O because below these levels future normalized financial wealth ω_{t+1}^O after a bad shock reaches the lower bound 0 and cannot fall further.

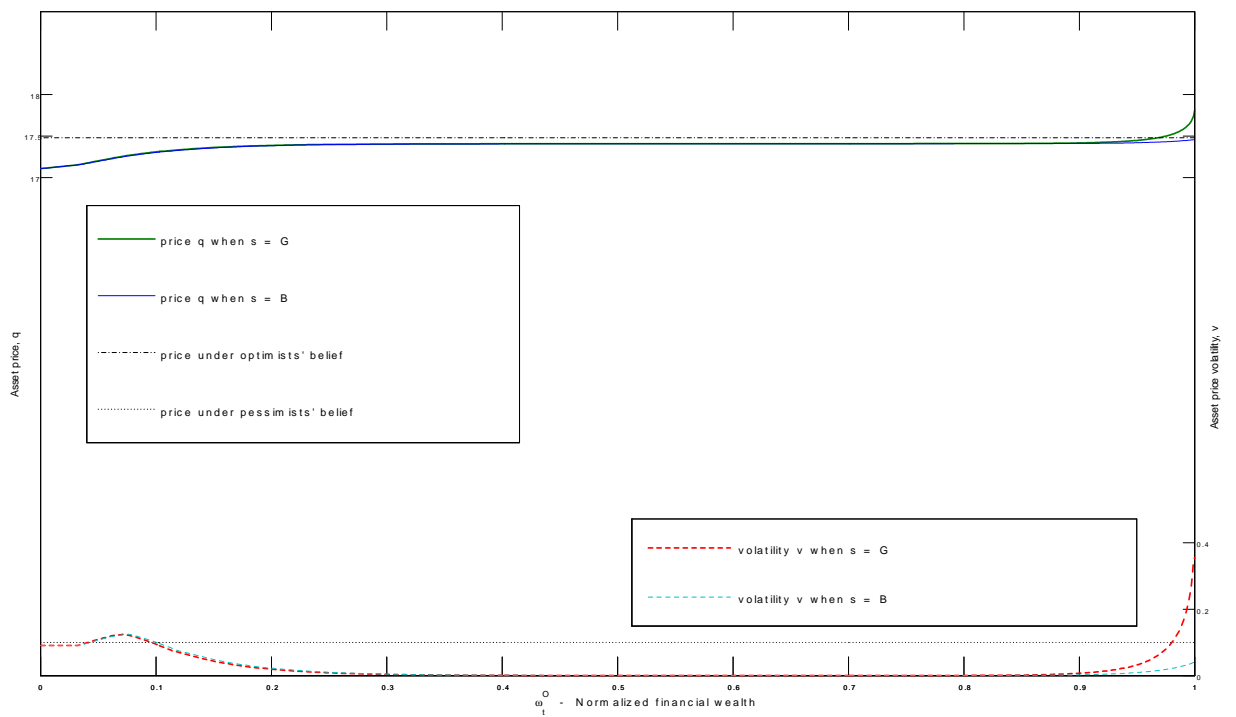


Figure 2: Asset price and asset price volatility under full leverage

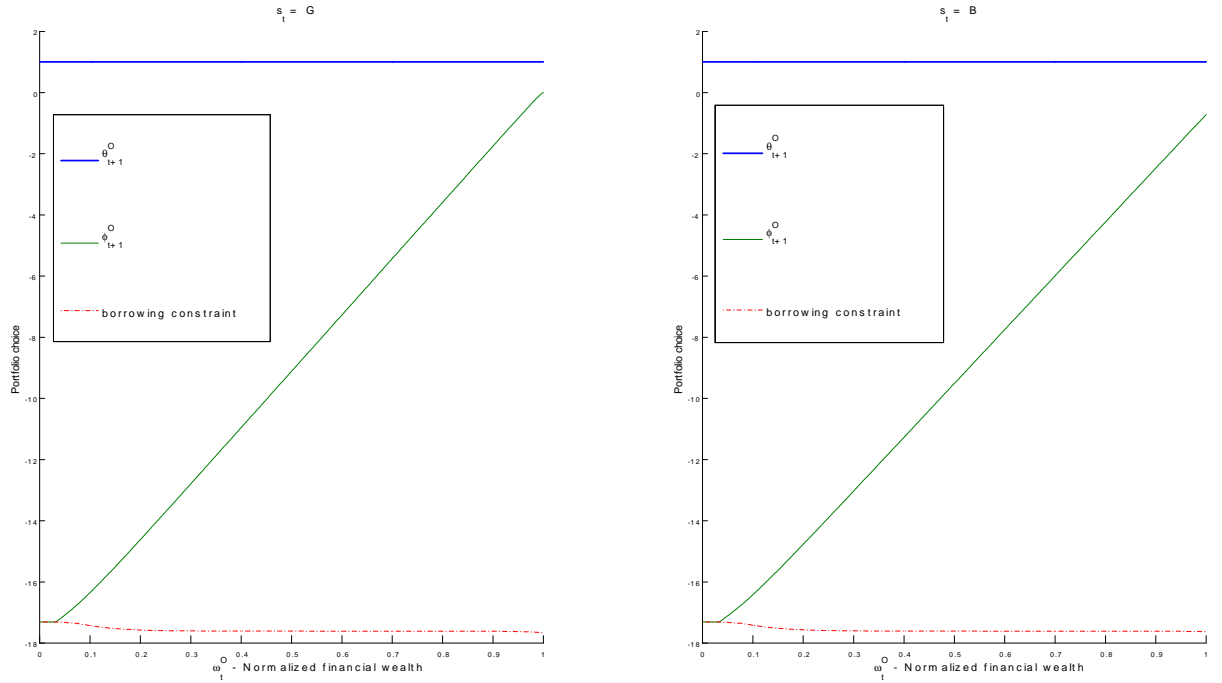


Figure 3: Portfolio choice under full leverage

and the evolution of the normalized financial wealth distribution, ω_t^O . Figure 4 shows the evolution of ω_t^O . The left and right panels represent the transition of the normalized financial wealth when the current state is G and B respectively. The bold green lines represent next period normalized financial wealth of the optimists as a function of the current normalized financial wealth if the good shock realizes next period. The blue lines represent the same function when the bad shock realize next period. I also plot the 45 degree lines for comparison (the dotted black lines). This figure shows that, in general, good shocks tend to increase - and bad shocks tend to decrease - the normalized wealth of the optimists. This is because the optimists bet more on the likelihood that the good state realizes next period (buy borrowing collateralized and investing into the real asset).³⁴

We can also think of normalized financial wealth, ω_t^O , as the fraction of the real asset that the optimists owns. When the current state is good, and the fraction is high, the optimists will receive high dividend from their real asset holding. Due to consumption smoothing, they will not consume all the dividend, but will use some part of the dividend to buy more real asset. Thus as we see from the left panel of Figure 4, the real asset holding of the optimists normally increases at high ω_t^O . Similarly, when the current state is bad and the fraction is high, the optimists will sell off some of their real asset holding to smooth consumption. As a

³⁴A similar evolution of the wealth distribution holds for complete markets, see Cao (2010).

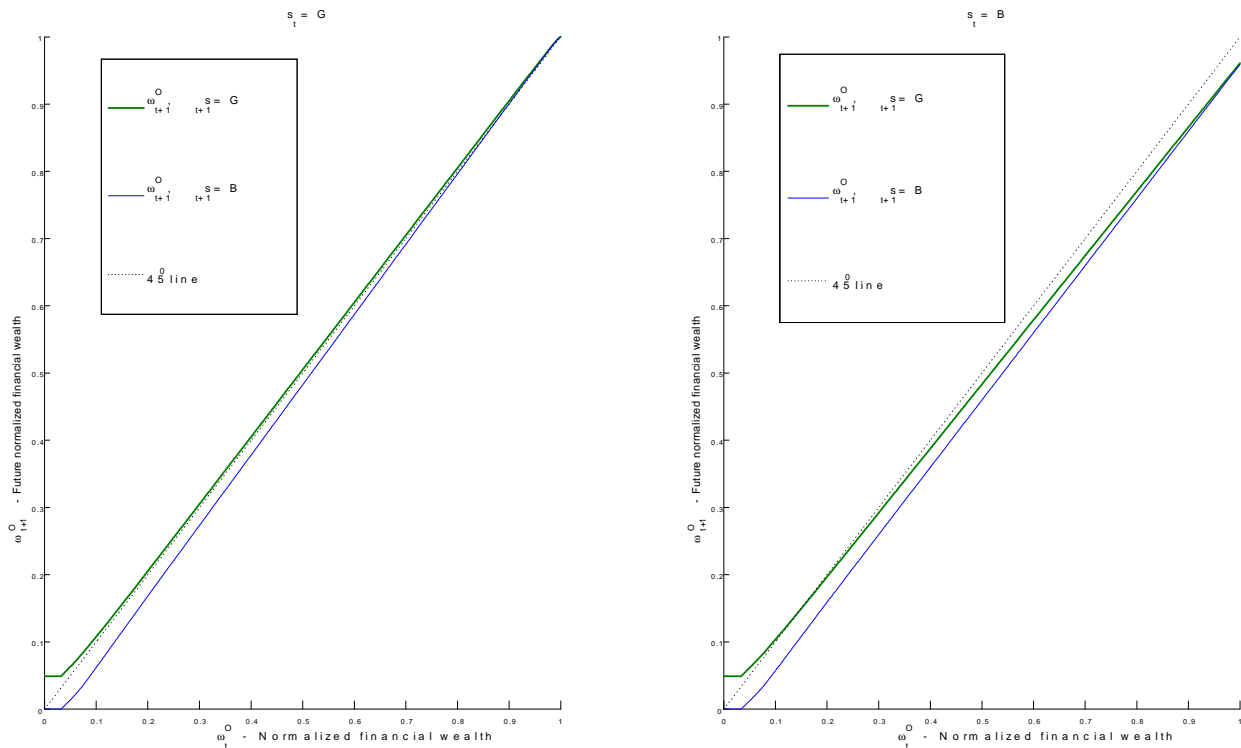


Figure 4: Wealth dynamics under full leverage

result, we see on the right panel that the real asset holding of the optimists tends to decrease at high ω_t^O .

The wealth dynamics in Figure 4 imply that, as opposed to the case without leverage in Subsection 4.2, in the stationary distribution the optimists end up with low normalized financial wealth (Figure 9).³⁵ As we see from Figure 3, at low levels of ω_t^O , the optimists always leverage up to the collateral constraint to buy the real asset, and when bad shock hits they will lose all their asset holdings (they have to sell off their asset holdings to pay off their debt), in which case, their financial wealth reverts to zero. However, they can always use their non-financial endowment to return to the financial markets by investing in the real asset again (with leverage). Sometimes they are lucky - that is, when the asset pays high dividend and its price appreciates - their financial wealth increases rapidly. But due to the fact that the optimists have an incorrect belief, they are unlucky more often than they think, such that most of the time they end up with low financial wealth. The ability to lever actually hurts these optimists because they can suffer larger financial losses. So leverage

³⁵This result is due to belief heterogeneity. In Brunnermeier and Sannikov (2013) without belief heterogeneity, the stationary distribution has a U-shape form: the economy spends most of the time in the non-binding constraint region and is occasionally pushed toward the binding constraint region.

brings the equilibrium close to the complete markets equilibrium in Sandroni (2000), where the optimists' total wealth goes to zero over time.³⁶

Lastly, by combining Figures 2 and 4, we can choose sequences of shock realizations to generate booms and busts documented in Burnside, Eichenbaum, and Rebelo (2011). In that paper, to generate the booms and busts in asset prices, the authors rely on the slow propagation of beliefs but shut down the effect of wealth distribution by assuming risk-neutrality.

4.4 Regulating Leverage and Asset Price Volatility

Remark 1 in Section 3 suggests that the variations in the wealth distribution drive up asset price volatility relative to the long run complete markets benchmark. It is thus tempting to conclude that by restricting leverage, we can reduce the variation of wealth of the optimists, therefore reduce asset price volatility. However, this simple intuition is not always true. Similar to collateral constraints, financial regulation acts as another device to protect the agents with incorrect beliefs from making wrong bets and from disappearing. Worse yet, regulation may help these agents prosper by speculation as in Subsection 4.2. The higher wealth of the agents with incorrect beliefs increases their impact on real asset price, thus making real asset price more volatile.

To show this result, I consider a financial regulation that chooses a margin requirement $m \in [0, 1]$. When $m = 0$, the equilibrium is the unregulated one in Subsection 4.3. When $m = 1$, collateral requirement is infinite, so there are no bonds traded. The equilibrium is that agents only trade in the real asset subject to no short-selling in Subsection 4.2. Figure 5 shows that the long run asset return volatility³⁷ is increasing in collateral requirement, m .

To explain this increasing relationship, I study in detail an economy with an intermediate $m = 0.5$. Figures 6, 7, and 8 are the counterparts of Figure 2, 3, and 4 when $m = 0.5$. The portfolio choice in Figure 7, shows that due to restricted leverage the optimists are able to purchase all the supply of the real asset only when they have sufficiently large financial wealth, i.e., $\omega_t^O \geq 0.5$. Below that threshold, the pessimists are also the marginal buyers of the asset, so the price of the asset is closer to the pessimists' valuation, as depicted in Figure 6. The same figure also shows that asset price is the most volatile around $\omega_t^O = 0.5$, at that point, when the bad shock hits, the optimists have to give up some of the real asset. Asset price falls as the pessimists become the marginal buyers. In addition, the *debt deflation*

³⁶See Cogley, Sargent, and Tsyrennikov (2012) for a similar point when Arrow securities are introduced in addition to bonds.

³⁷This long run return volatility is computed using the long run stationary distribution of normalized financial wealth.

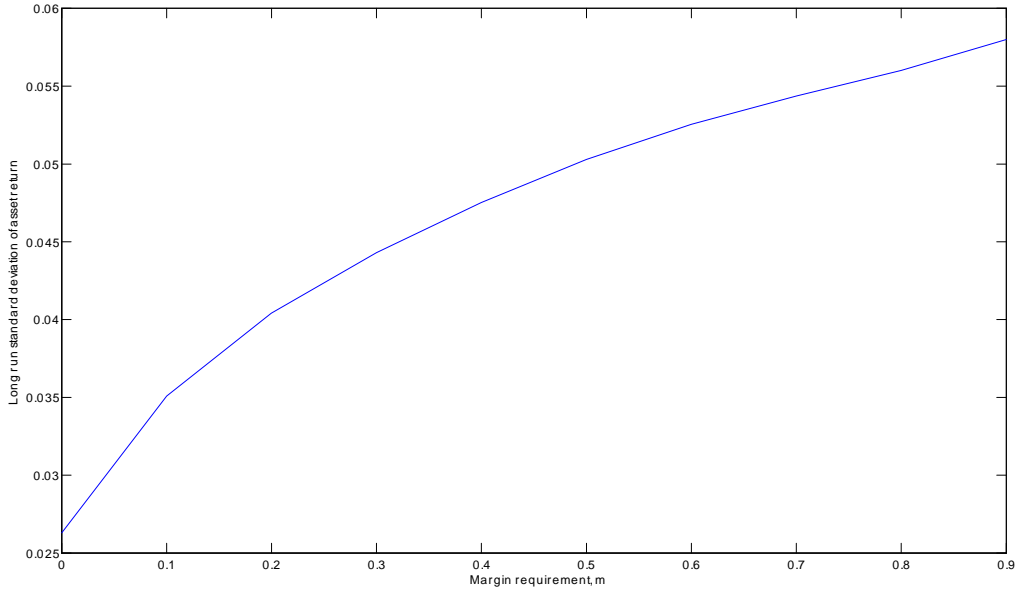


Figure 5: Asset price volatility and margin requirement

channel kicks in because of the collateral constraint on the optimists. Debt deflation further depresses the price of the real asset. The wealth dynamics are described by Figure 8. When $\omega_t^O < 0.5$, the optimists are protected by the regulation, as they cannot borrow as much as they can, most of their investment is in the real asset, so they accumulate wealth through speculation as in Subsection 4.2. However, above $\omega_t^O = 0.5$, their borrowing constraint is not binding, so they tend to borrow too much, relative to when they have correct belief, and lose wealth over time as in Subsection 4.3.

Figure 9 shows the stationary distributions of the normalized financial wealth of the optimists under three different financial structures, $m = 0$ (bold blue line), 0.5 (red circled line) and 1 (black crossed line). The stationary mean level of financial wealth of the optimists is increasing in m . As suggested by the analysis of volatility above, stricter financial regulation, which corresponds to higher collateral requirement, i.e., higher margin, higher m , leads to higher stationary financial wealth of the optimists. Consequently, asset price volatility is higher due to their greater influence on asset price. Indeed that the long run standard deviations of asset return is 0.0263 for $m = 0$, 0.0475 for $m = 0.5$, and 0.0712 for $m = 1$.

This increasing relationship between collateral requirement and asset returns volatility is in contrast with the result obtained in Brumm et al. (2011) (Figure 2) that collateral requirement decreases asset return volatility at high levels of margin requirement. This difference comes from the fact that Brumm et al. (2011) assume that agents trade due

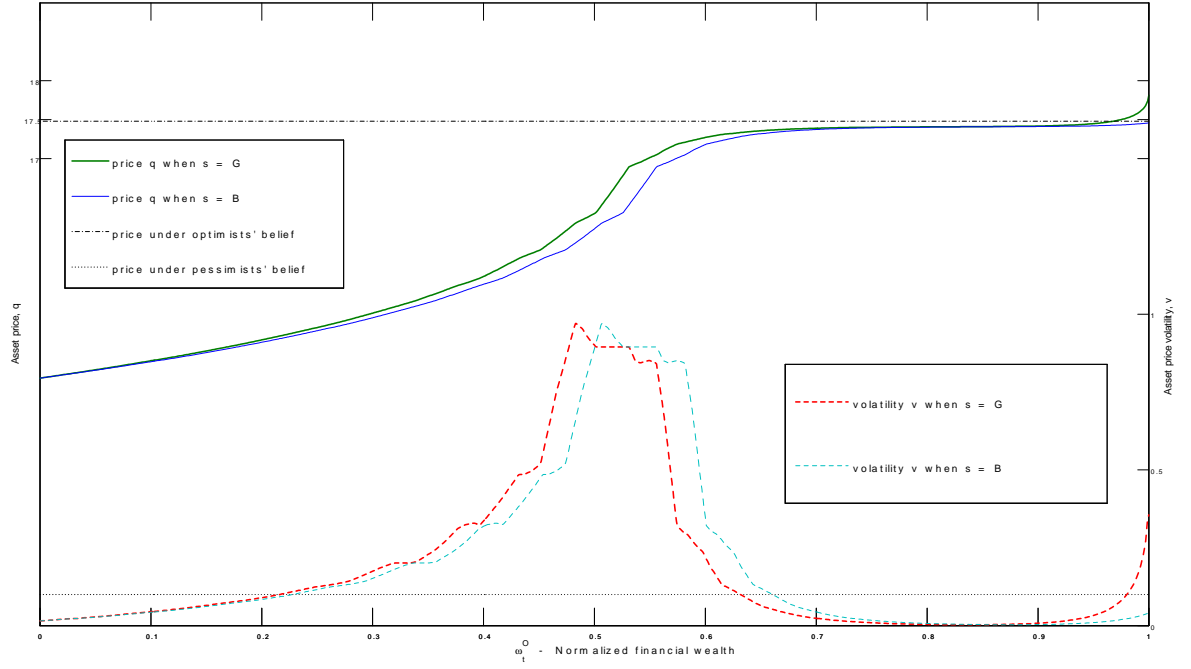


Figure 6: Asset price and asset price volatility under regulated leverage

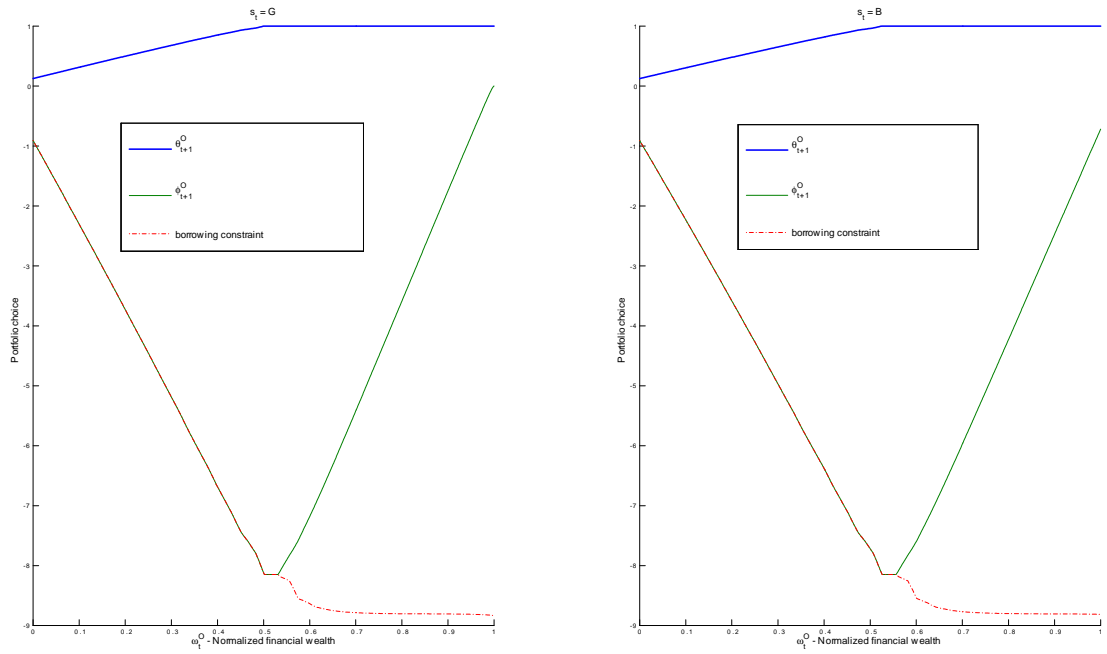


Figure 7: Portfolio choice under regulated leverage

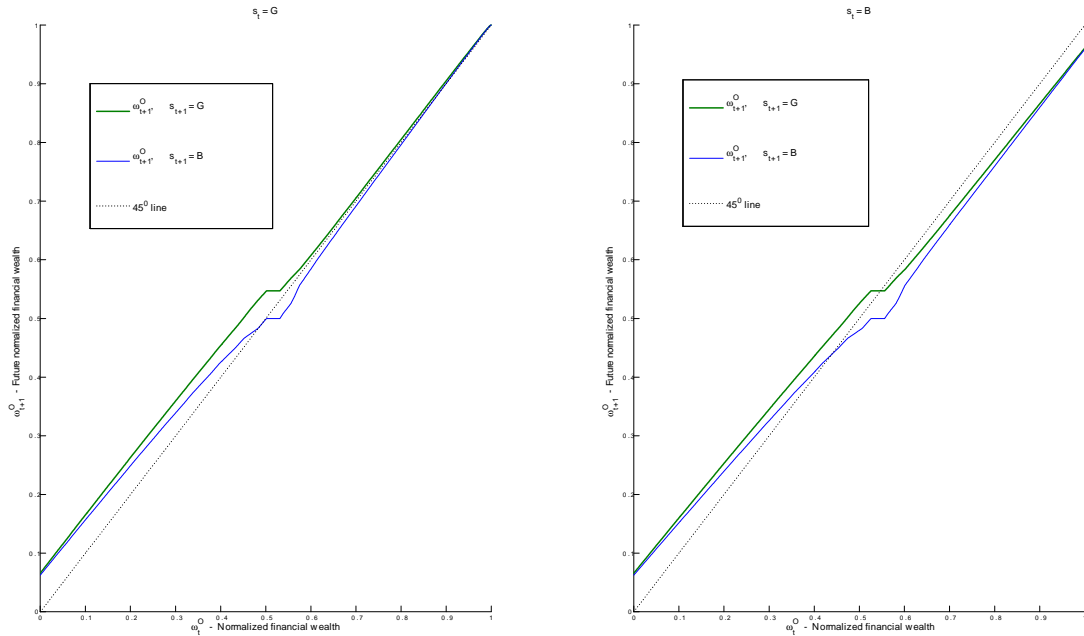


Figure 8: Wealth dynamics under regulated leverage

to their difference in risk-aversion (under Epstein-Zin recursive preferences), while in my paper agents trade due to their difference in beliefs. Under belief heterogeneity, there is a new mechanism that does not exist in Brumm et al. (2011): a higher margin requirement actually protects the agents with incorrect beliefs. Thus they are financially wealthier in the long run and their trading activity drives up asset price volatility.

5 Conclusion

In this paper I develop a dynamic general equilibrium model to examine the effects of belief heterogeneity on the financial wealth of agents and on asset price dynamics and asset price volatility under different financial market structures. I show that when financial markets are collateral constrained (endogenously incomplete markets) agents with incorrect beliefs sometime prosper, in the long run. The prosperity of these agents leads to higher asset price volatility. This result contrasts with the frictionless complete markets case in which agents holding incorrect beliefs are eventually driven out and, as a result, asset price exhibits lower volatility. The prosperity of agents with incorrect beliefs also generates rich dynamics of asset price and financial wealth distribution.

In the earlier version of this paper - Cao (2010) - I show the existence of stationary

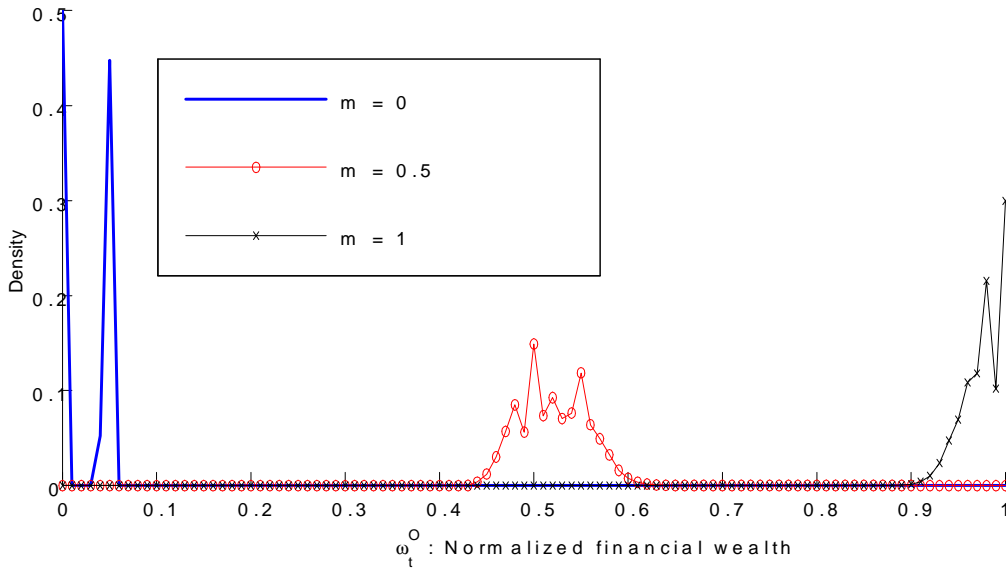


Figure 9: Stationary distributions of normalized financial wealth under different market structures

Markov equilibria in this framework with collateral constrained financial markets and with general production and capital accumulation technology. I also develop an algorithm for computing the equilibria. As a result, the framework can be readily used to investigate questions about asset pricing and about the interaction between financial markets and the macroeconomy. For instance, it would be interesting in future work to apply these methods in calibration exercises using more rigorous quantitative asset pricing techniques, such as in Alvarez and Jermann (2001). This could be done by allowing for uncertainty in the growth rate of dividends rather than uncertainty in the levels, as modeled in this paper, in order to match the rate of return on stock markets and the growth rate of aggregate consumption. Such a model would provide a set of moment conditions that could be used to estimate relevant parameters using GMM as in Chien and Lustig (2009). One challenge in such work, however, is the computational demands of finding Markov equilibria. I start this exercise in the online appendix and further follow that path in Cao, Chen, and Scott (2011).

A second avenue for further research is to examine more normative questions in the framework developed in this paper. My normative results suggest, for example, that financial regulation aimed at reducing asset price volatility should be state-dependent, as conjectured by Geanakoplos (2009).

A Markov Equilibrium

In this Appendix, I argue that collateral constrained equilibrium exists with a stationary structure. The equilibrium prices and allocation depend on the exogenous state of the economy and a measure of the financial wealth distribution. The detailed proofs are in Cao (2010). I also show an numerical algorithm to compute the equilibrium in Subsection A.3.

A.1 The state space and definition

Let ω_t denote the normalized financial wealth distribution and Ω denote the (H-1)-dimensional simplex in Subsection 3.3. I show that, under the conditions detailed in Subsection A.2 below, there exists a Markov equilibrium over the compact space $\mathcal{S} \times \Omega$ defined in Subsection 3.3.

In particular, for each $(s_t, \omega_t) \in \mathcal{S} \times \Omega$, we need to find a vector of prices and allocation

$$\nu_t \in \widehat{\mathcal{V}} = \mathbf{R}_+^H \times \mathbf{R}_+^H \times \mathbf{R}^{JH} \times \mathbf{R}_+ \times \mathbf{R}_+ \quad (13)$$

that consists of the consumers' decisions: consumption of each consumer $(c^h(\sigma))_{h \in \mathcal{H}} \in \mathbf{R}_+^H$, real and bond holdings $(\theta^h(\sigma), \phi^h(\sigma))_{h \in \mathcal{H}} \in \mathbf{R}_+^H \times \mathbf{R}^H$, the price of the real asset $q(\sigma) \in \mathbf{R}_+$ and the price of bonds $p(\sigma) \in \mathbf{R}_+$ must satisfy the market clearing conditions and the budget constraint of consumers bind. Moreover, for each future state $s_{t+1}^+ \in \mathcal{S}$ succeeding s_t , we need to find a corresponding normalized financial wealth distribution ω_{t+1}^+ and equilibrium allocation and prices $\nu_{t+1}^+ \in \widehat{\mathcal{V}}$ such that for each household $h \in \mathcal{H}$ the following conditions hold:

a) For each $s^+ \in \mathcal{S}$ succeeding s

$$\omega_s^{h+} = \frac{\theta^h(q_s^+ + d_s^+) + \phi^h}{q^+ + d^+}. \quad (14)$$

b) There exist multipliers $\mu^h \in \mathbf{R}^+$ corresponding to collateral constraints such that

$$\begin{aligned} 0 &= \mu^h (1 - m) \theta^h \min(q^+ + d^+) - qU'_h(c^h) + \beta E^h \{(q^+ + d^+) U'_h(c^{h+})\} \\ 0 &= \mu^h - pU'_h(c^h) + \beta E^h \{U'_h(c^{h+})\} \\ 0 &= \mu^h (\phi^h + (1 - m) \theta^h \min(q^+ + d^+)) \\ 0 &\leq \phi^h + (1 - m) \theta^h \min(q^+ + d^+). \end{aligned} \quad (15)$$

c) There exists multipliers $\eta^h \in \mathbf{R}^+$ corresponding to no-short sale constraint such that

such that

$$\begin{aligned} 0 &= \eta^h \theta^h \\ 0 &\leq \theta^h. \end{aligned}$$

Condition a) guarantees that the future normalized wealth distributions are consistent with the current equilibrium decision of the consumers. Conditions b) are the first-order conditions with respect to real asset and bond holdings from the maximization problem (2) of the consumers. Given that the maximization problem is convex, the first-order conditions are sufficient for a maximum.

Before continue, let me briefly discuss the prices of real asset and bond in a Markov equilibrium. We can rewrite the first-order condition with respect to real asset holding (15) as

$$\begin{aligned} qU'_h(c^h) &= \mu^h (1 - m) \theta^h \min(q^+ + d^+) + \beta E^h \{(q^+ + d^+) U'_h(c^{h+})\} \\ &\geq \beta E^h \{(q^+ + d^+) U'_h(c^{h+})\}. \end{aligned}$$

By re-iterating this inequality we obtain

$$q_t \geq E_t^h \left\{ \sum_{r=1}^{\infty} \beta^r d_{t+r} \frac{U'_h(c_{t+r}^h)}{U'_h(c_t^h)} \right\}.$$

We have a strict inequality if there is a strict inequality $\mu_{t+r}^h > 0$ in the future. So the asset price is higher than the present discounted value of the stream of its dividend because in future it can be sold to other agents, as in Harrison and Kreps (1978) or it can be used as collateral to borrow as in Fostel and Geanakoplos (2008). In Cao (2010), I show that under belief heterogeneity collateral constraints will eventually be binding for every agent when they strictly differ in their belief. As a result, the price of the real asset is strictly higher than the present discounted value of its dividend.³⁸

Equation (15) also shows that the real asset will have collateral value when some $\mu^h > 0$, in addition to the asset's traditional pay-off value weighted at the appropriate discount factors. Unlike in Alvarez and Jermann (2000), attempts to find a pricing kernel which prices assets using their pay-off value might prove fruitless because assets with the same payoffs but different collateral values will have different prices. This point is also emphasized in Geanakoplos' papers. Following Fostel and Geanakoplos (2008), I define the collateral value

³⁸We can also derive a formula for the equity premium that depends on the multipliers μ similar to the equity premium formula in Mendoza (2010)

of the real asset (as a proportion of the total value of the asset) by the cross-sectional expected value

$$CV_h = \mathbf{E}^h \left[\frac{\mu^h (1 - m) \theta^h \min(q^+ + d^+)}{qU'_h(c^h)} \right] \quad (16)$$

in the long run stationary distribution. Online appendix E offers an estimate of this value in a calibration for the U.S. economy.

A.2 Existence and Properties of Markov equilibrium

The existence proof is similar to the ones in Kubler and Schmedders (2003) and Magill and Quinzii (1994). We approximate the Markov equilibrium by a sequence of equilibria in finite horizon. There are three steps in the proof. First, using Kakutani fixed point theorem to prove the existence proof of the truncated T-period economy. Second, show that all endogenous variables are bounded. And lastly, show that the limit as T goes to infinity is the equilibrium of the infinite horizon economy.

For the second step, we need the following assumption:

Assumption 4 *There exist $\bar{c}, \underline{c} > 0$ such that³⁹*

$$\begin{aligned} & U_h(\underline{c}) + \max \left\{ \frac{\beta}{1-\beta} U_h(\bar{c}), 0 \right\} \\ \leq & \min \left\{ \frac{1}{1-\beta} \min_{s \in \mathcal{S}} U_h(\underline{e}), \min_{s \in \mathcal{S}} U_h(\underline{e}) \right\} \quad \forall h \in \mathcal{H}. \end{aligned}$$

and

$$\begin{aligned} & U_h(\bar{c}) + \min \left\{ \frac{\beta}{1-\beta} U_h(\underline{e}), 0 \right\} \\ \geq & \max \left\{ \frac{1}{1-\beta} U_h(\bar{c}), U_h(\bar{c}) \right\} \quad \forall h \in \mathcal{H}. \end{aligned}$$

The intuition for first inequality is detailed in the existence proof in Cao (2010); it ensures a lower bound for consumption. The second inequality ensures that price of the real asset is bounded from above. Both inequalities are obviously satisfied by CRRA functions with CRRA coefficients exceeding 1.

Lemma 1 *Suppose that Assumption 4 is satisfied. Then there is a compact set that contains the equilibrium endogenous variables constructed in Cao (2010) for every T and every initial condition lying inside the set.*

³⁹This assumption is slightly different from the one in Kubler and Schmedders (2003) as U_h might be negative, for example with log utility.

Proof. Cao (2010) ■

Theorem 1 *Under the same conditions, a Markov equilibrium exists.*

Proof. In Cao (2010), I show the existence of Markov equilibria for a general model also with capital accumulation. As in Kubler and Schmedders (2003), we extract a limit from the T-finite horizon equilibria. Lemma 1 guarantees that equilibrium prices and quantities are bounded as T goes to infinity. The proof follows the Appendix of Kubler and Schmedders (2003) but uses an alternative definition of attainable sets in order to accommodate production. ■

A.3 Numerical Method

The construction of Markov equilibria in the last section also suggests an algorithm to compute them. The following algorithm is based on Kubler and Schmedders (2003). There is one important difference between the algorithm here and the original algorithm. The future normalized financial wealth distributions are included in the current mapping ρ instead of solving for them using sub-fixed-point loops. This innovation reduces significantly the computing time, given that solving for a fixed-point is time consuming in MATLAB. Moreover, in section 4, as agents face the borrowing constraint (4), we need to know the future prices of the real asset. As we know the future normalized financial wealth distributions, these future prices can be computed easily.

As in the existence proof, we look for the following correspondence

$$\begin{aligned} \rho : \mathcal{S} \times \Omega &\longrightarrow \widehat{\mathcal{V}} \times \Omega^S \times \mathcal{L} \\ (s, \omega) &\longmapsto (\widehat{v}, \omega_s^+, \mu, \eta) \end{aligned}$$

$\widehat{v} \in \widehat{\mathcal{V}}$ is the set of endogenous variables excluding the normalized financial wealth distribution as defined in (13). $(\omega_s^+)_{s \in \mathcal{S}}$ are normalized financial the wealth distributions in the S future states and $(\mu, \eta) \in \mathcal{L}$ are Lagrange multipliers as defined in subsection A.1.

From a given continuous initial mapping $\rho^0 = (\rho_1^0, \rho_2^0, \dots, \rho_S^0)$, we construct the sequence of mappings $\{\rho^n = (\rho_1^n, \rho_2^n, \dots, \rho_S^n)\}_{n=0}^\infty$ by induction. Suppose we have obtained ρ^n , for each state variable (s, ω) , we look for

$$\rho_s^{n+1}(\omega) = (\widehat{v}_{n+1}, \omega_{s,n+1}^+, \mu_{n+1}, \eta_{n+1})$$

that solves the first-order conditions (15), market clearing conditions, and the consistency of the future normalized financial wealth distribution (14).

We construct the sequence $\{\rho^n\}_{n=0}^\infty$ on a finite discretization of $\mathcal{S} \times \Omega$. So from ρ^n to ρ^{n+1} , we will have to extrapolate the values of ρ^n to outside the grid using extrapolation methods in MATLAB. Fixing a precision δ , the algorithm stops when $\|\rho^{n+1} - \rho^n\| < \delta$.

There are two important details in implementing this algorithm: First, in order to calculate the $(n + 1)$ -th mapping ρ^{n+1} from the n -th mapping, we need to only keep track of the consumption decisions c^h and asset prices q and p_j . Even though other asset holding decisions and Lagrange multipliers might not be differentiable functions of the normalized financial wealth distribution, the consumption decisions and asset prices normally are.⁴⁰ Relatedly, when there are redundant assets, there might be multiple asset holdings that implement the same consumption policies and asset prices.⁴¹ Second, if we choose the initial mapping ρ^0 as an equilibrium of the 1-period economy as in Subsection A.2, then ρ^n corresponds to an equilibrium of the $(n + 1)$ -period economy. I follow this choice in computing an equilibrium of the two agent economy presented below.

B No non-financial wealth

In this section, I show the importance of non-financial wealth for the dynamics of financial wealth distribution and asset prices by studying a model in Section 4 in the absence of non-financial wealth. Specifically, the budget constraints of the agents become

$$c_t^h + q_t \theta_{t+1}^h + p_t \phi_{t+1}^h \leq (q_t + d_t) \theta_t^h + \phi_t^h$$

and under the collateral constraint (4). When the margin $m = 1$, agents trade only in the real asset. So the budget constraints of the agents become

$$c_t^h + q_t \theta_{t+1}^h \leq (q_t + d_t) \theta_t^h.$$

Proposition 3 *Suppose that agents have log utility and agents trade only in the real asset, i.e. $m = 1$. Regardless of the beliefs of the agents, the equilibrium price of the real asset is*

$$q_t = \frac{\beta}{1 - \beta} d_t. \tag{17}$$

and the asset holdings of the agents do not change over time, i.e. $\theta_{t+1}^h = \theta_t^h$ for all t, s^t , and $h \in \mathcal{H}$.

⁴⁰See Brumm and Grill (2010) for an algorithm with adapted grid points that deals directly with non-differentiabilities in the policy functions.

⁴¹See Cao, Chen, and Scott (2011) for such an example.

Proof. Given log utility

$$c_t^h = (1 - \beta)(q_t + d_t)\theta_t^h. \quad (18)$$

So the market clearing condition in the consumption-market at time t implies that

$$\begin{aligned} d_t &= \sum_{h \in \mathcal{H}} c_t^h \\ &= (1 - \beta)(q_t + d_t) \sum_{h \in \mathcal{H}} \theta_t^h \\ &= (1 - \beta)(q_t + d_t). \end{aligned}$$

Thus we have the expression (17) for the price of the real asset. Plug this price into the budget constraint of the agents and use the solution for consumption (18), we obtain the constant asset holdings of the agents over time. ■

This proposition shows that the *prosper by speculation* mechanism in Subsection 4.2 is driven by the non-financial wealth of the agents in the economy.

However when agents are allowed to trade also in bonds then the standard disappearance result applies.

Proposition 4 *Suppose that agents are allowed use the real asset as collateral to borrow, i.e. $m = 0$ and in each history s^t , there are only two possible future exogenous states s_{t+1} . Then only the agents with correct beliefs survive, i.e. almost surely:*

$$\lim_{t \rightarrow \infty} c_t^h = 0.$$

and

$$\lim_{t \rightarrow \infty} \omega_t^h = 0$$

for all $h \in \mathcal{H}$ and $P^h \neq P$, where P is the correct belief and some agent has the correct belief.

Proof. *Financial markets are dynamically complete so this proposition is a direct application of Sandroni (2000).* ■

C Robustness checks

In this Appendix, I show that the numerical results in Subsection 4 do not change even if agents differ less in their beliefs and dividends change less across exogenous states. More

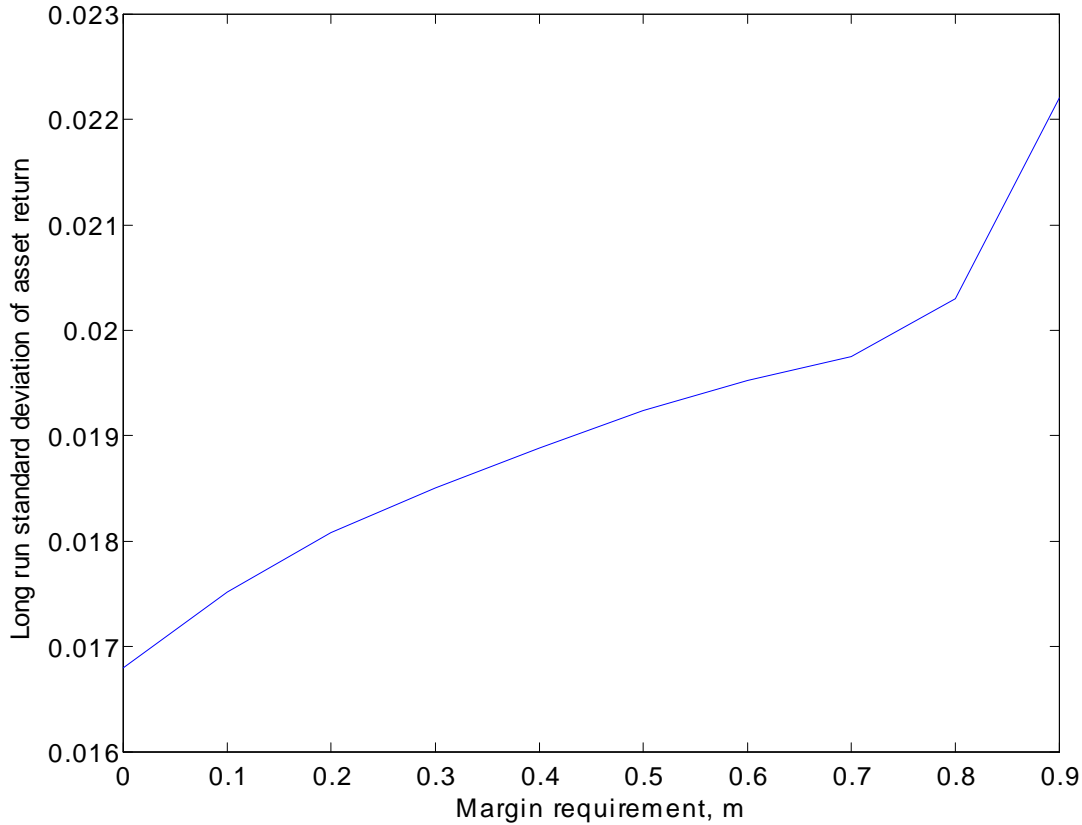


Figure 10: Robustness check - Asset price volatility and margin requirement

specifically, I replicate the analysis in that Subsection for the following set of parameters:

$$\begin{aligned}
 \beta &= 0.95 \\
 d(G) &= 1 > d(B) = 0.5 \\
 U(c) &= \log(c) \\
 \pi^O &= 0.6 > \pi^P = 0.5 \\
 e^P &= \begin{bmatrix} 100 & 100.5 \end{bmatrix} \\
 e^O &= \begin{bmatrix} 3 & 3 \end{bmatrix}.
 \end{aligned}$$

Figure 10 is the counterpart of Figure 5 for the current set of parameters. The increasing relationship between the long run asset return volatility and collateral requirement holds similarly.

Figure 11 is the counterpart of Figure 9 for the current set of parameters. Again, the stationary mean level of financial wealth of the optimists is increasing in m .

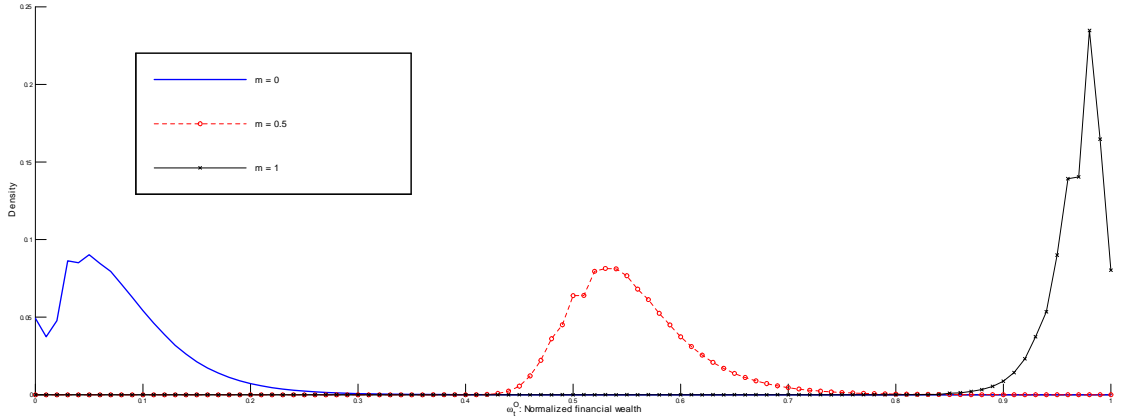


Figure 11: Robustness check - Stationary distributions under different market structures

D Collateral constrained equilibrium and collateral equilibrium

In this appendix, I show that the collateral constrained equilibrium defined in Section 3.1 is the same as the collateral equilibrium in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003) under the condition that in each history there are only two possible exogeneous future states. As in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003), I assume that agents have access to a wide range of financial assets instead of only one collateralized bond in Section 3.1.

Financial asset: In each history s^t , there are also (collateralized) financial assets, $j \in \mathcal{J}_t(s^t)$. $\mathcal{J}_t(s^t)$ have a finite number of elements, $J_t(s^t)$. Each financial asset j (or financial security) is characterized by a vector and a scalar, (b_j, k_j) , of promised payoffs and collateral requirement. Promises are a standard feature of financial asset similar to Arrow's securities, i.e., asset j traded in history s^t promises next-period pay-off $b_j(s_{t+1}) > 0$ in terms of final good at the successor history $s^{t+1} = (s^t, s_{t+1})$. The non-standard feature is the collateral requirement. Agents can only sell the financial asset j if they hold shares of the real asset as collateral. If an agent sells one unit of security j , she is required to hold k_j units of the real asset as collateral.⁴²

There are no penalties for default except for the seizure of collateral. Thus, a seller of the financial asset can default in history s^{t+1} whenever the total value of collateral falls below the promise at that state at the cost of losing the collateral. By individual rationality, the

⁴²Notice that, there are only one-period ahead financial assets. See He and Xiong (2011) for a motivation why longer term collateralized financial assets are not used in equilibrium.

effective pay-off of security j in history s^t is always given by

$$f_j(s^t, s_{t+1}) = \min \{b_j(s_{t+1}), (q(s^{t+1}) + d(s_{t+1})) k_j\}. \quad (19)$$

Let $p_{j,t}(s^t)$ denote the price of security j in history s^t .

Assumption 5 *Each financial asset requires a strictly positive level of collateral*

$$\min_{j \in \mathcal{J}_t(s^t)} k_j > 0$$

If a financial asset j requires no collateral then its effective pay-off, determined by (19), will be zero. Hence it will be easy to show that in equilibrium its price, p_j , will be zero as well. We can thus ignore these financial assets.

Remark 2 *Selling one unit of financial asset j is equivalent to purchasing k_j units of the real asset and at the same time pledging these units as collateral to borrow $p_{j,t}$. It is shown in Cao (2010) that*

$$k_j q_t - p_{j,t} > 0,$$

which is to say that the seller of the financial asset always has to pay some margin. So the decision to sell financial asset j using the real asset as collateral corresponds to the desire to invest in the real asset at margin rather than the simple desire to borrow.

Remark 3 *We can then define the leverage ratio associated with the transaction as*

$$L_{j,t} = \frac{k_j q_t}{k_j q_t - p_{j,t}} = \frac{q_t}{q_t - \frac{p_{j,t}}{k_j}}. \quad (20)$$

Even though there are many financial assets available, in equilibrium only some financial asset will be actively traded, which in turn determines which leverage levels prevail in the economy in equilibrium. In this sense, both asset price and leverage are simultaneously determined in equilibrium, as emphasized in Geanakoplos (2009). Online appendix F uses this definition of leverage to investigate how leverage varies over the business cycles.

Consumer h takes the sequences of prices $\{q_t, p_{j,t}\}$ as given and solves

$$\max_{\{c_t^h, \theta_{t+1}^h, \phi_{j,t+1}^h\}} \mathbf{E}_0^h \left[\sum_{t=0}^{\infty} \beta^t U_h(c_t^h) \right] \quad (21)$$

subject to the budget constraint

$$c_t^h + q_t \theta_{t+1}^h + \sum_{j \in \mathcal{J}_t} p_{j,t} \phi_{j,t+1}^h \leq e_t^h + \sum_{j \in \mathcal{J}_{t-1}} f_{j,t} \phi_{j,t}^h + (q_t + d_t) \theta_t^h \quad (22)$$

and the collateral constraint

$$\theta_{t+1}^h + \sum_{j \in \mathcal{J}_t: \phi_{j,t}^h < 0} \phi_{j,t+1}^h k_j \geq 0 \quad (23)$$

The collateral constraint (23) implies the no-short sale constraint. I define a collateral equilibrium as in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003):

Definition 3 *A collateral equilibrium for an economy with initial asset holdings*

$$\{\theta_0^h\}_{h \in \{1, 2, \dots, H\}}$$

and initial shock s_0 is a collection of consumption, real and financial asset holdings and prices in each history s^t ,

$$\left(\{c_t^h(s^t), \theta_{t+1}^h(s^t), \phi_{j,t+1}^h(s^t)\}_{h \in \mathcal{H}} \right. \\ \left. q_t(s^t), \{p_{j,t}(s^t)\}_{j \in \mathcal{J}_t(s^t)} \right)$$

satisfying the following conditions:

i) The markets for final good, real and financial assets in each period clear:

$$\begin{aligned} \sum_{h \in \mathcal{H}} \theta_{t+1}^h(s^t) &= 1 \\ \sum_{h \in \mathcal{H}} \phi_{j,t+1}^h(s^t) &= 0 \quad \forall j \in \mathcal{J}_t(s^t) \\ \sum_{h \in \mathcal{H}} c_t^h(s^t) &= \sum_{h \in \mathcal{H}} e^h(s_t) + d(s_t) \end{aligned}$$

ii) For each consumer h , $\{c_t^h(s^t), \theta_{t+1}^h(s^t), (\phi_{j,t+1}^h(s^t))_{j \in \mathcal{J}_t(s^t)}\}$ solves the individual maximization problem subject to the budget constraint (3), and the collateral constraint (4).

Now, to study the standard debt contracts, I consider the sets \mathcal{J}_t of financial assets which promise state-independent payoffs in the next period. I normalize these promises to $b_j = 1$.

Asset j also requires k_j units of the real asset as collateral. The effective pay-off is therefore

$$f_{j,t+1}(s^{t+1}) = \min \{1, k_j (q(s^{t+1}) + d(s^{t+1}))\}. \quad (24)$$

Due to the finite supply of the real asset, in equilibrium only a subset of the financial assets in \mathcal{J}_t are traded. It turns out that in some special cases we can determine exactly which financial assets are traded. For example, Fostel and Geanakoplos (2008) and Geanakoplos (2009) argue that if we allow for the set \mathcal{J}_t to be dense enough, then in equilibrium the only financial asset traded in equilibrium is the one with the minimum collateral level to avoid default. This statement also applies to the infinite-horizon set-up in this paper under the condition that in any history, there are only two succeeding future exogenous states. I formalize the result in Proposition 5 below. The proposition uses the following definition

Definition 4 *Two collateral equilibria are equivalent if they have the same allocation of consumption to the consumers and the same prices of real and financial assets. The equilibria might differ in the consumers' portfolios of real and financial assets.*

Proposition 5 *Consider a collateral equilibrium and suppose in a history s^t , there are only two possible future exogenous states s_{t+1} . Let*

$$\begin{aligned} u_t &= \max_{s^{t+1}|s^t} (q(s^{t+1}) + d(s_{t+1})) \\ d_t &= \min_{s^{t+1}|s^t} (q(s^{t+1}) + d(s_{t+1})) \end{aligned}$$

and

$$k_t^* = \frac{1}{d_t}.$$

We can find an equivalent collateral equilibrium such that only the financial assets with the collateral requirements

$$\max_{j \in \mathcal{J}_t, k_j \leq k_t^*} k_j \text{ and } \min_{j \in \mathcal{J}_t, k_j \geq k_t^*} k_j$$

are actively traded. In particular, when $k_t^ \in \mathcal{J}_t$, we can always find an equivalent equilibrium in which only financial asset with the collateral requirement exactly equal to k_t^* is traded.*

Proof. Intuitively, given only two future states, using two assets - a financial asset and the real asset - we can effectively replicate the pay-off of all other financial assets. But we need to make sure that the collateral constraints, including the no short-selling constraint, are satisfied in the replications. To simplify the notations, I will suppress the subscripts t from the variables. Let $k_d = \max_{j \in \mathcal{J}, k_j \leq k^*} k_j$. Let p_d denote the price of the financial assets with

collateral requirement k_d . Suppose that there is another financial asset in \mathcal{J} that is actively traded and have collateral requirement $k \leq k^*$. Then by definition $k < k_d$. Let p_k denote the price of the financial asset. We have two cases:

Case 1) $\frac{1}{d} \geq k_d > k > \frac{1}{u}$: Consider the optimal portfolio choice of a seller of financial asset k . The pay-off from selling the asset is $(ku - 1, 0)$ and she has to pay $kq - p_k$ in cash: she buys k units of the real asset but she get p_k from selling the financial asset. So the return on the financial asset is $\frac{ku-1}{kq-p_k}$ (when good state realizes next period). Similarly, the return from selling financial asset k_d is $\frac{k_d u - 1}{k_d q - p_d}$. If there are sellers for financial asset k , it then implies that

$$\frac{ku - 1}{kq - p_k} \geq \frac{k_d u - 1}{k_d q - p_d}.$$

or equivalently

$$p_k \geq \frac{ku - 1}{k_d u - 1} p_d + \frac{k_d - k}{k_d u - 1} q, \quad (25)$$

otherwise, sellers will strictly prefer selling financial asset k_d to financial asset k . Now from the perspective of the buyers of financial assets, the pay-off of financial asset k is $(1, kd)$. We can write this pay-off as a portfolio of financial asset k_d and the real asset:

$$\begin{bmatrix} 1 \\ kd \end{bmatrix} = \frac{ku - 1}{k_d u - 1} \begin{bmatrix} 1 \\ k_d d \end{bmatrix} + \frac{k_d - k}{k_d u - 1} \begin{bmatrix} u \\ d \end{bmatrix}. \quad (26)$$

As a result, if there are buyers for financial asset k , we must have

$$p_k \leq \frac{ku - 1}{k_d u - 1} p_d + \frac{k_d - k}{k_d u - 1} q \quad (27)$$

otherwise the buyers will buy the portfolio (26) of financial asset k_d and the real asset instead. Thus we have both (25) and (27) happen with equality if financial asset k is actively traded. Armed with the equality, we can now prove the proposition. Consider each pair of seller and buyer of a unit of financial asset k : the buyer buys one unit, and the seller who sells one unit of financial asset k , at the same time is required to buy k units of the real asset. We alter the their portfolios in the following way: instead of buying one unit of financial asset k , the buyer buys $\frac{ku-1}{k_d u - 1}$ units of financial asset k_d from the seller and $\frac{k_d - k}{k_d u - 1}$ of the real asset. Given (26) and the equality (27), this changes in portfolio let the consumption and future wealth of the buyer unchanged. Now the seller instead of selling one unit of financial asset k , sells $\frac{ku-1}{k_d u - 1}$ units of financial asset k_d and holds $\frac{ku-1}{k_d u - 1} k_d$ units of the real asset as collateral. Similarly, due to the equality (25), this transaction costs the same and yields the same returns to the seller compared to selling one unit of financial asset k . So the consumption and the future

wealth of the seller remains unchanged. Now we just need to verify that the total quantity of the real asset used remain unchanged. Indeed this is true because

$$\frac{k_d - k}{k_d u - 1} + \frac{k u - 1}{k_d u - 1} k_d = k.$$

Case 2) $\frac{1}{u} \geq k$: Financial asset k 's pay-off to the buyers is $k(u, d)$ and to the seller is 0. So the financial asset has exactly the same payoffs as the real asset. This implies, $p_k = kq$. The proposition follows immediately.

Now let $k_u = \min_{j \in \mathcal{J}, k_j \geq k^*} k_j$. Let p_u denote the price of the financial asset k_u . The proof of the proposition is similar. First, we show that the price of any actively traded financial asset k with $k \geq k^*$ is $p_k = p_u$ and we can alter the portfolio of the buyers and sellers of financial asset k to transfer all the trade in the financial asset to financial asset k_u . ■

Unless specified otherwise, I suppose that the set of financial assets, \mathcal{J}_t , includes all collateral requirements $k \in \mathbf{R}^+, k > 0$.⁴³ Proposition 5 implies that, for any collateral equilibrium, we can find an equivalent collateral equilibrium in which the only financial asset with the collateral requirement exactly equal to $k^*(s^t)$ is traded in equilibrium. Therefore in such an equilibrium the only actively traded financial asset is riskless to its buyers. Let $p(s^t)$ denote the price of this financial asset. The endogenous interest rate is therefore $r(s^t) = \frac{1}{p(s^t)} - 1$.^{44,45}

We can also establish the following corollary which corresponds to Proposition 1:

Corollary 1 (Formalization of Proposition 1) *When \mathcal{J}_t includes all collateral requirements $k \in \mathbf{R}^+, k > 0$, a collateral equilibrium is equivalent to a collateral constrained equilibrium in which agents can borrow ϕ_{t+1}^h but are subject to the collateral constraint (4) with $m = 0$:*

$$\phi_{t+1}^h + \theta_{t+1}^h \min_{s^{t+1}|s^t} (q(s^{t+1}) + d(s_{t+1})) \geq 0. \quad (28)$$

Proof. Proposition 5 shows that we can find an equivalent collateral equilibrium in which only financial asset with collateral level k_t^* is traded. The collateral constraint (23) at this collateral level can be re-written as the collateral constraint (28). ■

⁴³To apply the existence theorem 1 I need \mathcal{J} to be finite. But we can think of \mathcal{J} as a fine enough grid.

⁴⁴The uniqueness of actively traded financial assets established in Geanakoplos (2009) and He and Xiong (2011) is in the "equivalent" sense in Definition 4.

⁴⁵In a two-period economy, Araujo et al. (2012) show that if there are S future states, for any collateral equilibrium, we can find an equivalent collateral equilibrium in which only $S - 1$ collateralized bonds are actively traded. This Proposition can be extended for the infinite-horizon economy in this paper as in Propostion 5.

By assuming that lenders can seize only the real asset but not the current dividend in (19), we can also show that - under the same conditions as in Proposition 5 - the collateral equilibrium is equivalent to a collateral constrained economy in which the agents face the collateral constraint

$$\phi_{t+1}^h + \theta_{t+1}^h \min_{s^{t+1}|s^t} q(s^{t+1}) \geq 0.$$

This is constraint used in Kiyotaki and Moore (1997). The authors microfound this condition by assuming that human capital is inalienable. This paper thus provides an alternative microfoundation using the endogeneity of the set of actively traded financial assets.⁴⁶

By restricting the set \mathcal{J}_t , we can also find a collateral equilibrium equivalent to a collateral constrained equilibrium in which the margin $m \in (0, 1)$.

⁴⁶Mendoza (2010) and Kocherlakota (2000) use a similar collateral constraint but $\min_{s^{t+1}|s^t} (q(s^{t+1}) + d(s_{t+1}))$ is replaced by $q(s^t)$. I have not seen a microfoundation for such a collateral constraint but the quantitative implications of the constraint should be similar if asset prices do not vary too much across immediate future histories.

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Speculation and Financial Wealth Distribution under Belief Heterogeneity: Online Appendix

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February 19, 2014

Abstract

In Appendix E, I apply the model in the main paper to the U.S. economy using the parameters estimated in Heaton and Lucas (1995). I find that 1) the equity premium is higher under binding collateral constraints than under non-binding collateral constraints; 2) due to risk-aversion, the equilibrium portfolio choice of the agents is such that the collateral constraints are not often binding in the stationary distribution; 3) consequently, the unconditional moments of asset prices are not very different from the unconditional moments when there are no collateral constraints. These findings are consistent with Mendoza (2010). In Appendix F, I make changes to the structure of the shock in Section 4 in order to match the pattern of leverage over the business cycles.

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1 Online Appendix E: Quantitative assessment

In this Appendix, I apply the numerical solution method to a more seriously calibrated setup used in Heaton and Lucas (1995). In order to do so, I need to modify the economy in Section 3 to allow for the possibility that aggregate endowment grows overtime. As in Heaton and Lucas (1995), I assume that the aggregate endowment $\bar{e}(s^t)$ evolves according to the process

$$\frac{\bar{e}(s^{t+1})}{\bar{e}(s^t)} = 1 + g(s^t).$$

There is only one Lucas tree that pays off the aggregate dividend income at time, D_t and

$$\delta(s^t) = \frac{D(s^t)}{\bar{e}(s^t)},$$

the remaining endowment is distributed among the consumers under the form of labor income

$$\sum_{h \in \mathcal{H}} e^h(s^t) = \bar{e}(s^t) - D(s^t).$$

Consumer h 's labor income as a fraction of aggregate labor income is given by

$$\eta^h(s^t) = \frac{e^h(s^t)}{\sum_{h \in \mathcal{H}} e^h(s^t)}.$$

The following Proposition shows that, by working with the normalized variables, $\hat{c}_t^h = \frac{c_t^h}{\bar{e}_t}$, $\hat{e}_t^h = \frac{e_t^h}{\bar{e}_t}$, $\hat{d}_t = \frac{d_t}{\bar{e}_t}$, and $\hat{q}_t = \frac{q_t}{\bar{e}_t}$, we can find and compute collateral constrained equilibria in which these normalized variables depend only on the normalized financial wealth distribution.

Proposition 6. Suppose that the per-period utility functions are CRRA $U_h(c) = \frac{c^{1-\sigma_h}}{1-\sigma_h}$, and (g, δ, η) depend only on the current state of the economy. There exists a collateral constrained equilibrium in this growth economy. ■

Proof. To apply the existence proof and the numerical method used in Subsection 3.3, I use the following normalized variables

$$\hat{c}_t^h = \frac{c_t^h}{\bar{e}_t}, \hat{e}_t^h = \frac{e_t^h}{\bar{e}_t}, \hat{d}_t = \frac{d_t}{\bar{e}_t},$$

and

$$\hat{q}_t = \frac{q_t}{\bar{e}_t}.$$

Assuming CRRA for the agents, $U_h(c) = \frac{c^{1-\sigma_h}}{1-\sigma_h}$, the expected utility can also be re-written

using the normalized variables

$$\begin{aligned} \mathbf{E}_t^h \left[\sum_{r=0}^{\infty} \beta^r U_h(c_{t+r}^h) \right] &= (\bar{e}_t)^{1-\sigma_h} \mathbf{E}_t^h \left[\sum_{r=0}^{\infty} \beta^r U_h(\hat{c}_{t+r}^h) \left(\frac{\bar{e}_{t+r}}{\bar{e}_t} \right)^{1-\sigma_h} \right] \\ &= (\bar{e}_t)^{1-\sigma_h} \mathbf{E}_t^h \left[\sum_{r=0}^{\infty} \beta^r \prod_{r'=1}^r (1 + g(s^{t+r'}))^{1-\sigma_h} U_h(\hat{c}_{t+r}^h) \right] \end{aligned}$$

These collateral constraints is

$$\phi_{t+1}^h + (1 - m) \theta_{t+1}^h \min_{s^{t+1}|s^t} \{q(s^{t+1}) + d(s^{t+1})\} \geq 0,$$

where $m \in [0, 1]$ and when $m = 1$, no borrowing is possible, the agents are allowed to only trade on the real asset. In these environment, I also use the normalized variables for the choice of debt holding $\hat{\phi}_t^h = \frac{\phi_t^h}{\bar{e}_t}$.

We rewrite the optimization of the consumer as

$$\max_{\{c_t^h, \theta_{t+1}^h, \phi_{t+1}^h\}} \mathbf{E}_0^h \left[\sum_{t=0}^{\infty} \beta^t U_h(\hat{c}_t^h) \prod_{r=1}^t (1 + g(s^r))^{1-\sigma_h} \right]$$

and in each history s^t , she is subject to the budget constraint

$$\hat{c}_t^h + \hat{q}_t \theta_{t+1}^h + \hat{\phi}_{t+1}^h \leq \hat{e}_t^h + \frac{1}{1 + g(s^t)} \hat{\phi}_t^h + (\hat{q}_t + \hat{d}_t) \theta_t^h,$$

the collateral constraints

$$\hat{\phi}_{t+1}^h + (1 - m) \theta_{t+1}^h \min_{s^{t+1}|s^t} \left\{ \left(\hat{q}(s^{t+1}) + \hat{d}(s^{t+1}) \right) (1 + g(s^{t+1})) \right\} \geq 0 \quad (29)$$

and finally the no short-sale constraint in the real asset, $\theta_{t+1}^h \geq 0$, as before. We can just use the same analysis in Subsection 3.3 for this economy with the hat variables. ■

The following subsections show that, under reasonable calibrated parameters of the utility functions and exogenous shock processes, 1) the moments of asset prices such as equity premium behave differently under binding collateral constraints than under non binding collateral constraints; 2) due to risk-aversion, the equilibrium portfolio choice of the agents are such that the collateral constraints are not binding often in the stationary distribution; 3) consequently, the unconditional moments of asset prices are not very different from the unconditional moments when there are no collateral constraints. These findings hold both with and without belief heterogeneity. These findings are also consistent with the ones in

Mendoza (2010), in which the author finds that "precautionary saving makes sudden stops low probability events nested within normal cycles, as observed in the data."

1.1 Homogeneous beliefs

Table 1 corresponds to Table in Heaton and Lucas (1995). The authors use the annual aggregate labor income and dividend data from NIPA and individual income from PSID to calibrate $g(\cdot)$, $\delta(\cdot)$, $\eta(\cdot)$ and the transition matrix π . There are two representative agents in the economy and η corresponds to the endowment share of agent 1. In Panel A, the growth rate of the economy, g , fluctuates between 0.9904 and 1.0470, and the share of tradable income, δ , stays around 15% of the total endowment. Given the importance of income heterogeneity shown in column 4 of Panel A, I start the analysis with the benchmark in which both agents have the correct estimate of the transition matrix in Panel B. Despite the common belief, income heterogeneity gives the two agent strong incentives to trade with each other.

I use the equilibrium existence and computation procedure developed in Proposition 6 with the calibrated parameters in Table 1. Notice that, the economy in Heaton and Lucas (1995) is the same as in my economy except for the fact that the collateral constraint (29) is replaced by the exogenous borrowing constraint $\phi_t^h \geq -B^h$. Table 2 shows that collateral constraints do not alter significantly the quantitative result in Heaton and Lucas (1995). Despite the collateral constraints, the standard deviations of consumption and stock returns and bond returns are very similar in the economies with collateral constraints to the economy in Heaton and Lucas (1995).¹ The reason for this similarity is that the collateral constraints are not binding often, even though the behavior of the equilibrium variables are very different when the collateral constraints are binding from when they are not. Indeed, Figures 2 and 1 show the equity premium and portfolio choice of agent 1 as functions of her normalized financial wealth, ω_t^1 . When the collateral constraint for this agent is binding, i.e. $\omega_t^1 < 0.1$, the equity premium is significantly higher than it is when the constraint is not binding. However, the stationary distribution of wealth shows that the constraints are binding in less than 5% in the distribution (See column 5, the second to last row of Table 3). In addition, the collateral value as a fraction of the price of the real asset is less 0.2% (See column 5, the last row of Table 3).

¹The last column in Table 2, $m = 1$, corresponds to the economy in which the agents can only trade in the real asset, subject to the no short selling. Thus there are no bonds traded.

A. Markov chain model for exogenous state variables							
State Number	States						
	g	δ			η		
1	0.9904	0.1402			0.3772		
2	1.0470	0.1437			0.3772		
3	0.9904	0.1561			0.3772		
4	1.0470	0.1599			0.3772		
5	0.9904	0.1402			0.6228		
6	1.0470	0.1437			0.6228		
7	0.9904	0.1561			0.6228		
8	1.0470	0.1599			0.6228		
B. Transition probability matrix $[\pi_{ij}]_{8 \times 8}$							
0.3932	0.2245	0.0793	0.0453	0.1365	0.0799	0.0275	0.0157
0.3044	0.3470	0.0425	0.0484	0.1057	0.1205	0.0147	0.0168
0.0484	0.0425	0.3470	0.3044	0.0168	0.0147	0.1205	0.1057
0.0453	0.0793	0.2245	0.3932	0.0157	0.0275	0.0779	0.1365
0.1365	0.0779	0.0275	0.0157	0.3932	0.2245	0.0793	0.0453
0.1057	0.1205	0.0147	0.0168	0.3044	0.3470	0.0425	0.0484
0.0168	0.0147	0.1205	0.1057	0.0484	0.0425	0.3470	0.3044
0.0157	0.0275	0.0779	0.1365	0.0453	0.0793	0.2245	0.3932

Table 1: Summary Statistics for Baseline Case

Moments	Data	CM	HL	$m = 0$	$m = 0.9$	$m = 1$
Consumption Growth						
Average	0.020	0.018	0.018	0.019	0.019	0.019
Standard deviation	0.030	0.028	0.044	0.041	0.045	0.045
Bond return						
Average	0.008	0.080	0.077	0.078	0.077	
Standard deviation	0.026	0.009	0.012	0.012	0.013	
Stock return						
Average	0.089	0.082	0.079	0.079	0.080	0.080
Standard deviation	0.173	0.029	0.032	0.030	0.035	0.036

Table 2: Summary Statistics for Baseline Case

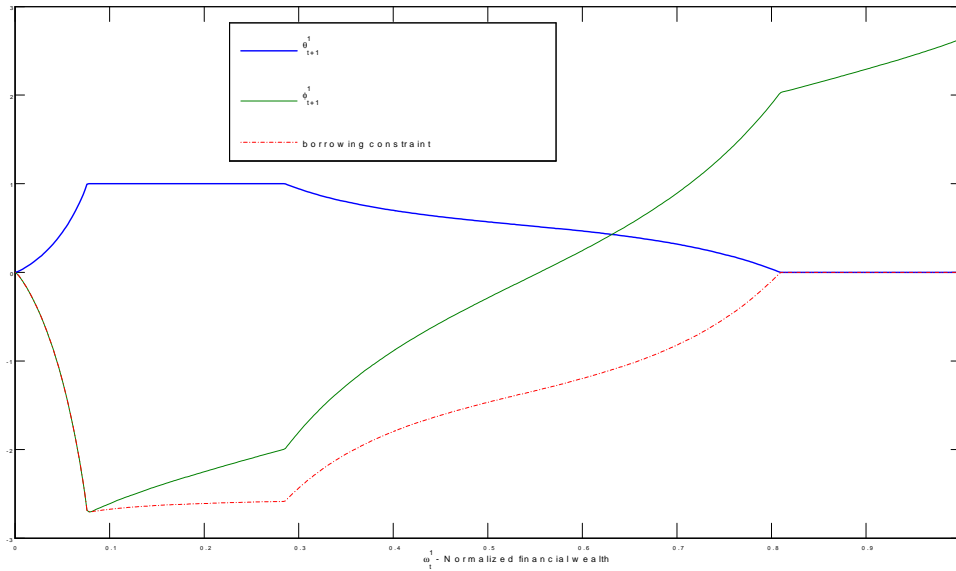


Figure 1: Portfolio choice

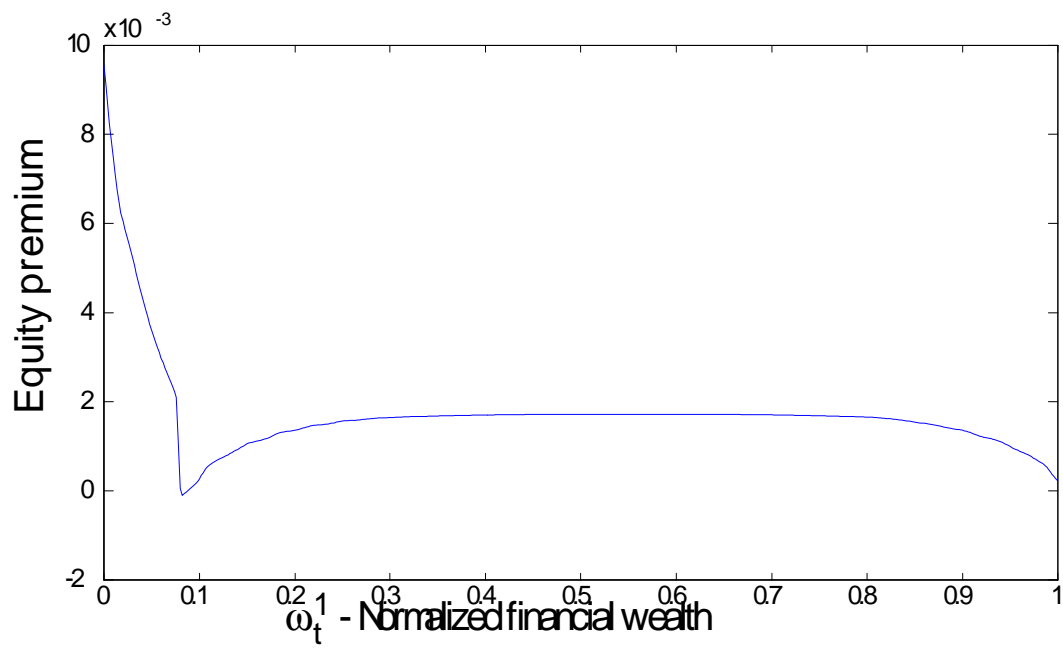


Figure 2: Equity premium

1.2 Belief heterogeneity

Now, I introduce belief heterogeneity by assuming the first agent in Heaton and Lucas (1995) are more optimistic about the growth rate of the economy, while the second agents have the correct estimate of the transition matrix: $\pi^2 = \pi$ and

$$\begin{aligned}\pi^1(1, 1) &= \pi(1, 1) - p \\ \pi^1(1, 2) &= \pi(1, 2) + p \\ \pi^1(3, 3) &= \pi(3, 3) - p \\ \pi^1(3, 4) &= \pi(3, 4) + p \\ \pi^1(5, 5) &= \pi(5, 5) - p \\ \pi^1(5, 6) &= \pi(5, 6) + p \\ \pi^1(7, 7) &= \pi(7, 7) - p \\ \pi^1(7, 8) &= \pi(7, 8) + p,\end{aligned}$$

where $p = 0.3$.

Table 3 shows that the standard deviation of stock returns are still similar to when beliefs homogeneous. The reason is again that the collateral constraints are not binding often due to risk-aversion. Figures 4 and 3 plot the equity premium and portfolio choice of agent 1 as functions of agent 1's normalized financial wealth. Equity premium is very high when the collateral constraint is binding. The last two rows of Table 3 shows the small probabilities of binding collateral constraints and the small collateral value as a fraction of asset price under either agent 1's belief or agent 2's belief.

2 Online Appendix F: Dynamic Leverage Cycles

This appendix uses this definition of leverage in Remark 3 to investigate how leverage varies over the business cycles. Subsections 4.2 through 4.4 capture some realistic behavior of asset price including the *debt deflation* channel, leverage defined in equation (20) is not consistent with what we observe in financial markets: high leverage in good times and low leverage in bad times, as documented in Geanakoplos (2009). In order to generate the procyclicality of leverage, I use the insight from Geanakoplos (2009) regarding aggregate uncertainty: bad news must generate more uncertainty and more disagreement in order to reduce equilibrium leverage significantly. The economy also constantly moves between low uncertainty and high uncertainty regimes. To formalize this idea, I assume that in the good state, $s = G$,

Moments	Data	CM	HL	$m = 0$	$m = 0, \pi^1$	$m = 0, \pi^2$
Consumption Growth						
Average	0.020	0.018	0.018	0.019	0.027	0.020
Standard deviation	0.030	0.028	0.044	0.041	0.048	0.057
Bond return						
Average	0.008	0.080	0.077	0.078	0.084	0.080
Standard deviation	0.026	0.009	0.012	0.012	0.011	0.018
Stock return						
Average	0.089	0.082	0.079	0.079	0.086	0.076
Standard deviation	0.173	0.029	0.032	0.030	0.029	0.033
Collateral constraint						
Pr(binding constraint)				0.042	0.085	0.001
Collateral value				0.002	0.006	0.005

Table 3: Summary Statistics with Belief Heterogeneity

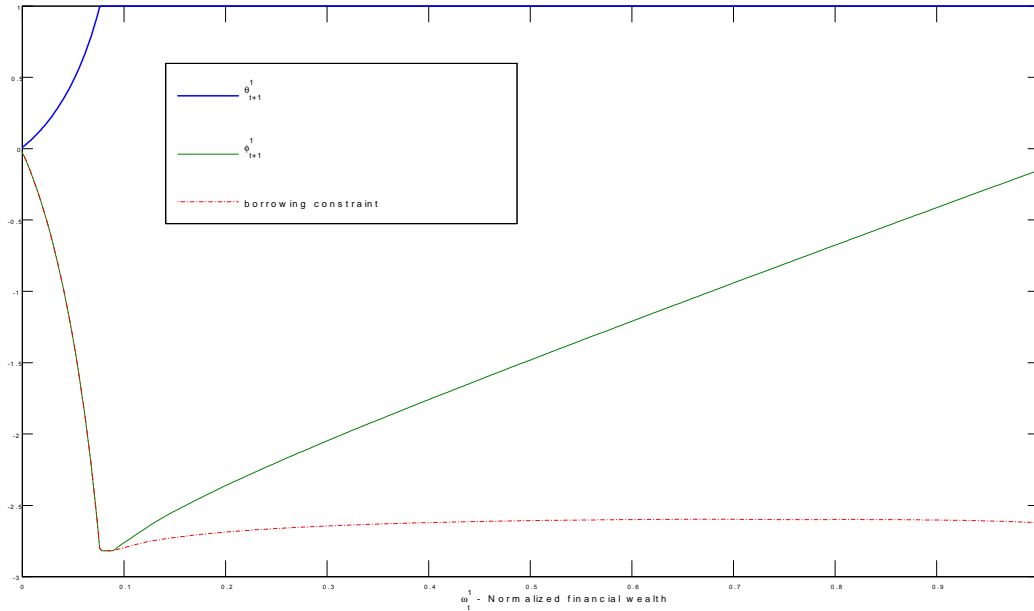


Figure 3: Portfolio choice

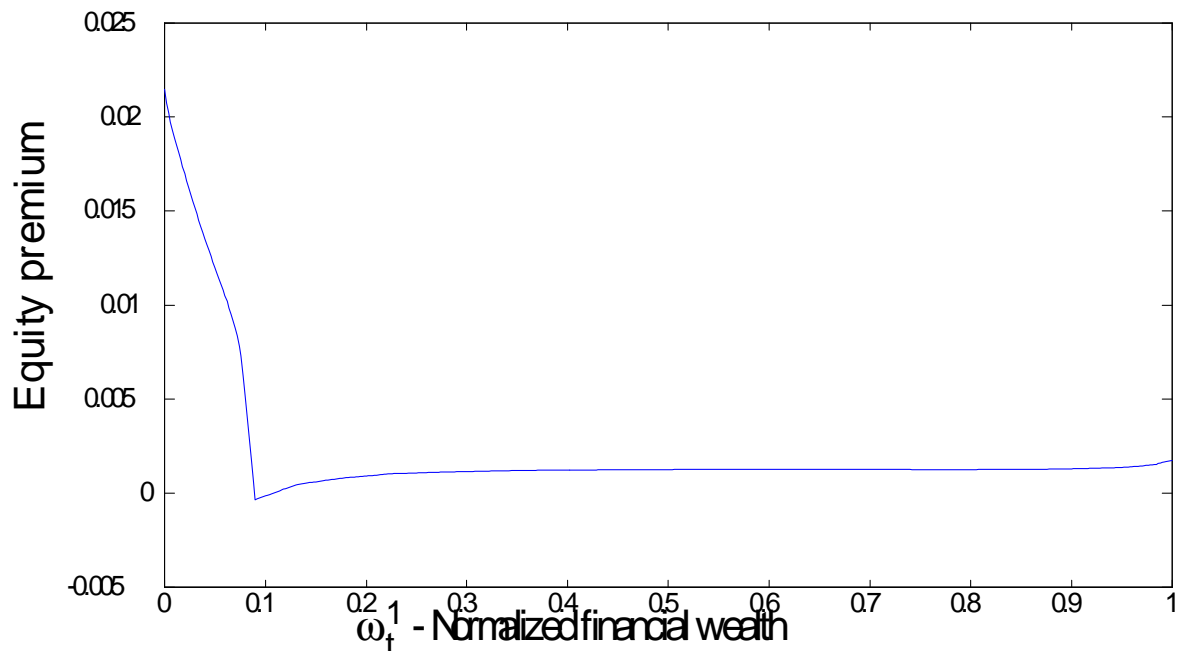


Figure 4: Equity premium

next period's dividend has low variance. However, when a bad shock hits the economy - $s = GB$ or BB - the variance of next period dividend increases. In this dynamic setting, the formulation translates to a dividend process that depends not only on current exogenous shock but also on the last period exogenous shock: if last period shock is good, dividend is 1 for current good shock and 0.8 for current bad shock; if last period shock is bad, dividend is 1 for current good shock and 0.2 for current bad shock. Therefore, in the Markov chain, we need to use three exogenous states instead of the two exogenous states in the Section 4:

$$s \in \{G, GB, BB\}.$$

Figure 5, left panel, shows that in good state the variance of next period dividend is low: $d = 1$ or 0.8 . However in bad states, the variance of next period dividend is higher: $d = 1$ or 0.2 . The right panel of the figure shows the evolution through time of the exogenous states using Markov chain representation. Even though we have three exogenous states in this set-up, each state has only two immediate successors. So we can still use Proposition 5 to show that, in any history, there is only one leverage level in the economy. Moreover, in order to generate realistic levels of leverage observed in financial markets, i.e., around 20, we need to set $m = 0.05$ in the collateral constraint (4):

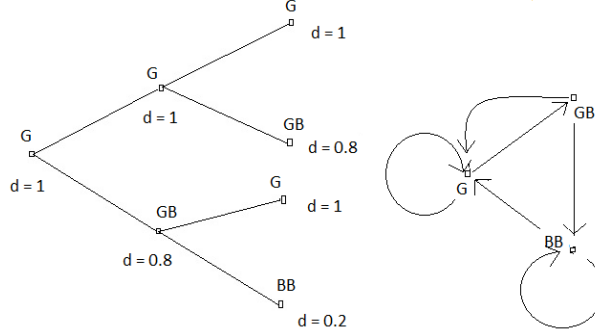


Figure 5: Evolution of the Aggregate States

$$\phi_{t+1}^h + 0.95\theta_{t+1}^h \min_{s^{t+1}|s^t} (q(s^{t+1}) + d(s_{t+1})) \geq 0.$$

The uncertainty structure generates high leverage in good state, $s_t = G$ and low leverage in bad states $s_t = GB$ or BB . Figure 6 shows this pattern of leverage. The bold green line represents leverage level in good states $s = G$ as a function of the normalized financial wealth distribution. The two lines, blue and dashed-dotted red, represent leverage level in bad states $s = GB$ or BB respectively.

In addition to the fact that increased uncertainty significantly decreases leverage emphasized in Geanakoplos (2009), we also learn from Figure 6 that financial wealth distribution is another determinant of leverage. Figure 6 shows that leverage decreases dramatically from good states to bad states. However, in contrast to the static version in Geanakoplos (2009), we can quantify the relative contributions of the changes in wealth distribution and the changes in uncertainty to the changes in leverage over the business cycles. Figure 6 shows that changes in the wealth distribution contribute relatively little to the changes in leverage at higher levels of ω_t^O . At lower levels of ω_t^O , this version of dynamic leverage cycles generates a pattern of leverage build-up in good times. As shown in Figure 7, good shocks increase leverage as they increase the wealth of the optimists relative to the wealth of the pessimists and leverage is increasing in the wealth of the optimists when $\omega_t^O > 0.1$.

Notice, however, that even though leverage decreases significantly from 17 to 12 when a bad shock hits the economy, the leverage level is still too high compared to what was observed during the last financial crisis. Gorton and Metrick (2010) show that leverage in some class of assets actively declined to almost 1. Admittedly, there are some other channels, such as counter-party risk, absent from this paper that might have caused the rapid decline in leverage. This rapid change in leverage is another important question for future research.

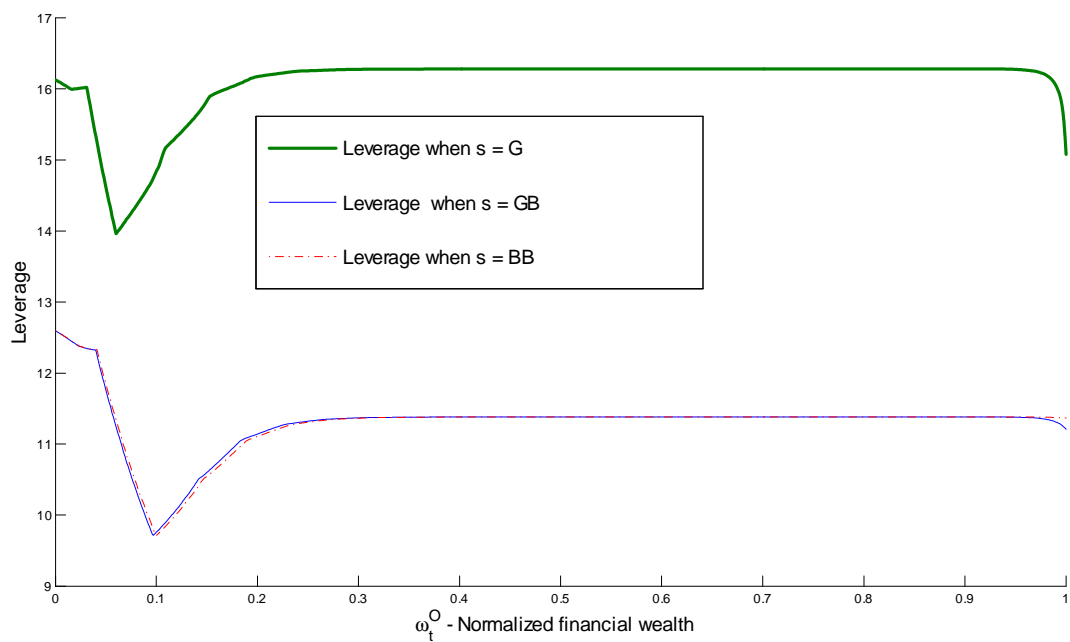


Figure 6: Leverage Cycles

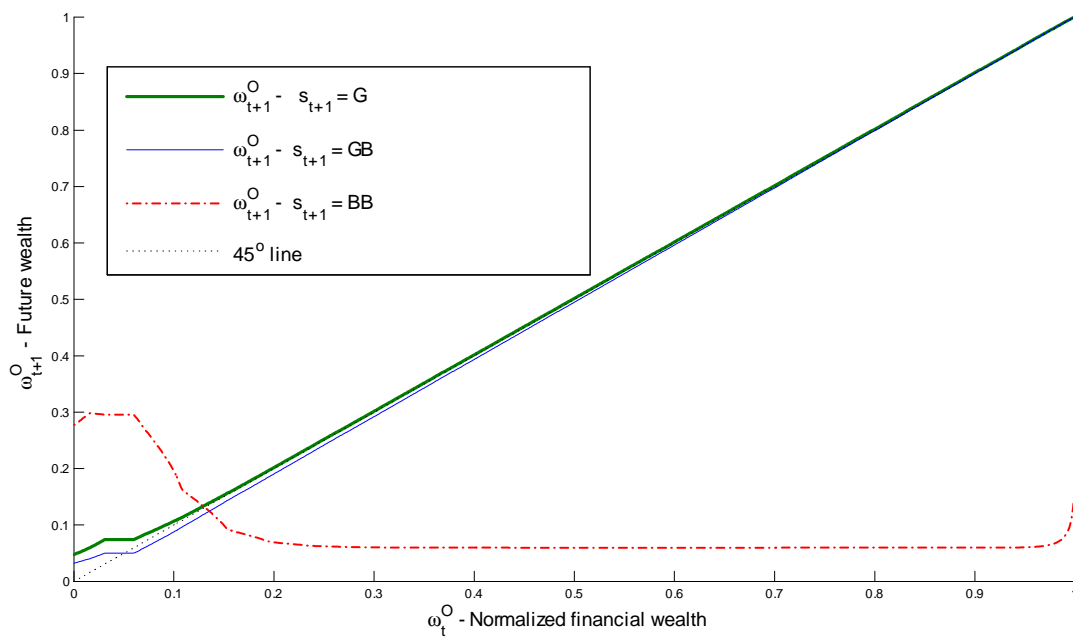


Figure 7: Wealth dynamics when $s_t = G$

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