Belief Heterogeneity, Collateral Constraint, and Asset Prices[*]

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Abstract

The recent economic crisis highlights the role of financial markets in allowing economic agents, including prominent banks, to speculate on the future returns of different financial assets, such as mortgage-backed securities. This paper introduces a dynamic general equilibrium model with aggregate shocks, endogenously incomplete markets and heterogeneous agents to investigate this role of financial markets. In addition to their risk aversion and endowments, agents differ in their beliefs about the future exogenous states (aggregate and idiosyncratic) of the economy. This difference in beliefs induces them to take large bets under frictionless complete financial markets, which enable agents to leverage their future wealth. Consequently, as hypothesized by Friedman (1953), under complete markets, agents with incorrect beliefs will eventually be driven out of the markets. In this case, they also have no influence on asset prices in the long run. In contrast, I show that under incomplete markets generated by collateral constraints, agents with heterogeneous (potentially incorrect) beliefs survive in the long run and the movement in the financial wealth distribution between agents with different beliefs permanently drive up asset price volatility. The movement in wealth distribution between agents with different beliefs also generates complex dynamics of asset prices, such as debt-deflation and fire-sale, and booms, busts, and bubbles. Laslty, tighter financial regulation in this framework surprisingly increases the long run financial wealth of agents with incorrect beliefs as well as asset price volatility.

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1 Introduction

The events leading to the financial crisis of 2007-2008 have highlighted the importance of belief heterogeneity and how financial markets create opportunities for agents with different beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds profited from the securities by short-selling them.

One reason for why there has been relatively little attention, in economic theory, paid to heterogeneity of beliefs and how these interact with financial markets is the *market selection hypothesis.* The hypothesis, originally formulated by Friedman (1953), claims that in the long run, there should be limited differences in beliefs because agents with incorrect beliefs will be taken advantage of and eventually be driven out of the markets by those with the correct belief. Therefore, agents with incorrect beliefs will have no influence on economic activity in the long run. This hypothesis has been formalized and extended in recent work by Blume and Easley (2006) and Sandroni (2000). However these papers assume that financial markets are complete and this assumption plays a central role in allowing agents to pledge all their wealth.

In this paper, I present a dynamic general equilibrium framework in which agents differ in their beliefs but markets are endogenously incomplete because of collateral constraints. Collateral constraints limit the extent to which agents can pledge their future wealth and ensure that agents with incorrect beliefs never lose so much as to be driven out of the market. Consequently, all agents, regardless of their beliefs, survive in the long run and continue to trade on the basis of their heterogeneous beliefs. Their survival leads to complex dynamics of financial wealth distribution and asset prices.

To shed light on these dynamics, the dynamic general equilibrium, infinite horizon approach adopted here provides a transparent mapping between the financial wealth distribution and economic variables such as asset prices and portfolio choices. It also allows for a comprehensive study of leverage and a characterization of the effects of financial regulation on economic fluctuations.[1]

More specifically, I study an economy in dynamic general equilibrium with both aggregate shocks and idiosyncratic shocks and heterogeneous, infinitely-lived agents. The shocks follow a Markov process. Consumers differ in terms of their beliefs on the transition matrix of the Markov process. For simplicity, these belief differences are never updated as there is no learning; in other words agents in this economy agree to disagree.[2] There is a unique final

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[1] In Cao (2010), I show that the dynamic stochastic general equilibrium model with endogenously incomplete markets presented here also includes well-known models as special cases, including recent models, such as those in Fostel and Geanakoplos (2008) and Geanakoplos (2009), as well as more classic models including those in Kiyotaki and Moore (1997) and Krusell and Smith (1998). For instance, a direct generalization of the current model allows for capital accumulation with adjustment costs in the same model in Krusell and Smith (1998) and shows the existence of a recursive equilibrium. The generality is useful in making this framework eventually applicable to a range of questions on the interaction between financial markets, heterogeneity, aggregate capital accumulation and aggregate activity.

[2] Alternatively, one could assume that even though agents differ with respect to their initial beliefs, they partially update them. In this case, similar results would apply provided that the learning process is sufficiently slow (as will be the case when individuals start with relatively firm priors). In the paper, I also show how to incorporate learning into the framework.

consumption good, one real and several financial assets. The real asset, which I sometimes refer to as *Lucas tree* as in Lucas (1978), is in fixed supply. I assume that agents cannot short sell the real asset.

Endogenously incomplete (financial) markets are introduced by assuming that all loans have to use financial assets as collateralized promises as in Geanakoplos and Zame (2002). Selling a financial asset is equivalent to borrowing and in this case, agents need to put up some real asset as collateral. Loans are non-recourse and there is no penalty for defaulting. Consequently, whenever the face value of the security is higher than the value of its collateral, the seller of the security can choose to default without further consequences. In this case, the security buyer seizes the collateral instead of receiving the face value of the security. I refer to equilibria of the economy with these financial assets as *collateral constrained equilibria*[3,4].

Households (consumers) can differ in many aspects, such as risk-aversion and endowments. Most importantly, they differ in their beliefs concerning the transition matrix governing transitions across the exogenous states of the economy. Given the consumers' subjective expectations, they choose their consumption and real and financial asset holdings to maximize their intertemporal expected utility.

The framework delivers several set of results. The first set of results, already mentioned above, is related to the survival of agents with incorrect beliefs. As in Blume and Easley (2006) and Sandroni (2000), with perfect, complete markets, in the long run, only agents with correct beliefs survive. Their consumption is bounded from below by a strictly positive number. Agents with incorrect beliefs see their consumption go to zero, as uncertainties realize overtime. However, in any collateral constrained equilibrium, every agent survives because of the constraints. When agents lose their bets, they can simply walk away from their collateral while keeping their current and future endowments to come back and trade in the financial markets in the same period.[5] They cannot do so under complete markets because they can commit to delivering all their future endowments.

As a corollary of the survival result, I show that collateral constraints will eventually be binding for some agent in any collateral constrained equilibrium provided that the face values of the financial assets with collateral span the complete set of state-contingent Arrow-Debreu securities, i.e., markets are complete in the spanning sense but endogenously incomplete due to collateral constraints. Intuitively, if this was not the case, as shown in Sandroni (2000) for complete markets, the unconstrained asset holdings would imply arbitrarily low levels of consumption at some state of the world for every agent, contradicting the result that consumption is bounded from below. This result sharply contrasts with those obtained when agents have homogenous beliefs but still have reasons to trade due to differences in endowments or utility functions. In these cases, if the economy has enough collateral, then

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[3]I avoid using the term *incomplete markets equilibria* to avoid confusion with economies with missing markets. Markets can be complete in the sense of having a complete spanning set of financial assets. But the presence of collateral constraints introduces *endogenously incomplete* markets because not all positions in these financial assets can be taken.

[4]Collateral constrained equilibria are closer to liquidity constrained equilibria than to debt-constrained equilibria in Kehoe and Levine (2001), in which the authors show that the dynamics of the former is much more complex than the one of the latter. Liquidity constrained economies are special cases of collateral constrained economies when the set of financial assets is chosen to be empty.

[5]The collateral constraints are a special case of limited commitment because there will be no need for collateral if agents can fully commit to their promises.

collateral constraints may not bind and the complete markets allocation is achieved. Heterogeneous beliefs, therefore, guarantee binding collateral constraints, but other dimensions of heterogeneity, such as heterogeneity in risk-aversion or endowment, might not.

Even though the survival mechanism above is relatively simple (but realistic), characterizing equilibrium variables such as asset prices and leverage in this environment is not an easy exercise. The second set of results establishes the existence of collateral constrained equilibria with a stationary structure. I look for *Markov equilibria*, i.e., in which equilibrium prices and quantities depend only on the distribution of *normalized financial wealth*. I show the existence of the equilibria under standard assumptions.

Given that a Markov equilibrium is a special case of collateral constrained equilibrium, in any Markov equilibrium, all agents including agents with incorrect beliefs survive. The survival of agents with incorrect beliefs affects asset price volatility. Under complete markets, agents with incorrect beliefs will eventually be driven out of the markets in the long run. The economies converge to economies with homogeneous beliefs, i.e., the correct belief. Market completeness then implies that asset prices in these economies are independent of the past realizations of the exogenous shocks. In addition, asset prices are the net present discounted values of the dividend processes with appropriate discount factors. As a result, asset price volatility is proportional to the volatility of dividend if the aggregate endowment, or equivalently the equilibrium stochastic discount factor, only varies by a limited amount over time and across states. These properties no longer hold in collateral constrained economies, especially in Markov equilibria. Given that agents with incorrect beliefs survive in the long run, they exert permanent influence on asset prices. Asset prices are not only determined by the exogenous shocks as in the complete markets case, but also by the evolution of the normalized financial wealth distribution across agents. This also implies that asset prices are history-dependent as the realizations of the past exogenous shocks affect the current normalized financial wealth distribution. In the long run, the additional dependence on the normalized financial wealth distribution raises asset price volatility under collateral constraints above the volatility level under complete markets.[6]

In addition to establishing the existence and properties of Markov equilibria, I also develop an algorithm to compute these equilibria The numerical solutions of these equilibria exhibits complex joint equilibrium dynamics of economic variables including wealth distribution, asset prices and leverage. The third set of results use the algorithm to solve for collateral constrained equilibria and present these dynamics in a reasonable calibrated example. In this example, I assume that there are two types of agents, the pessimists have correct beliefs, and the optimists are over-optimistic about the return of the real asset.

First, I start the numerical analysis by studying a collateral constrained economy in which agents are allowed to trade only in the real asset subject to the no-short-selling constraint. This economy corresponds to the liquidity constrained economy in Kehoe and Levine (2001),

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[6]I establish this result more formally using a special case in which the aggregate endowment is constant and the dividend processes are I.I.D. Under complete markets, asset prices are asymptotically constant. Asset price volatility, therefore goes to zero in the long run. In contrast, asset price volatility stays above zero under collateral constraints as the wealth distribution changes constantly, and asset prices depend on the wealth distribution. Although this example is extreme, numerical simulations show that its insight carries over to less special cases.

but with heterogeneous beliefs.[7] In this economy, not only agents with incorrect beliefs (the optimists) survive but they also prosper by postponing consumption to invest into the real asset. Their speculative activities combined with their increasing financial wealth constantly push up the price of the real asset. Interestingly, the increasing price dynamics is such that the pessimists do not always want to short-sell the asset. They start trying to short-sell the asset only when the price of the real asset is too high, at that point their short-selling constraint is strictly binding.

Then, I allow the agents to borrow using the real asset as collateral. The dynamic general equilibrium of the economy captures the "*debt-deflation*" channel as in Mendoza (2010), which models a small open economy. The economy in this example also follows two different dynamics in different times, "*normal business cycles*" and "*debt-deflation cycles,*" depending on whether the collateral constraints are binding for any of the agents. In a debt-deflation cycle, some collateral constraint binds. Then, when a bad shock hits the economy, the constrained agents are forced to liquidate their real asset holdings. This *fire sale* of the real asset reduces the price of the asset and tightens the constraints further and starting a vicious circle of falling asset prices. This example shows that the debt-deflation channel still operates when we are in a closed-economy with endogenous interest rate, as opposed to exogenous interest rates as in Mendoza (2010). Moreover, due to this mechanism, asset price volatility also tends to be higher at low levels of asset price near the debt-deflation region. This pattern has been documented in several empirical studies, including Heathcote and Perri (2011). A similar nonlinear dynamics of the economy is also emphasized in a recent paper by Brunnermeier and Sannikov (2010).

The third set of results answers some normative questions in this framework. In the numerical example above, simple and extreme forms of financial regulations such as shutting down financial markets or uniformly restricting leverage surprisingly do not reduce asset price volatility. The intuition for the greater volatility under such regulations is similar to the intuition for why long run asset price volatility is higher under collateral constrained economies than under complete markets economies explained earlier. Financial regulations act as further constraints protecting the agents with incorrect beliefs. Thus, in the long run these agents hold most of the assets which they believe, incorrectly, to have high rates of return. The shocks to the rates of return on these assets then create large movements in the marginal utilities of the agents, hence large volatility of the prices of the assets.[8]

Finally, in the Appendix, I apply the model to the U.S. economy using the parameters estimated in Heaton and Lucas (1995). I find that 1) the moments of asset prices such as equity premium behave differently under binding collateral constraints than under non binding collateral constraints; 2) due to risk-aversion, the equilibrium portfolio choice of the agents are such that the collateral constraints are not binding often in the stationary distribution; 3) consequently, the unconditional moments of asset prices are not very different from the unconditional moments when there are no collateral constraints. These findings are consistent with the ones in Mendoza (2010).

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[7]This is also Harrison and Kreps (1978) with risk-aversion.

[8]When financial markets are completely shut down, agents can only trade in the real asset subject to no-short sale constraint; collateral constrained equilibrium becomes liquidity constrained equilibrium as in Kehoe and Levine (2001). The numerical solution in this paper completely characterizes the equilibrium which Kehoe and Levine (2001) conjecture that the dynamics can be very complicated.

The rest of the paper proceeds as follow. The next section reviews the related literature. In Section 3, I present the general model of an endowment economy and preliminary analysis of survival, asset price volatility under the complete markets benchmark as well as under collateral constraints. In Section 4, I define and show the existence of collateral constrained equilibria under the form of Markov equilibria. In this section, I also prove important properties of Markov equilibria in this model and derive a general numerical algorithm to compute these equilibria. Section 5 focuses on an example with only one asset to illustrate the ideas in Sections 3 and 4 and the equilibrium dynamics of wealth distribution, asset prices, and leverage. Section 8 presents a preliminary assessment of the quantitative significant of belief heterogeneity, collateral constraint and wealth distribution on asset prices using the parameters used in Heaton and Lucas (1995). In Cao (2010), I develop the most general model with capital accumulation, labor supply and production. Section 6 concludes with potential applications of the framework in this paper. Lengthy proofs and constructions are in the appendix and in Cao (2010).

2   Related literature

This paper is related to the growing literature studying collateral constraints, started with a series of papers by John Geanakoplos. The dynamic analysis of collateral constrained equilibria is related to Kubler and Schmedders (2003). They pioneer the introduction of financial markets with collateral constraints into a dynamic general equilibrium model with aggregate shocks and heterogeneous agents. The technical contribution of this paper relative to Kubler and Schmedders (2003) is to introduce heterogeneous beliefs using Radner (1972) rational expectations equilibrium concept: even though agents assign different probabilities to both aggregate and idiosyncratic shocks, they agree on the equilibrium outcomes, including prices and quantities, once a shock is realized. This rational expectations concept differs from the standard rational expectation concept, such as the one used in Lucas and Prescott (1971), in which subjective probabilities should coincide with the true conditional probabilities given all the available information.[9]

Related to the survival of agents with incorrect beliefs, Coury and Sciubba (2005) and Beker and Chattopadhyay (2009) suggest a mechanism for agents' survival based on explicit debt constraints as in Magill and Quinzii (1994). These authors do not consider the effects of the agents' survival on asset prices. My framework is tractable enough for a simultaneous analysis of survival and its effects on asset prices. As mentioned in footnote 5, collateral constraints are a special case of limited commitment. However, this special case of limited commitment is different from the usual limited commitment literature where agents are assumed to be banned from trading in financial markets after their defaults such as in Kehoe and Levine (1993) and Alvarez and Jermann (2001). In this paper, agents can always come back to the financial markets and trade starting with their endowment after defaulting and

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[9]Another technical contribution in Cao (2010), is to introduce capital accumulation and production in a tractable way. Capital accumulation or physical investment is modelled through intermediate asset producers with convex adjustment costs that convert old units of assets into new units of assets using final good. Lorenzoni and Walentin (2009) models capital accumulation with adjustment cost using used capital markets. Through asset producers, I assume markets for both used and new capital.

loosing all their financial wealth. Given this better outside option, the financial constraints are more stringent then they are in the other papers. Beker and Espino (2010) and Tsyrennikov (2012) has a similar survival mechanism to mine based on the limited commitment framework in these papers. However, my approach to asset pricing is different because asset prices are computed explicitly as a function of wealth distribution. Moreover, my approach also allows a comprehensive study of asset-specific leverage. Kogan, Ross, Wang, and Westerfield (2006) and Borovicka (2010) explore yet another survival mechanism based on the preferences of agents but use complete markets. Also under complete markets, Kubler and Schmedders (2011) show the importance of beliefs heterogeneity and wealth distribution on asset prices in a model with overlapping-generations.

My paper is also related to the literature on the effect of heterogeneous beliefs on asset prices studied in Xiong and Yan (2009) and Cogley and Sargent (2008). These authors, however, consider only complete markets. The survival of irrational traders is studied Long, Shleifer, Summers, and Waldmann (1990) and Long, Shleifer, Summers, and Waldmann (1991) but they do not have a fully dynamic framework to study the long run survival of the traders. Simsek (2012) also studies the effects of belief heterogeneity on asset prices, but in a static setting. He assumes exogenous wealth distributions to investigate the question whether heterogeneous beliefs affect asset prices. In contrast, I study the effects of the endogenous wealth distribution on asset prices as well as asset price volatility.

The channel through which asset prices deviate from their fundamental values is different from the limited arbitrage mechanism in Shleifer and Vishny (1997). In their paper, the deviation arises because agents with correct beliefs hit their financial constraints before being able to arbitrage away the price anomalies. In this paper, agents with incorrect beliefs hit their financial constraint more often and are protected by the constraint. Moreover, in the equilibria computed in Section 5.2, agents with the correct belief (the pessimists) often do not hit their borrowing constraint.

One the normative question of how financial regulation affects asset price volatility, in a recent paper, Brumm, Grill, Kubler, and Schmedders (2011) show that at high levels of margin requirement (more than 80%) increasing margin requirement, i.e. restricting leverage, decreases asset price volatility. This result is different from the normative result in my paper. However, in their paper, agents trade due to their difference in risk-aversion (under Epstein-Zin recursive preferences) while in my paper, agents trade due to their difference in beliefs. Under belief heterogeneity, there is a new mechanism that does not exist in Brumm, Grill, Kubler, and Schmedders (2011): a higher margin requirement actually protects the agents with incorrect beliefs. Thus they are financially wealthier in the long run and their trading activity drives up asset price volatility.

When capital accumulation is introduced, in Cao (2010), the model presented here is a generalization of Krusell and Smith (1998) with financial markets and adjustment costs.[10] In particular, the existence theorem 1 shows that a recursive equilibrium in Krusell and Smith (1998) exists. Krusell and Smith (1998) derives numerically such an equilibrium, but they do not formally show its existence. My paper is also related to Kiyotaki and Moore (1997), although I provide a microfoundation for the financial constraint (3) in their paper using the

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[10]See Feng, Miao, Peralta-Alva, and Santos (2009) for an existence proof in an environment without financial markets.

endogeneity of the set of actively traded financial assets.

In a recent and breakthrough paper, Brunnermeier and Sannikov (2010) present an economy in continuous time with collateral constraint and large shocks. Instead of linearizing around the steady state, the authors are able to solve for the global equilibrium of the economy in which asset prices and aggregate economic activities depend on the financial wealth distribution. The long run stationary distribution of the economy has a U-shape form. Most of the time the economy spends in the linear region in which the collateral constraint is not binding. But occasionally, large shocks push the economy toward a highly nonlinear region in which the collateral constraint is nearly binding. My paper is a discrete time counter part of Cao (2010) in the sense that I also solve for the global nonlinear equilibrium of the economy. In contrast to Cao (2010), under belief heterogeneity, the stationary distribution of the economy concentrates on the nonlinear region in which the collateral constraint is binding or nearly binding.

3 General model

In this general model, there are heterogeneous agents who might differ in their beliefs about the future streams of dividends and the individual endowments. In order to study the effects of belief heterogeneity on asset prices, I allow for only one real asset in fixed supply. In the most general version of the model in Cao (2010), there can also be many types of assets (for examples trees, land, housing and machines) that differ in their dividend process and their collateral value. For example, some of the assets can be used as collateral to borrow and others cannot. Moreover, that version of the model can also allow for assets in flexible supply and production in order to study the effects of belief heterogeneity on the aggregate capital accumulation and aggregate economic activity. Assets in fixed supply are special cases of assets in flexible supply with adjustment costs approaching infinity.

3.1 The endowment economy

Consider an endowment, a single consumption (final) good economy in infinite horizon with infinitely-lived agents (consumers). Time runs from $t = 0$ to ∞. There are H types of consumers

$$h \in \mathcal{H} = \{1, 2, \ldots, H\}$$

in the economy with a continuum of measure 1 of identical consumers in each type. These consumers might differ in many dimensions including per period utility function $U_h(c)$ (i.e., risk-aversion), discount rate β_h, and endowment of good e_h. The consumers might also differ in their initial endowment of a real asset, Lucas' tree,[11] that pays off real dividend in terms of the consumption good. However, the most important dimension of heterogeneity is the heterogeneity in beliefs over the evolution of the exogenous state of the economy. There are S possible exogenous states (or equivalently exogenous shocks)

$$s \in \mathcal{S} = \{1, 2, \ldots, S\}.$$

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[11]See Lucas (1978)

The states capture both idiosyncratic uncertainty, i.e., uncertain individual endowments, and aggregate uncertainty i.e., uncertain aggregate dividends.[12]

The evolution of the economy is captured by the realizations of the shocks over time: $s^t = (s_0, s_1, \ldots, s_t)$ is the history of realizations of shocks up to time t. I assume that the shocks follow a Markov process with the transition probabilities $\pi(s, s')$. In order to rule out transient states, I make the following assumption:

Assumption 1 $\mathcal{S}$ *is ergodic.*

Now, in contrast to the standard rational expectation literature, I assume that the agents do not have the perfect estimate of the transition matrix π. Each of them has their own estimate of the matrix, π^h.[13] However, these estimates are not very far from the truth, .i.e., there exist u and U strictly positive such that

$$u < \frac{\pi^h(s, s')}{\pi(s, s')} < U \quad \forall s, s' \in S \text{ and } h \in \mathcal{H} \tag{1}$$

where $\pi(s, s') = 0$ if and only if $\pi^h(s, s') = 0$ in which case let $\frac{\pi^h(s,s')}{\pi(s,s')} = 1$. This formulation allows for time varying belief heterogeneity as in He and Xiong (2011). In particular, agents might share the same beliefs in good states, $\pi^h(s, .) = \pi^{h'}(s, .)$, but their beliefs can start diverging in bad states, $\pi^h(s, .) \neq \pi^{h'}(s, .)$.[14] Notice that (1) implies that every agent believes that $\mathcal{S}$ is ergodic.

Real Asset: As mentioned above, there is only one real asset that pays off state-dependent dividend $d(s)$ in final good. The asset can both be purchased and be used as collateral to borrow. This gives rise to the notion of leverage. The ex-dividend price of each unit of the asset in history s^t is denoted by $q(s^t)$. I assume that agents cannot short-sell the real asset.[15] The total supply 1 of the asset is given at the beginning of the economy, under the form of asset endowments to the consumers.

Financial Assets: In each history s^t, there are also (collateralized) financial assets, $j \in \mathcal{J}_t(s^t)$. $\mathcal{J}_t(s^t)$ have a finite number of elements, $J_t(s^t)$. Each financial asset j (or financial security) is characterized by a vector and a scalar, (b_j, k_j), of promised payoffs and collateral requirement. Promises are a standard feature of financial asset similar to Arrow's securities, i.e., asset j traded in history s^t promises next-period pay-off $b_j(s_{t+1}) > 0$ in terms of final good at the successor node $s^{t+1} = (s^t, s_{t+1})$. The non-standard feature is the collateral requirement. Agents can only sell the financial asset j if they hold shares of the

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[12]A state s can be a vector $s = (A, \epsilon_1, \ldots, \epsilon_H)$ where A consists of aggregate shocks and ϵ_h are idiosyncratic shocks.

[13]Learning can be easily incorporated into this framework as into this framework by allowing additional state variables which are the current beliefs of agents in the economy. As in Blume and Easley (2006) and Sandroni (2000), agents who learn slower will dissappear under complete markets. However they all survive under collateral constraints. The dynamics of asset prices describe here will corresponds to the short-run behavior of asset prices in the economy with learning.

[14]Simsek (2012) shows in a static model that only the divergence in beliefs about bad states matters for asset prices.

[15]I can relax this assumption by allowing limited short-shelling.

real asset as collateral. If an agent sells one unit of security j, she is required to to hold k_j units of the real asset as collateral.[16]

There are no penalties for default. Thus, a seller of the financial asset can default at a node s^{t+1} whenever the total value of collateral falls below the promise at that state at the cost of losing the collateral. By individual rationality, the effective pay-off of security j at node s^t is always given by

$$f_j\left(s^t, s_{t+1}\right) = \min\left\{b_j\left(s_{t+1}\right), \left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right)\right) k_j\right\}. \tag{2}$$

Let $p_{j,t}\left(s^t\right)$ denote the price of security j at node s^t.

Assumption 2 *Each financial asset requires a strictly positive level of collateral*

$$\min_{j \in \mathcal{J}_t(s^t)} k_j > 0$$

If a financial asset j requires no collateral then its effective pay-off, determined by (2), will be zero, it will be easy to show that in equilibrium its price, p_j, will be zero as well. We can thus ignore these financial assets.

Remark 1 *The financial markets are endogenously incomplete even if $\mathcal{J}_t$ are complete in the usual sense of spanning, i.e., $J_t \geq S-1$.[17] Because agents are constrained in the positions they can take due to the collateral requirement and the fact that the total supply of collateral is finite. The collateral requirement is a special case of limited commitment, because if the borrowers (the sellers of the financial assets) have full commitment, they will not be required to put up any collateral to borrow.[18]*

Remark 2 *Selling one unit of financial asset j is equivalent to purchasing k_j units of the real asset and at the same time pledging these units as collateral to borrow $p_{j,t}$. It is shown in Cao (2010) that*

$$k_j q_t - p_{j,t} > 0,$$

that is the seller of the financial asset always has to pay some margin. So the decision to sell financial asset j using the real asset as collateral corresponds to the desire to invest into the asset at margin rather than the simple desire to borrow.

Remark 3 *We can then define the leverage ratio associated with the transaction as*

$$L_{j,t} = \frac{k_j q_t}{k_j q_t - p_{j,t}} = \frac{q_t}{q_t - \frac{p_{j,t}}{k_j}}. \tag{3}$$

Even though there are many financial assets available, in equilibrium only some financial asset will be actively traded, which in turn determines which leverage levels prevail in the economy in equilibrium. In this sense, both asset price and leverage are simultaneously determined in equilibrium, as emphasized in Geanakoplos (2009). Subsubsection 5.2.4 uses this definition of leverage to investigate how leverage varies over the business cycles.

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[16]Notice that, there are only one-period ahead financial assets. See He and Xiong (2011) for a motivation why longer term collateralized financial assets are not used in equilibrium.

[17]The real asset is another asset that helps complete the markets.

[18]Alvarez and Jermann (2000) is another example of asset pricing under limited commitment.

Remark 4 *Notice that the set $\mathcal{J}_t$ of financial securities can be dependent on the history of the economy s^t. For example, $\mathcal{J}_t$ may contain a financial security j with the collateral requirement depending on the history s^t through prices*

$$k_j\left(s^t\right) = \max_{s^{t+1}|s^t}\left\{\frac{b_j\left(s_{t+1}\right)}{q\left(s^{t+1}\right)+d\left(s^{t+1}\right)}\right\}. \tag{4}$$

So in this example k_j is the minimum collateral level that ensures no default, that is

$$f_{j,t+1}\left(s^{t+1}\right) = b_j\left(s_{t+1}\right) \ \forall s_{t+1} \in \mathcal{S}.$$

This constraint is similar to the constraint in Kiyotaki and Moore (1997) in which agents can borrow only up to the minimum across future states of the future value of their land. With $S = 2$, and state non-contingent debts, i.e., $b_j\left(s_{t+1}\right) = b_j$, Geanakoplos (2009) argues that even if we allow for a wide range of collateral level, that is the unique collateral level that prevails in equilibrium (thus there is a unique level of leverage in each instance, according to the remark above). This statement for two future states still holds in this context of infinitely-lived agents as proved later in Section 5. However, this might not be true if we have more than two future states.

Consumers: Consumers are the most important actors in this economy. They can be hedge fund managers or banks' traders in financial markets. In each state s^t, each consumer is endowed with a potentially state dependent endowment $e_t^h = e^h\left(s_t\right)$ units of the consumption good. I suppose that there is a strictly positive lower bound on these endowments. This lower bound guarantees a lower bound on consumption if a consumer decides to default on all her debt.[19]

Assumption 3 $\min_{h,s} e_h\left(s\right) > \underline{e} > 0.$

An example of this assumption is that commercial banks receive deposits from their retail branches while these banks also have trading desks that trade independently in the financial markets.

Consumers maximize their intertemporal expected utility with the per period utility functions $U_h\left(.\right) : \mathbf{R}^+ \longrightarrow \mathbf{R}$ that satisfy

Assumption 4 $U_h\left(.\right)$ *is concave and strictly increasing.*

Even though the formulation and solution method in Cao (2010) allow for heterogeneity in the discount rates, to focus on beliefs heterogeneity, I assume from now on that agents have the same discount factor.

Assumption 5 *Agents have the same discount factor $\beta_h = \beta \ \forall h \in \mathcal{H}$.*

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[19]I also introduce the disutility of labor in the general existence proof in Cao (2010) in order to study employment in this environment. The existence of equilibria for finite horizon allows for labor choice decision. When we have strictly positive labor endowments, l^h, we can relax Assumption 3 on final-good endowments, e^h.

Notice that I do not require U_h to be strictly concave. This assumption allows for linear utility functions in Geanakoplos (2009) and Harrison and Kreps (1978).

Consumer h takes the sequences of prices $\{q_t, p_{j,t}\}$ as given and solves

$$\max_{\left\{c_t^h, \theta_{t+1}^h, \phi_{j,t+1}^h\right\}} \mathbf{E}_0^h \left[\sum_{t=0}^{\infty} \beta^t U_h \left(c_t^h \right) \right] \tag{5}$$

and in each history s^t, she is subject to the budget constraint

$$c_t^h + q_t \theta_{t+1}^h + \sum_{j \in \mathcal{J}_t} p_{j,t} \phi_{j,t+1}^h \leq e_t^h + \sum_{j \in \mathcal{J}_{t-1}} f_{j,t} \phi_{j,t}^h + (q_t + d_t) \theta_t^h \tag{6}$$

the collateral constraints

$$\theta_{t+1}^h + \sum_{j \in \mathcal{J}_t : \phi_{j,t} < 0} \phi_{j,t+1}^h k_j \geq 0 \tag{7}$$

One implicit condition from the assumption on utility functions is that consumptions are positive, i.e., $c_t^h \geq 0$. The collateral constraint implies the no-short sale constraint

$$\theta_{t+1}^h \geq 0. \tag{8}$$

The most important feature of the objective function is the superscript h in the expectation operator, $\mathbf{E}^h [.]$ that represents the subjective beliefs when agents estimate their future expected utility. Entering period t, agent h holds θ_t^h old units of real asset and $\phi_{j,t}^h$ units of financial asset j. She can trade old units of real asset at price q_t and buy new units of real asset θ_{t+1}^h for time $t + 1$ at the same price. She can also buy and sell financial securities $\phi_{j,t}^h$ at price $p_{j,t}$. If she sells financial securities she is subject to collateral requirement (7).

At the first sight, the collateral constraint (7) does not have the usual property of financial constraints in the sense that higher asset prices do not seem to enable more borrowing. However, using the definition of the effective pay-off, $f_{j,t+1}$, in (2), we can see that this effective pay-off is weakly increasing in the future prices of the real asset, q_{t+1}. As a result, financial asset prices, $p_{j,t}$, are also weakly increasing in the real asset's future prices. So borrowers can borrow more if q_{t+1}'s increase. Given that the borrowing constraints is effective through future asset prices and the current asset price depends directly on the future prices, when we embed this channel in a production economy in Cao (2010), this constraint creates a feed-back mechanism from the financial sector to the real sector similar to Kiyotaki and Moore (1997) and a more recent paper, Brunnermeier and Sannikov (2010).[20]

Equilibrium: In this environment, I define an equilibrium as follows

Definition 1 *An **collateral constrained equilibrium** for an economy with initial asset holdings*

$$\left\{ \theta_0^h \right\}_{h \in \{1, 2, \dots, H\}}$$

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[20]My paper is closer to Brunnermeier and Sannikov (2010) because I analyze the nonlinear dynamics of asst price using a global solution method instead of linearization method used in Kiyotaki and Moore (1997).

and initial shock s_0 is a collection of consumption, real and financial asset holdings and prices in each history s^t,

$$(\{c_t^h\left(s^t\right), \theta_{t+1}^h\left(s^t\right), \phi_{j,t+1}^h\left(s^t\right)\}_{h\in\mathcal{H}}$$
$$q_t\left(s^t\right), \{p_{j,t}\left(s^t\right)\}_{j\in\mathcal{J}_t(s^t)})$$

satisfying the following conditions:

i) The markets for final good, real and financial assets in each period clear:

$$\sum_{h\in\mathcal{H}} \theta_{t+1}^h\left(s^t\right) = 1$$

$$\sum_{h\in\mathcal{H}} \phi_{j,t+1}^h\left(s^t\right) = 0 \; \forall j \in \mathcal{J}_t\left(s^t\right)$$

$$\sum_{h\in\mathcal{H}} c_t^h\left(s^t\right) = \sum_{h\in\mathcal{H}} e^h\left(s_t\right) + d\left(s_t\right)$$

ii) For each consumer h, $\left\{ c_t^h\left(s^t\right), \theta_{t+1}^h\left(s^t\right), \left(\phi_{j,t+1}^h\left(s^t\right)\right)_{j\in\mathcal{J}_t(s^t)} \right\}$ solves the individual maximization problem subject to the budget constraint, (6), and the collateral constraint, (7).

Notice that by setting the sets of financial securities $\mathcal{J}_t$ empty, we obtain a model with no financial markets in which agents are only allowed to trade in real asset, but they cannot short-sell these assets. This case corresponds to Lucas (1978)'s model and the liquidity constrained economy in Kehoe and Levine (1993) with belief heterogeneity.

In the next subsection, I establish some properties of collateral constrained equilibrium.

3.2   General properties of collateral constrained equilibrium

Given the endowment economy, we can easily show that total supply of final good in each period is bounded by a constant $\bar{e}$. Indeed in each period, total supply of final good is bounded by

$$\bar{e} = \max_{s\in\mathcal{S}} \left(\sum_{h\in\mathcal{H}} e^h\left(s\right) + d\left(s\right) \right) \tag{9}$$

The first term on the right hand side is the total endowment of all consumers. The second term is the dividend from the real asset. In collateral constrained equilibrium, the market clearing condition for the final good implies that total consumption is bounded from above by $\bar{e}$.

We can show that in any collateral constrained equilibrium, the consumption of each consumer is bounded from below by a strictly positive constant $\underline{c}$. Two assumptions are important for this result. First, no-default-penalty allows consumers, at any moment in time, to walk away from their past debt and only lose their collateral. After defaulting, they can always keep their non-financial wealth (inequality (11) below). Second, increasingly large speculation by postponing current consumption is not an equilibrium optimal plan, because in equilibrium, consumption is bounded by $\bar{e}$, in inequality (12).[21] This assumption prevents

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[21]Even though an atomistic consumer may have unbounded consumption, in equilibrium, prices will adjust such that a consumption plan in which consumption sometime exceeds $\bar{e}$ will not be optimal.

agents from constantly postponing their consumption to speculating in the real assets and is main difference with the survival channel in Alvarez and Jermann (2000) used by Beker and Espino (2010) for heterogeneous beliefs which will be explained in detail below. Formally, we have the following Proposition

Proposition 1 *Suppose that there exists a $\underline{c}$ such that*

$$U_h\left(\underline{c}\right) < \underbrace{\frac{1}{1-\beta}U_h\left(\underline{e}\right)}_{endowment} - \underbrace{\frac{\beta}{1-\beta}U_h\left(\overline{e}\right)}_{speculation}, \ \forall h \in \mathcal{H}, \tag{10}$$

where $\overline{e}$ is defined in (9). Then in a collateral constrained equilibrium, consumption of each consumer in each history always exceeds $\underline{c}$.

Proof. This result is shown in an environment with homogenous beliefs (Lemma 3.1 in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994)). It can be done in the same way under heterogenous beliefs. However, I replicate the proof in order to provide the economic intuition in this environment. The consumption of each consumer in each future period is bounded by future aggregate endowment. Also, in each period, one of the feasible strategies of consumer h is to default on all her past debt at the only cost of losing all the collateral assets, but she can still at least consume her endowment from the current period on, therefore

$$U_h\left(c_{h,t}\right) + \mathbf{E}_t^h\left[\sum_{r=1}^{\infty}\beta^r U_h\left(c_{h,t+r}\right)\right] \geq \frac{1}{1-\beta}U_h\left(\underline{e}\right). \tag{11}$$

Notice that in equilibrium, $\sum_h c_{h,t+r} \leq \overline{e}$ therefore $c_{h,t+r} \leq \overline{e}$. So

$$U_h\left(c_h\right) + \frac{\beta}{1-\beta}U_h\left(\overline{e}\right) \geq \frac{1}{1-\beta}U_h\left(\underline{e}\right) \tag{12}$$

This implies

$$U_h\left(c_h\right) \geq \frac{1}{1-\beta}U_h\left(\underline{e}\right) - \frac{\beta}{1-\beta}U_h\left(\overline{e}\right) > U_h\left(\underline{c}\right).$$

Thus, $c_h \geq \underline{c}$. ∎

Condition (10) is satisfied immediately if $\lim_{c\longrightarrow 0}U_h\left(c\right) = -\infty$, for example, with log utility or CRRA utility with CRRA constant exceeding 1.

One immediate corollary of Proposition 1 is that every consumer survives in equilibrium. Sandroni (2000) shows that in complete markets equilibrium, almost surely the consumption of agents with incorrect beliefs converges to 0 at infinity. Therefore, collateral constrained equilibrium differs from complete markets equilibrium when consumers strictly differ in their beliefs.

The survival mechanism under collateral constrained is similar to the one in Alvarez and Jermann (2000) and Beker and Espino (2010). In particular, the first term on the right hand side of (10) captures the fact that the agents always have the option to default and go to autarky in which they only consume their endowment which exceeds $\underline{e}$ in each period, which is the lower bound for consumption in Alvarez and Jermann (2000) and Beker and Espino

(2010). However, the two survival mechanisms also differ because, in this paper, agents can always default on their promises and lose all their real asset holdings, but they can always go back to financial markets to trade right after defaulting. The second term in the right hand side of (10) captures the fact that, this possibility might hurt the agents if they have incorrect beliefs. The prospect of higher reward for speculation, i.e. high $\bar{e}$, will induce these agents to constantly postpone consumption to speculate. As a result, their consumption level might fall well below $\underline{e}$. Indeed, the lower bound of consumption, $\underline{c}$, is decreasing in $\bar{e}$: the more there is of the total available final good, the more profitable speculative activities are and the more incentives consumers have to defer current consumption to engage in these activities.

This survival mechanism is also different from the limited arbitrage mechanism in Shleifer and Vishny (1997), where asset prices differ from their fundamental values because agents with correct beliefs hit their financial constraints (or short-selling constraints) before being able to arbitrage away the difference between fundamental value and market price. In this paper, agents with incorrect beliefs hit their financial constraint more often than the agents with correct beliefs do and are protected by the constraint. In the equilibria computed in Section 5.2, agents with the correct belief (the pessimists) sometime do not hit their borrowing constraint (or short-selling constraints).

The difference between consumption allocations under complete markets and under collateral constraints when there is belief heterogeneity also implies the following proposition.

Proposition 2 *Assume that Proposition 1 holds: If financial markets are complete in terms of spanning, i.e., the set of the vectors of promises b_j together with the vector of returns from investing in the real asset $(q_{t+1} + d_{t+1})$ has full rank, then, for any given time t, with positive probability, the collateral constraints must be binding for some agent after time t.*

Proof. We prove this corollary by contradiction. Suppose none of the collateral constraints are binding after a certain date. Then we can take the first-order condition with respect to the state-contingent securities. This leads to consumption of some agents to approach zero at infinity, as shown in Sandroni (2000). This contradicts the conclusion of Proposition 1 that consumption of each agent is bounded away from zero. ∎

This proposition emphasizes the difference between belief heterogeneity and other forms of heterogeneity such as heterogeneity in endowments or in risk-aversion. The following proposition, in the same form as Theorem 5 in Geanakoplos and Zame (2007), shows that if consumers share the same belief and discount rate, there exist endowment profiles with which collateral constrained equilibria attain the complete markets allocations.

Proposition 3 *If consumers share the same belief, there is an open set of endowment profiles with the properties that the complete markets equilibrium can be supported by an collateral constrained equilibrium.*

Proof. We start with an allocation such that there is no trade in the complete markets equilibrium, then as we move to a neighborhood of that allocation, all trade can be collateralized. ∎

The properties in this section are established under the presumption that collateral constrained equilibria exist. The next section is devoted to showing the existence of these

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equilibria with a stationary structure. Then Section 4.3 presents an algorithm to compute the equilibria.

4 Markov Equilibrium

In this section, I argue that collateral constrained equilibrium exists with a stationary structure. The equilibrium prices and allocation depend on the exogenous state of the economy and a measure of the wealth distribution. The detailed proofs are in Cao (2010). I also show an numerical algorithm to compute the equilibrium in Subsection 4.3.

4.1 The state space and definition

I define the *normalized financial wealth* of each agent by

$$\omega_t^h = \frac{(q_t + d_t)\,\theta_t^h + \sum_j \phi_{j,t}^h f_{j,t}}{q_t + d_t}. \tag{13}$$

Let $\omega\left(s^t\right) = \left(\omega^1\left(s^t\right), ..., \omega^H\left(s^t\right)\right)$ denote the *normalized financial wealth distribution.* Then in equilibrium $\omega\left(s^t\right)$ always lies in the (H-1)-dimensional simplex Ω, i.e., $\omega^h \geq 0$ and $\sum_{h=1}^{H} \omega^h = 1$. ω^h's are non-negative because of the collateral constraint (7) that requires the value of each agent's asset holding to exceed the liabilities from their past financial assets holdings; and the sum of ω^h equals 1 because of the real asset market clearing and financial markets clearing conditions. In Cao (2010), I show that, under the conditions detailed in Subsection 4.2 below, there exists a Markov equilibrium over the compact state space $\mathcal{S} \times \Omega$, i.e., an equilibrium in which equilibrium prices and allocation depend only on the state $(s_t, \omega_t) \in \mathcal{S} \times \Omega$.

In particular, for each $(s_t, \omega_t) \in \mathcal{S} \times \Omega$, we need to find a vector of prices and allocation

$$\nu_t \in \widehat{\mathcal{V}} = \mathbf{R}_+^H \times \mathbf{R}_+^H \times \mathbf{R}^{JH} \times \mathbf{R}_+ \times \mathbf{R}_+^J \tag{14}$$

that consists of the consumers' decisions: consumption of each consumer $\left(c^h\left(\sigma\right)\right)_{h \in \mathcal{H}} \in \mathbf{R}_+^H$, real and financial asset holdings $\left(\theta^h\left(\sigma\right), \phi_j^h\left(\sigma\right)\right)_{h \in \mathcal{H}} \in \mathbf{R}_+^H \times \mathbf{R}^{JH}$, the price of the real asset $q\left(\sigma\right) \in \mathbf{R}_+$ and the prices of the financial assets $\left(p_j\left(\sigma\right)\right)_{j \in \mathcal{J}} \in \mathbf{R}_+^J$ must satisfy the market clearing conditions and the budget constraint of consumers bind. Moreover, for each future state $s_{t+1}^+ \in \mathcal{S}$ succeeding s_t, we need to find a corresponding wealth distribution ω_{t+1}^+ and equilibrium allocation and prices $\nu_{t+1}^+ \in \widehat{\mathcal{V}}$ such that for each household $h \in \mathcal{H}$ the following conditions hold:

a) For each $s^+ \in \mathcal{S}$ succeeding s

$$\omega_s^{h+} = \frac{\theta^h\left(q_s^+ + d_s^+\right) + \sum_{j \in \mathcal{J}} \phi_j^h \min\left\{b_j\left(s\right), k^j\left(q^+ + d^+\right)\right\}}{q^+ + d^+}. \tag{15}$$

b) There exist multipliers $\mu^h \in \mathbf{R}^+$ corresponding to collateral constraints such that

$$0 = \mu^h - q U_h' \left(c^h \right) + \beta E^h \left\{ \left(q^+ + d^+ \right) U_h' \left(c^{h+} \right) \right\} \tag{16}$$

$$0 = \mu^h \left(\theta^h + \sum_{j \in \mathcal{J}: \phi_j^h < 0} k^j \phi_j^h \right)$$

$$0 \leq \theta^h + \sum_{j \in \mathcal{J}: \phi_j^h < 0} k^j \phi_j^h.$$

c) Define $\phi_j^h(-) = \max\left(0, -\phi_j^h\right)$ and $\phi_j^h(+) = \max\left(0, \phi_j^h\right)$, there exist multipliers $\eta_j^h(-)$ and $\eta_j^h(+) \in \mathbf{R}^+$ such that

$$
\begin{aligned}
0 &= \mu^h k^j - p_j U_h'\left(c^h\right) + \beta E^h \left\{ f_j^+ U_h'\left(c^{h+}\right) \right\} - \eta_j^h(-) \\
0 &= -p_j U_h'\left(c^h\right) + \beta E^h \left\{ f_j^+ U_h'\left(c^{h+}\right) \right\} + \eta_j^h(+) \\
0 &= \phi_j^h(+) \eta_j^h(+) \\
0 &= \phi_j^h(-) \eta_j^h(-).
\end{aligned}
\tag{17}
$$

Condition a) guarantees that the future normalized wealth distributions are consistent with the current equilibrium decision of the consumers. Conditions b) and c) are the first-order conditions with respect to real and financial asset holdings from the maximization problem (5) of the consumers. Given that the maximization problem is convex, the first-order conditions are sufficient for a maximum. As a result, we have the following corollary, which can also be found in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) and Kubler and Schmedders (2003):

Corollary 1 *A Markov equilibrium is a collateral constrained equilibrium.*

Before continue, let me briefly discuss the prices of real and financial assets in a Markov equilibrium. We can rewrite the first-order condition with respect to real asset holding (16) as

$$q U_h'\left(c^h\right) = \mu^h + \beta E^h \left\{ \left(q^+ + d^+\right) U_h'\left(c^{h+}\right) \right\} \geq \beta E^h \left\{ \left(q^+ + d^+\right) U_h'\left(c^{h+}\right) \right\}.$$

By re-iterating this inequality we obtain

$$q_t \geq E_t^h \left\{ \sum_{r=1}^{\infty} \beta^r d_{t+r} \frac{U_h'\left(c_{t+r}^h\right)}{U_h'\left(c_t^h\right)} \right\}.$$

We have a strict inequality if there is a strict inequality $\mu_{t+r}^h > 0$ in the future. So the asset price is higher than the present discounted value of the stream of its dividend because in future it can be sold to other agents, as in Harrison and Kreps (1978) or it can be used as collateral to borrow as in Fostel and Geanakoplos (2008). Proposition 2 shows some condition under which collateral constraints will eventually be binding for every agent when they strictly differ in their belief. As a result, the price of the real asset is strictly higher than the present discounted value of its dividend.[22]

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[22] We can also derive a formula for the equity premium that depends on the multilipliers μ similar to the equity premium formula in Mendoza (2010)

Equation (16) also shows that the real asset will have collateral value when some $\mu^h > 0$, in addition to the asset's traditional pay-off value weighted at the appropriate discount factors. Unlike in Alvarez and Jermann (2000), attempts to find a pricing kernel which prices assets using their pay-off value might prove fruitless because assets with the same payoffs but different collateral values will have different prices. This point is also emphasized in Geanakoplos' papers. Following Fostel and Geanakoplos (2008), I define the collateral value of the real asset (as a proportion of the total value of the asset) by the cross-sectional expected value

$$CV_h = \mathbf{E}^h \left[\frac{\mu^h}{qU_h'(c^h)} \right] \tag{18}$$

in the long run stationary distribution. In Section 8, I offer a preliminary estimate of this value in a calibration for the U.S. economy. Cao and Gete (2012) further explore this value.

The first-order condition with respect to financial asset holdings, equation (17), implies that

$$p_j = \frac{\mu^h k^j}{U_h'(c^h)} + \beta E^h \left[f_j^+ \frac{U_h'(c^{h+})}{U_h'(c^h)} \right].$$

As in Garleanu and Pedersen (2010), the price of financial asset j does not only depend on its promised payoffs in future states $\beta E^h \left[f_j^+ \frac{U_h'(c^{h+})}{U_h'(c^h)} \right]$ but also on its collateral requirements $\frac{\mu^h k^j}{U_h'(c^h)}$.

4.2   Existence and Properties of Markov equilibrium

The existence proof is similar to the ones in Kubler and Schmedders (2003) and Magill and Quinzii (1994). We approximate the Markov equilibrium by a sequence of equilibria in finite horizon. There are three steps in the proof. First, using Kakutani fixed point theorem to prove the existence proof of the truncated T-period economy. Second, show that all endogenous variables are bounded. And lastly, show that the limit as T goes to infinity is the equilibrium of the infinite horizon economy.

For the second step, we need the following assumption:

Assumption 6 *There exist $\bar{c}, \underline{c} > 0$ such that*[23]

$$U_h(\underline{c}) + \max\left\{ \frac{\beta}{1-\beta} U_h(\bar{e}), 0 \right\}$$

$$\leq \min\left\{ \frac{1}{1-\beta} \min_{s \in \mathcal{S}} U_h(\underline{e}), \min_{s \in \mathcal{S}} U_h(\underline{e}) \right\} \quad \forall h \in \mathcal{H}.$$

and

$$U_h(\bar{c}) + \min\left\{ \frac{\beta}{1-\beta} U_h(\underline{e}), 0 \right\}$$

$$\geq \max\left\{ \frac{1}{1-\beta} U_h(\bar{e}), U_h(\bar{e}) \right\} \quad \forall h \in \mathcal{H}.$$

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[23] This assumption is slightly different from the one in Kubler and Schmedders (2003) as U_h might be negative, for example with log utility.

The intuition for first inequality is detailed in the existence proof in Cao (2010); it ensures a lower bound for consumption. The second inequality ensures that price of the real asset is bounded from above. Both inequalities are obviously satisfied by CRRA functions with CRRA coefficients exceeding 1.

Lemma 1 *Suppose that Assumption 6 is satisfied then there is a compact set that contains the equilibrium endogenous variables constructed in Cao (2010) for every T and every initial condition lying inside the set.*

Proof. Cao (2010) ∎

Theorem 1 *Under the same conditions, a Markov equilibrium exists.*

Proof. In Cao (2010), I show the existence of Markov equilibria for a general model also with capital accumulation. As in Kubler and Schmedders (2003), we extract a limit from the T-finite horizon equilibria. Lemma 1 guarantees that equilibrium prices and quantities are bounded as T goes to infinity. The proof follows the Appendix of Kubler and Schmedders (2003) but uses an alternative definition of attainable sets in order to accommodate production. ∎

By corollary 1, any Markov equilibrium is a collateral constrained equilibrium. So Markov equilibria inherit all properties of collateral constrained equilibria. In particular, in a Markov equilibrium, every consumer survives, Proposition 1. Regarding asset prices, the construction of Markov equilibria shows that asset prices can be history-dependent in the long run through the evolution of the normalized financial wealth distribution defined in (13).

This result is in contrast with the one in Sandroni (2000) that under complete markets, asset prices do depend on the wealth distribution but the wealth distribution converges in the long run, so asset prices only depend on the current exogenous state s_t. In a Markov equilibrium, the normalized financial wealth distribution constructed in (13) constantly moves over time even in the long run. For example, if some agent h with incorrect belief lose all her real asset holding due to leverage, next period, she can always use her endowment to speculate in the real asset again. In this case, $\widehat{\omega}_t^h$ will jump from 0 to a strictly positive number. So asset prices depend on the past realizations of the exogenous shocks, which determine the evolution of the normalized financial wealth distribution $\widehat{\omega}_t$.

Proposition 4 *When the aggregate endowment is constant across states $s \in \mathcal{S}$, and shocks are I.I.D., long run asset price volatility is higher in Markov equilibria than it is in complete markets equilibria.*

Proof. As shown in Sandroni (2000), in the long run, under complete markets, the economy converges to the one with homogenous beliefs because agents with incorrect beliefs will eventually be driven out of the markets and real asset price $q(s^t)$ converges to a price independent of time and state. Hence, due to I.I.D. shocks and constant aggregate endowment, under complete markets, asset price volatility converges to zero in the long run. In Markov equilibrium, asset price volatility remains above zero as the exogenous shocks constantly change the normalized financial wealth distribution, which, in turn, changes real asset price. ∎

There are two components of asset price volatility. The first and standard one comes from the volatility in the dividend process and the aggregate endowment. The second one comes from wealth distribution, when agents strictly differ in their beliefs. In general, it depends on the correlation of the two components, that we might have asset price volatility higher or lower under collateral constrained versus under complete markets. However, the second component disappears under complete markets because only agents with correct belief survive in the long run. Whereas, under collateral constraints, this component persists. As a result, when we shut down the first component, asset price is more volatile under collateral constraints than it is under complete markets in the long run. In general, the same comparison holds or not depending on the long-run correlation between the first and the second volatility components.

4.3   Numerical Method

The construction of Markov equilibria in the last section also suggests an algorithm to compute them. The following algorithm is based on Kubler and Schmedders (2003). There is one important difference between the algorithm here and the original algorithm. The future wealth distributions are included in the current mapping ρ instead of solving for them using sub-fixed-point loops. This innovation reduces significantly the computing time, given that solving for a fixed-point is time consuming in MATLAB. Moreover, in section 5, as we seek to find the set of actively traded financial assets and the equilibrium leverage level in the economy, we need to know the future prices of the real asset. As we know the future wealth distributions, these future prices can be computed easily.

As in the existence proof, we look for the following correspondence

$$\rho : \mathcal{S} \times \Omega \longrightarrow \widehat{\mathcal{V}} \times \Omega^S \times \mathcal{L}$$
$$(s, \omega) \longmapsto \left(\widehat{v}, \omega_s^+, \mu, \eta\right)$$

$\widehat{v} \in \widehat{\mathcal{V}}$ is the set of endogenous variables excluding the wealth distribution as defined in (14). $(\omega_s^+)_{s \in \mathcal{S}}$ are the wealth distributions in the S future states and $(\mu, \eta) \in \mathcal{L}$ are Lagrange multipliers as defined in subsection 4.1.

From a given continuous initial mapping $\rho^0 = (\rho_1^0, \rho_2^0, \ldots, \rho_S^0)$, we construct the sequence of mappings $\{\rho^n = (\rho_1^n, \rho_2^n, \ldots, \rho_S^n)\}_{n=0}^{\infty}$ by induction. Suppose we have obtained ρ^n, for each state variable (s, ω), we look for

$$\rho_s^{n+1}(\omega) = \left(\widehat{v}_{n+1}, \omega_{s,n+1}^+, \mu_{n+1}, \eta_{n+1}\right)$$

that solves the first-order conditions (16), (17), market clearing conditions, and the consistency of the future wealth distribution (15).

We construct the sequence $\{\rho^n\}_{n=0}^{\infty}$ on a finite discretization of $\mathcal{S} \times \Omega$. So from ρ^n to ρ^{n+1}, we will have to extrapolate the values of ρ^n to outside the grid using extrapolation methods in MATLAB. Fixing a precision δ, the algorithm stops when $\|\rho^{n+1} - \rho^n\| < \delta$.

There are two important details in implementing this algorithm: First, in order to calculate the $(n + 1)$-th mapping ρ^{n+1} from the n−th mapping, we need to only keep track of the consumption decisions c^h and asset prices q and p_j. Even though other asset holding decisions

and Lagrange multipliers might not be differentiable functions of the normalized financial wealth distribution, the consumption decisions and asset prices normally are.[24]Relatedly, when there are redundant assets, there might be multiple asset holdings that implement the same consumption policies and asset prices.[25] Second, if we choose the initial mapping ρ^0 as an equilibrium of the $1-$period economy as in Subsection 4.2, then ρ^n corresponds to an equilibrium of the $(n+1)-$period economy. I follow this choice in computing an equilibrium of the two agent economy presented below.

5 Asset price volatility and leverage

In this section, I focus on a special case of the general framework analyzed in sections 3 and 4. I assume that only financial assets with state-independent future promises, i.e. collateralized bonds, are traded. Indeed, most of the collateralized financial assets traded in the financial markets are collateralized bonds. However, despite the state-independency of future promises, the effective payoffs, as defined in (2) can be state-dependent due to the possibility to default. The restriction on collateral bonds still leave many options of financial assets to the consumers: bonds with different levels of collateral. Proposition 5 in Subsubsection 5.1.1 shows special conditions in which we can identify exactly which bonds are actively traded in equilibrium. This proposition significantly simplify the computation procedure described in Subsection 4.3. I further restrict myself to an economy with only two types of agents. In such an economy, the normalized financial wealth distribution can be summarized by only one number, which is the fraction of the normalized financial wealth held by one of the two agents. Under these two simplifications, I can compute collateral constrained equilibria in order to study asset price and leverage.

In Subsubsection 5.2.1, I start the numerical analysis by calculating the equilibrium of a benchmark model in which agents can only trade in the real asset. This economy corresponds to the liquidity constrained economy in Kehoe and Levine (1993) with belief heterogeneity. The numerical result shows that agents with incorrect beliefs (the optimists) not only survive but prosper in the stationary equilibrium by constantly postponing consumption to speculate in the real asset. Moreover, in contrast to the limits to arbitrage theory of Shleifer and Vishny (1997), the price dynamics is such that the agents with correct beliefs (the pessimists) do not always want to short sell the asset to take advantage of the over-valuation created by the optimists.

Then in Subsubsection 5.2.2, I allow for the agents to borrow using the asset as collateral. Endogenous financial assets interestingly generates a lot of volatility in the financial wealth distribution as agents borrow to the maximum and lose most of their financial wealth as they lose their bets but their wealth increases largely when they win. This volatility in the wealth distribution in turn feeds into asset price volatility. As in Mendoza (2010), the economy undergoes episodes of non-binding collateral constraint, with low asset price volatility and episodes of binding collateral constraint, i.e. *debt-deflation*, with high asset price volatility.

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[24]See Brumm and Grill (2010) for an algorithm with adapted grid points that deals directly with non-differentiabilities in the policy functions.

[25]See Cao, Chen, and Scott (2011) for such an example.

To answer questions related to collateral requirements asked in the introduction, in Subsubsection 5.2.3, I allow regulators to control the sets of financial assets that can be traded. Given the restricted set, the endogenous active assets can still be determined. One special case is the extreme regulation that shuts down financial markets. There are surprising consequences of these regulations on the equilibrium wealth distribution and on asset prices. In contrast to conventional wisdoms, tighter financial regulations increase the long run financial wealth of agents with incorrect beliefs as well as asset price volatility.

An endogenous set of traded assets also implies endogenous leverage which has been an object of interest during the current financial crisis. In order to match the observed pattern of leverage, i.e., high in good states and low in bad states, I introduce the possibility for changing volatility from one aggregate state to others. This feature is introduced in Subsubsection 5.2.4.

5.1 Collateralized bond economy

Sections 3 and 4 analyze general economies with an arbitrary number of agents and general forms of financial assets. In order to compute collateral constrained equilibrium following the algorithm in Subsection 4.3, we need to know which financial assets are traded. Subsubsection 5.1.1 show a special case in which only financial asset is traded in equilibrium. In Subsubsection 5.1.2, I further simplify the computation problem by assuming that there are only two representative agents in the economy, thus the normalized financial wealth distribution can be captured by only a single number which is the normalized financial wealth of one representative agent.

5.1.1 Collateralized bonds

As in the general model in Section 3, the supply of the real asset is exogenous and normalized to 1 and $q\left(s^t\right)$ again denotes the ex-dividend price of the asset in each history $s^t = (s_0, s_1, \ldots, s_t)$. To study the standard debt contracts, I consider the sets $\mathcal{J}_t$ of financial assets which promise state-independent payoffs next period. I normalize these promises to $b_j = 1$. Asset j also requires k_j units of the real asset as collateral. The effective pay-off is therefore

$$f_{j,t+1}\left(s^{t+1}\right) = \min\left\{1, k_j\left(q\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right)\right\}.$$

Due to the finite supply of the real asset, in equilibrium only a subset of the financial assets in $\mathcal{J}_t$ are traded. Determining which asset are traded allows us to understand the evolution of leverage in the economy (Remark 3). This is also important for computing collateral constrained equilibria in Subsection 4.3. It turns out that in some special cases, we can determine exactly which financial assets are traded. For example, Fostel and Geanakoplos (2008) and Geanakoplos (2009) argue that if we allow for the set $\mathcal{J}_t$ to be dense enough then in equilibrium the only financial asset traded in equilibrium is the one with the minimum collateral level to avoid default.[26] This statement also applies to the infinite horizon set up in this paper under the condition that in any history, there are only two succeeding future

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[26] As noticed in Remark 3, the leverage level corresponding to this unique financial asset can be called the leverage level in the economy.

exogeneous states. Proposition 5 below makes it clear. The proposition uses the following definition

Definition 2 *Two collateral constrained equilibria are* equivalent *if they have the same allocation of consumption to the consumers and the same prices of real and financial assets. They might differ in the consumers' portfolios of real and financial assets.*

Proposition 5 *Consider a collateral constrained equilibrium and suppose in a history s^t, there are only two possible future exogenous states s_{t+1}. Let*

$$u_t = \max_{s^{t+1}|s^t} \left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right) \right)$$

$$d_t = \min_{s^{t+1}|s^t} \left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right) \right)$$

and

$$k_t^* = \frac{1}{d_t}.$$

We can find another collateral constrained equilibrium equivalent to the initial one such that only the financial assets with the collateral requirements

$$\max_{j \in \mathcal{J}_t, k_j \leq k_t^*} k_j \quad and \quad \min_{j \in \mathcal{J}_t, k_j \geq k_t^*} k_j$$

are actively traded. In particular, when $k_t^* \in \mathcal{J}_t$, we can always find an equivalent equilibrium in which only financial asset with the collateral requirement exactly equal to k_t^* is traded.

Proof. Intuitively, this proposition is true because with only two future states, two assets, a financial asset and the real asset, we can effectively replicate the pay-off all other financial assets. But we need to make sure that the collateral constraints, including the no short-selling constraint, are satisfied in the replications. The detail of the proof is in the appendix. ∎

Unless specified otherwise, I suppose that the set of financial assets, $\mathcal{J}_t$, includes all collateral requirements $k \in \mathbf{R}^+, k > 0$.[27] Proposition 5 says that, for any collateral constrained equilibrium, we can find an equivalent collateral constrained equilibrium in which the only financial asset with the collateral requirement exactly equals to $k^*(s^t)$ are traded in equilibrium. Therefore in such an equilibrium the only actively traded financial asset is riskless to its buyers. Let $p(s^t)$ denote the price of this financial asset. The endogenous interest rate is therefore $r(s^t) = \frac{1}{p(s^t)} - 1$. Proposition 5 is also useful to study financial regulations which correspond to choosing the sets $\mathcal{J}_t$ in Subsection 5.2.3.[28]

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[27]To apply the existence theorem 1 I need $\mathcal{J}$ to be finite. But we can think of $\mathcal{J}$ as a fine enough grid.

[28]The uniqueness of actively traded financial assets established in Geanakoplos (2009) and He and Xiong (2011) is in the "equivalent" sense in Definition 2.

5.1.2 Belief heterogeneity

Consider a special case of the general model presented in Section 3. There are two exogenous states $\mathcal{S} = \{G, B\}$ and one real asset of which the dividend depends on the exogeneous state:

$$d(G) > d(B).$$

The exogenous state follows an I.I.D. process, with the probability of high dividends, π, unknown to agents in this economy.

There are two types of consumers (measure one in each type), the optimists, O, and the pessimists, P, who differ in their belief: They have different estimates the probability of high dividend π_G^h, $h \in \{O, P\}$. I suppose that $\pi_G^O > \pi_G^P$, i.e. optimists always think that good states are more likely than the pessimists think they are. In each history s^t there are only two future exogenous states, thus Proposition 5 allows us to focus on only one level of collateral requirement k_t^*. As a result, the collateral constraint (7) can be replaced by the following two constraints: no short-sale constraint

$$\theta_{t+1}^h \geq 0,$$

and collateral constraint

$$\theta_{t+1}^h + \phi_{t+1}^h k_t^* \geq 0.$$

Each agent then maximizes the inter-temporal utility (5) given their belief of the evolution of the exogeneous state, and is subject to the budget constraint:

$$c_t^h + q_t \theta_{t+1}^h + p_t \phi_{t+1}^h \leq e_t^h + (q_t + d_t) \theta_t^h + f_t \phi_t^h \tag{19}$$

and the two additional constraints above.

Given prices q and p, this program yields solution $c_t^h(s^t), \theta_{t+1}^h(s^t), \phi_{t+1}^h(s^t)$. In equilibrium prices $\{q_t(s^t)\}$ and $\{p_t(s^t)\}$ are such that asset and financial markets clear, i.e.,

$$\theta_{t+1}^O + \theta_{t+1}^P = 1$$
$$\phi_{t+1}^O + \phi_{t+1}^P = 0$$

for each history s^t.

With only one real asset and one financial asset, the general formula for normalized financial wealth (13) translates into

$$\omega_t^h = \frac{(q_t + d_t) \theta_t^h + f_t \phi_t^h}{q_t + d_t}.$$

Again, due to the collateral constraint, in equilibrium, ω_t^h must always be positive and

$$\omega_t^O + \omega_t^P = 1.$$

The pay-off relevant state space

$$\{(\omega_t^O, s_t) : \omega_t^O \in [0, 1] \text{ and } s_t \in \{G, B\}\}$$

is compact.[29] Subsection 1 shows the existence of collateral constrained equilibria under the form of Markov equilibria in which prices and allocations depend solely on that state defined above. Subsection 4.3 provides an algorithm to compute such equilibria. As explained in Subsection 4.2, the equilibrium asset prices depend not only on the exogenous state but also on the normalized financial wealth ω_t^O.

5.2 Numerical Results

In this subsection, I will chose plausible parameters to quantify the magnitude of the properties presented in Sections 3 and 4. These parameters are chosen such that the size of the investment banking sector is about 3% (e^O) of the U.S. economy, and the size of the mortgage-backed security market in the U.S. (the real asset) is about 20% of the U.S. annual GDP. In particular, let

$$\beta = 0.95$$
$$d(G) = 1 > d(B) = 0.2$$
$$U(c) = \log(c),$$

and the beliefs are $\pi^O = 0.9 > \pi^P = 0.5$. The algorithm takes about 15 minutes to converge. To study the issue of survival and its effect on asset prices, I assume that the pessimists have the correct belief, i.e., $\pi = \pi^P = 0.5$. Thus the optimists are over-optimistics.

I fix the endowments of the pessimists and the optimists at

$$e^P = \begin{bmatrix} 100 & 100.8 \end{bmatrix}$$
$$e^O = \begin{bmatrix} 3 & 3 \end{bmatrix}.$$

The aggregate endowment is kept constant by choosing the pessimists' endowment to be state dependent.

In the next Subsubsection, I study the properties of equilibria under different market structures (different set $\mathcal{J}_t$'s) using the numerical method developed in Subsection 4.3.

5.2.1 Liquidity constrained equilibrium

As a benchmark, I assume that the set $\mathcal{J}_t$ is empty, then a collateral constrained economy is equivalent to an economy in which agents can only trade in the real asset

$$c_t^h + q_t \theta_{t+1}^h \leq e_t^h + (q_t + d_t) \theta_t^h \tag{20}$$

and $\theta_{t+1}^h \geq 0$. This economy corresponds the liquidity constrained economy studied in Kehoe and Levine (2001).

The left panel in Figure 1 shows the relationship between the price of the asset as functions of the current fraction of the real asset, θ_t^O, held by the optimists when the current exogeneous state $s_t = G$ (blue line) and $s_t = B$ (green line) respectively. The two dotted black bands

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[29]Given that the optimists prefer holding the real asset, i.e., $k_t^O > 0$, ω_t^O corresponds to the fraction of the asset owned by the optimists.
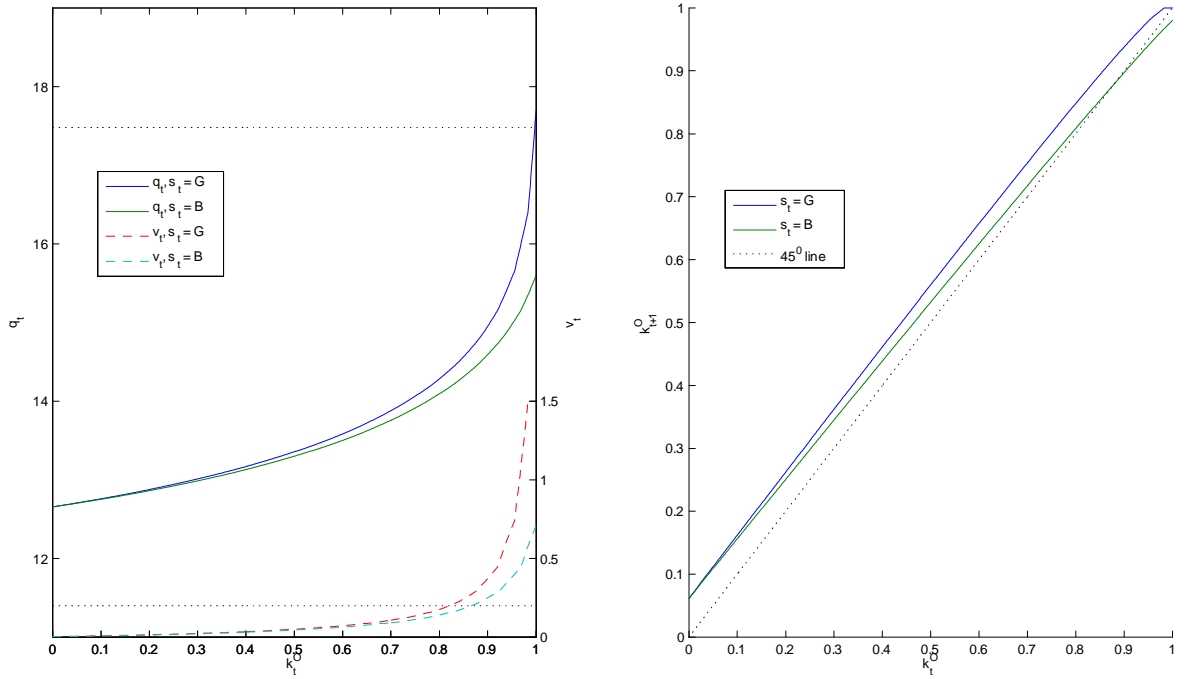
Figure 1: Asset prices and asset holdings without leverage

are the prices of the real asset evaluated using the belief the optimists (upper band) and the belief of the pessimist (lower band).[30] The scale is on the left axis. Similarly, the right panel shows the future fraction of the real asset held by the optimists, θ_{t+1}^O. We can easily see that when θ_t^O is far away enough from 1, $0 < \theta_{t+1}^O < 1$, so both sets of agents are marginal buyer of the real asset.[31] Given that the consumption of the pessimists changes relatively little due to their high level of non-tradable endowment, the price of the asset is close to the pessimists' estimate (the lower band). This is true if the non-short selling constraint of the pessimists are not going to bind in the near future, i.e., θ_t^O, sufficiently far away from 1. When θ_t^O is close to 1, θ_{t+1}^O is equal to 1, thus $\theta_{t+1}^P = 0$, i.e., the optimists are the only marginal buyers of the real asset, thus its price is close to the optimists' valuation (the upper band). Notice also that when $s_t = G$ and θ_t^O close to 1, the price of the asset exceeds the valuation of any of the agents in the economy. This phenomenon is similar to the one in Harrison and Kreps (1978), where agents expect to be able to sell their assets to other agents in the economy. Interestingly, only when θ_t^O is close to 1 and the price of the real asset is too high, the pessimists start trying to short-sell the real asset but they are constrained by the short-selling constraint.

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[30] $P^h = \frac{\beta}{1-\beta} \left(\pi^h(G) d(G) + \left(1 - \pi^h(G)\right) d(B) \right)$

[31] In constrast to the limits to arbitrage channel in Shleifer and Vishny (1997), the agents with correct beliefs, i.e. the pessimists, are not constrained by the short-selling constraint all the time. The dynamics of asset price is such that the pessimists are happy to hold the asset as well. Only when k_t^O is sufficiently, or the asset is sufficiently over-valued, the pessimits start their attempt to short-sell the asset and hit the no-short-selling constraint.

I define the volatility of asset price as one period ahead volatility[32]

$$v_t\left(s^t\right) = std_{s^{t+1}|s^t}\left(q_{t+1}\left(s^{t+1}\right)\right). \tag{21}$$

The dashed lines in Figure 1 shows asset price volatility as functions of k_t^O (red dashed line for $s_t = G$ and blue-green dashed line for $s_t = B$, the scale is on the right axis). Asset price volatility is the highest when $s_t = G$ and θ_t^O close two 1. The budget constraint (20) and the policy function θ_{t+1}^O in the right panel of Figure 1 show that, the consumption of the optimists c_t^O is close to $e_t^O + d_t\theta_t^O$. As θ_t^O is close to 1, the optimists' consumption changes more with the changes in d_t, which translates into higher asset price volatility because the optimists are always the marginal buyer of the real asset. The same logics explains why asset price volatility is increasing in θ_t^O.

Another implication of the policy functions in the right panel is that in the stationary distribution, the optimists will end up holding most of the real asset (see Figure 8). The optimists therefore not only are not driven out of the markets as suggested in Sandroni (2000) but also prospers. They survive by forgoing current consumption, even if the asset pays off low dividend, to hold on to an ever-increasing share of the asset. This survival by "speculation" channel is complementary to the survival by precautionary saving channel suggested by Cogley, Sargent, and Tsyrennikov (2011). The stationary distribution combined with the fact that volatility is increasing in θ_t^O, shows that asset price tend to be very volatile in the long run, as the optimists hold most of the real asset. We will go back to this result when we study the effect of financial regulation on volatility in Subsubsection 5.2.3.

5.2.2 Collateral Constrained Equilibrium

Now let $\mathcal{J}_t$ be the whole positive interval of collateral requirements. Proposition 5 shows that we can restrict ourselves to the economy in which the only financial asset with the collateral level $k_t^*\left(s^t\right) = \frac{1}{\min_{s^{t+1}|s^t}(q(s^{t+1})+d(s_{t+1}))}$ is traded. Collateral constraint (7) can then also be rewritten under a more familiar form[33]

$$\phi_{t+1}^h + \theta_{t+1}^h \min_{s^{t+1}|s^t}\left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right)\right) \geq 0. \tag{22}$$

I rewrite the budget constraint of the optimists (19) using normalized financial wealth, ω_t^O,

$$c_t^O + q_t\theta_{t+1}^O + p_t\phi_{t+1}^O \leq e_t^O + (q_t + d_t)\omega_t^O.$$

Therefore, their total wealth $e_t^O + (q_t + d_t)\omega_t^O$ affects their demand for the asset. If non-financial endowment e_t^O of the optimists is small relative to the price of the asset, their demand for the asset is more elastic with respect to their financial wealth $(q_t + d_t)\omega_t^O$. In Cao (2010), I investigate this relationship by varying the endowment of the optimists.

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[32]This definition corresponds to the instantaneous volality in continuous time asset pricing models. See Xiong and Yan (2009) for the most recent use of the application in the context of belief heterogeneity and complete markets.

[33]Under this form, this economy is a mixture of liquidity constrained and debt constrained economies studied in Kehoe and Levine (2001).

Figure 2 plots the price of the real asset as a function of the optimists' normalized financial wealth ω_t^O for the good state $s_t = G$ (blue solid line) and for the bad state $s_t = B$ (green solid line). The scale is on the left axis and the two black dotted bands are the valuations of the optimists and the pessimists. Interest rate r is also endogenously determined in this economy, however most of the time it hovers around the common discount factors of the two agents, i.e. $r(s^t) \approx \frac{1}{\beta} - 1$ because, as the pessimists have relatively large endowments, they are the marginal buyers of the collateralized bonds.

To understand the shape of the price functions in Figure 2, it is helpful to look at the equilibrium portfolio choice of the agents in Figure 3. The blue lines on the left and right panels are the real asset holdings of the optimists, θ_{t+1}^O, as functions of the normalized financial wealth ω_t^O when the state s_t is G or B respectively. Similarly, the green lines correspond to the bond holdings of the optimists, ϕ_{t+1}^O and the red lines correspond to the borrowing constraint $-\theta_{t+1}^O \min_{s^{t+1}|s^t} (q(s^{t+1}) + d(s_{t+1}))$. Negative ϕ_{t+1}^O means the optimists borrow from the pessimists to invest into the real asset. Leverage induces the optimists to hold the total supply of the real asset. The pessimists are no longer the marginal buyers of the real asset as in Subsubsection 5.2.1, so the price of the real asset is very close to the valuation of the optimists. Figure 3 also shows that the collateral constraint (22) is binding when $\omega_t^O < 0.1$. Due to this binding constraint, the optimists are restricted in their ability to smooth consumption, so their consumption becomes relatively low. Low consumption drives up marginal utility and consequently lower the price of the real asset. Thus the slope of the price function in normalized financial wealth is higher when $\omega_t^O < 0.1$. As price and portfolio choice are all endogenous in equilibrium, we can re-interpret the high slope under the light of the "*debt-deflation*" channel: In the region in which the borrowing constraint binds, or is going to bind in the near future, when a bad shock hits the economy, that is $s_{t+1} = B$, the optimists are forced to liquidate their real asset holdings. This *fire sale* of the real asset reduces its price and tightens the constraints further, thus setting off a vicious cycle of falling asset price and binding collateral constraint. This dynamics of asset price under borrowing constraint is called the *debt-deflation* channel in a small-open economy in Mendoza (2010). This example shows that the channel still operates when we are in a closed-economy with endogenous interest rate, $r(s^t)$ as opposed to exogenous interest rates in Mendoza (2010) or in Kocherlakota (2000).

On the right side of Figure 2, when ω_t^O is close to 1, the *debt-deflation* channel is not present as the collateral constraint is not binding or going to be binding in the near future. High asset price elasticity with respect to the normalized financial wealth of the optimists is due to their high exposure to the asset. As the optimists own most of the real asset without too much borrowing from the pessimists (Figure 3), the dividend from the real asset directly impacts the consumption of the optimists. Higher dividend implies higher consumption, lower marginal utility and higher asset price. The behavior of asset price when ω_t^O close to 1 is thus similar to the case without leverage in Subsubsection 5.2.1.

Figure 2 also plots asset price volatility, v_t in (21) as functions of the normalized financial wealth of the optimists, the red dashed line for $s_t = G$ and the green-blue dashed line for $s_t = B$. The scale on the right axe. These functions show that at low levels of financial wealth of the optimists, asset price is mostly invertedly related to asset price volatility. This negative relationship between price level and price volatility has been observed in several empirical studies, for example, Heathcote and Perri (2011). High asset price volatility at
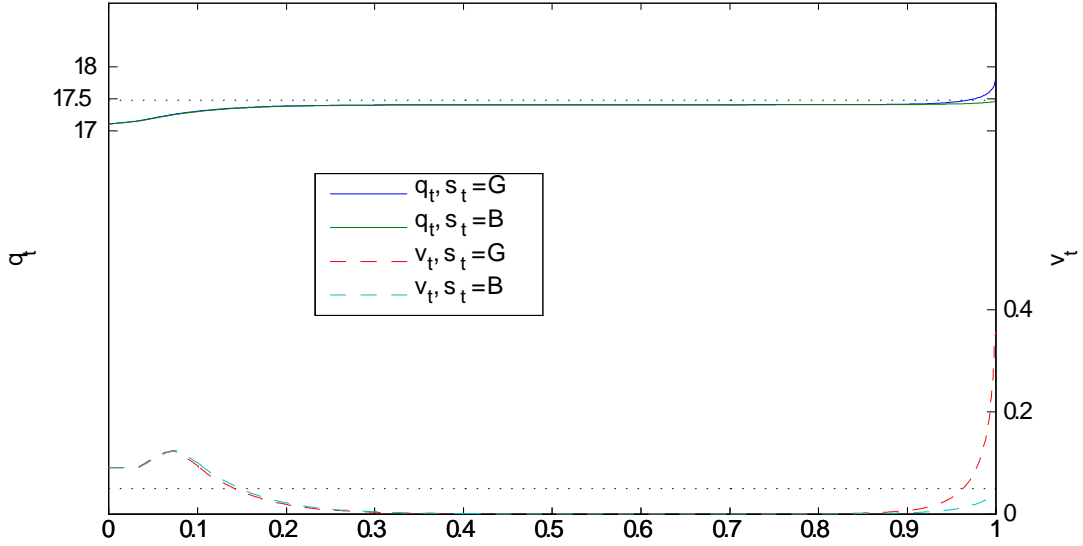
28

Figure 2: Asset price and asset price volatility under full leverage

low levels of ω_t^O is due to the *debt-deflation* channel described above.[34] High asset price volatility at high levels of ω_t^O is due to large changes in the marginal utility of the optimists as in Subsubsection 5.2.1. Comparing Figure 2 and the left panel of Figure 1, for higher ω_t^O, asset price volatility is also lower in this economy than it is in the no leverage economy in Subsubsection 5.2.1 because the ability to borrow allows the optimists to mitigate the drop in consumption when the bad shock with low dividend realizes.

In order to study the dynamics of asset price, we need to combine the fact that asset price as a function of the normalized financial wealth shown in Figure 2 with the evolution of the exogenous state and the evolution of the "normalized financial wealth" distribution, ω_t^O. Figure 4 shows the evolution of ω_t^O. The left and right panels represent the transition of the normalized financial wealth when the current state is G and B respectively. The blue solid lines represent next period normalized wealth of the optimists as a function of the current normalized wealth if the good shock realizes next period. The green solid lines represent the same function when the bad shock realize next period. I also plot the 45 degree lines for comparison (the dotted black lines). This figure shows that, in general, good shocks tend to increase and bad shocks tend to decrease the normalized wealth of the optimists. This is because the optimists bet more on the likelihood of the good state realizes next period (buy borrowing collateralized and investing into the real asset).[35]

We can also think of normalized financial wealth, ω_t^O, as the fraction of the real asset that the optimists owns. When the current state is good, and the fraction is high, the optimists will get a lot of dividends from their real asset holding, due to consumption smoothing they will not consume all the dividends, but will use some part of the dividends to buy more

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[34]Asset price volatility flattens below certain level of ω_t^O because below these levels future normalized financial wealth ω_{t+1}^O after a bad shock reaches the lower bound 0 and cannot fall further.

[35]A similar evolution of the wealth distribution holds for complete markets, see Cao (2010).
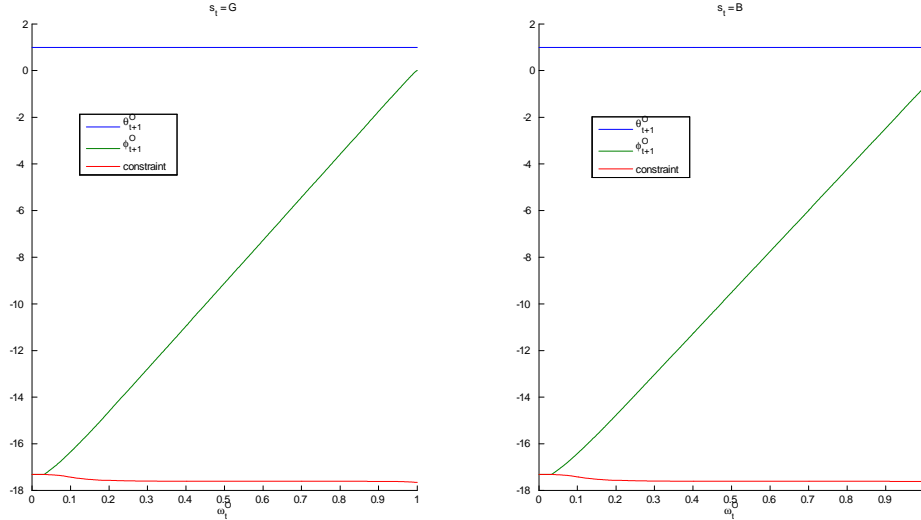
Figure 3: Portfolio choice under full leverage

real asset, so we see that on the left panel, the real asset holding of the optimists normally increases at high ω_t^O. Similarly when the current state is bad, and the fraction is high, the optimists will sell off some of their real asset holding to smooth consumption. As a result, we see on the right panel that the real asset holding of the optimists tends to decrease at high ω_t^O.

The wealth dynamics in Figure 4 implies that, as opposed to the case without leverage in Subsubsection 5.2.1, in the stationary distribution the optimists end up with low normalized financial wealth (Figure 8).[36] At low levels of ω_t^O, the optimists always lever up to the collateral constraint to buy the real asset and when bad shock hits they will lose all their asset holdings (they have to sell off their asset holdings to pay off their debt), in which case, their financial wealth reverts to zero. However, they can always use their non-financial endowment to come back to the financial markets by investing in the real asset again (with leverage). Sometime they are lucky, that is, when the asset pays high dividend and its price appreciates, their financial wealth increases rapidly. But due to the fact that the optimists have incorrect belief, they are unlucky more often than they think, so most of the time they end up with low financial wealth. The ability to lever actually hurts these optimists because they can suffer larger financial losses. So leverage brings the equilibrium close to the complete markets equilibrium in Sandroni (2000), where the optimists' total wealth goes to zero over time.[37]

Lastly, by combining Figures 2 and 4, we can choose sequences of shock realizations to generate booms and busts documented in Burnside, Eichenbaum, and Rebelo (2011). In that paper, to generate the booms and busts in asset prices, the authors rely on the

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[36]This result is due to belief heterogeneity. In Brunnermeier and Sannikov (2010) without belief heterogeneity, the stationary distribution has a U-shape form: the economy spends most of the time in the non-binding constraint region and is occasional pushed toward the binding constraint region.

[37]See Cogley, Sargent, and Tsyrennikov (2012) for a similar point when Arrow securities are introduced in addition to bonds.
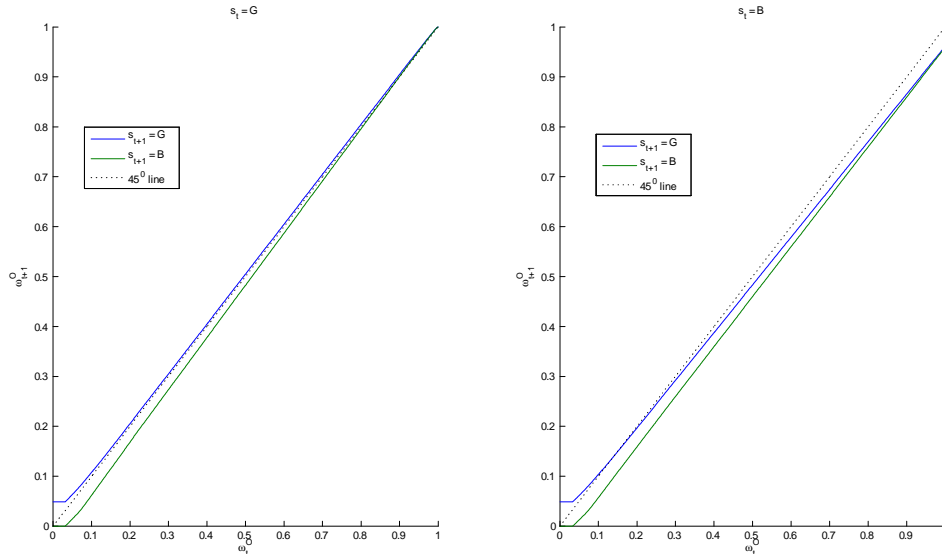
Figure 4: Wealth dynamics under full leverage

slow propagation of beliefs but shoot down the effect of wealth distribution by assuming risk-neutrality.

5.2.3 Regulating Leverage and Asset Price Volatility

Proposition 4 suggests that the variations in the wealth distribution drive up asset price volatility relative to the long run complete markets benchmark. It is then tempting to conclude that by restricting leverage, we can reduce the variation of wealth of the optimists, therefore reduce asset price volatility. However, this simple intuition is not always true. The reason is that, similar to collateral constraints, financial regulation may act as another device to protect the agents with incorrect beliefs from making wrong bets and from disappearing. Or even worse, regulation may help these agents prosper by speculation as in Subsubsection 5.2.1. The higher wealth of the agents with incorrect beliefs increases their impact on real asset price, thus makes real asset price more volatile.

To show this result, I consider a financial regulation that requires a minimum level of real asset holding for each financial asset

$$ k_t^j \geq k_{r,t} = \frac{1}{(1-m)\min_{s^{t+1}|s^t}\left(q\left(s^{t+1}\right)+d\left(s_{t+1}\right)\right)}, $$

where $m \in [0,1]$ stands for margin. Since $k_{r,t} \geq k_t^*$, Proposition 5 implies that for any collateral constrained equilibrium under this regulation, we can find an equivalent equilibrium in which the only financial asset with collateral requirement of exactly $k_{r,t}$ is traded. When $m = 0$, the equilibrium is the unregulated one in Subsubsection 5.2.2. When $m = 1$, collateral requirement is infinite, so there are no financial assets traded, the equilibrium is then the one in which agents only trade in the real asset subject to no short-selling in Subsubsection
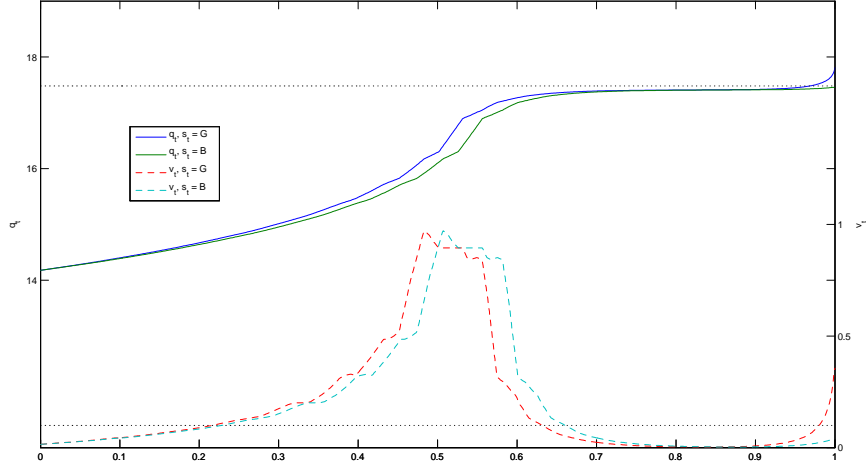
31

Figure 5: Asset price and asset price volatility under regulated leverage

5.2.1. In this Subsubsection, I study another economy with an intermediate $m = 0.5$. Notice that the collateral can also be re-written using a more familiar form:

$$\phi_{t+1}^h + (1-m)\,\theta_{t+1}^h \min_{s^{t+1}|s^t} \left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right) \right) \geq 0. \tag{23}$$

Figures 5, 6, and 7 are the counterparts of Figure 2, 3, and 4 when $m = 0.5$. The portfolio choice in Figure 6, shows that due to restricted leverage the optimists are able to purchase all the supply of the real asset only when they have sufficiently large financial wealth, i.e., $\omega_t^O \geq 0.5$. Below that threshold, the pessimists are also the marginal buyers of the asset, so the price of the asset is closer to the pessimists' valuation, as depicted in Figure 5. The same figure also shows that asset price is the most volatile around $\omega_t^O = 0.5$, at that point, when the bad shock hits, the optimists have to give up some of the real asset. Asset price falls as the pessimists become the marginal buyers. In addition, the *debt deflation* channel kicks in because the collateral constraint on the optimists. Debt deflation further depresses the price of the real asset. The wealth dynamics is described by Figure 7. When $\omega_t^O < 0.5$, the optimists are protected by the regulation, as they cannot borrow as much as they can, most of their investment is in the real asset, so they accumulate wealth through speculation as in Subsubsection 5.2.1. However, above $\omega_t^O = 0.5$, their borrowing constraint is not binding, so they tend to borrow too much, relative to when they have correct belief, and lose wealth over time as in Subsubsection 5.2.2.

Figure 8 shows the stationary distributions of the normalized financial wealth of the optimists under three different financial structures, $m = 0$ (blue solid line), 0.5 (red dashed line)and 1 (black dotted line). The stationary mean level of financial wealth of the optimists is increasing in m. As suggested by the analysis of volatility above, stricter financial regulation, which corresponds to higher collateral requirement, i.e., higher margin, higher m, leads to higher stationary financial wealth of the optimists. Consequently, asset price volatility is higher due to their greater influence on asset price. Indeed that the long run standard deviations of asset return is 0.0263 for $m = 0$, 0.0475 for $m = 0.5$, and 0.0712 for $m = 1$.
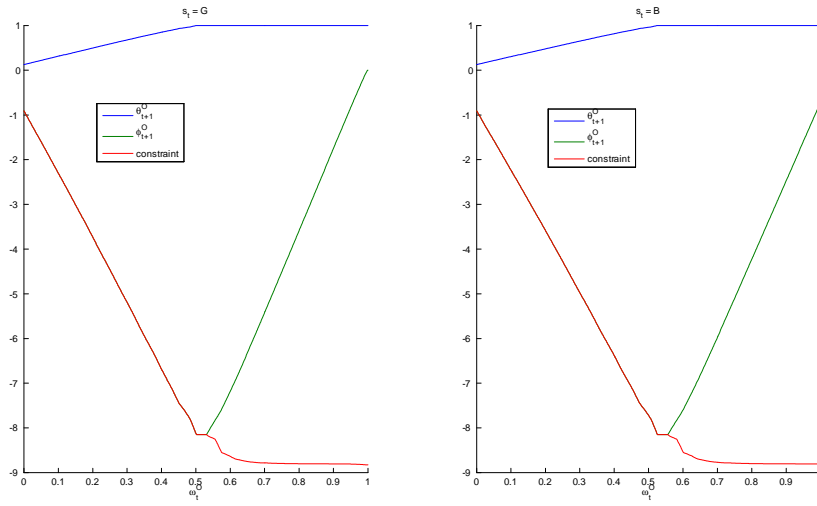
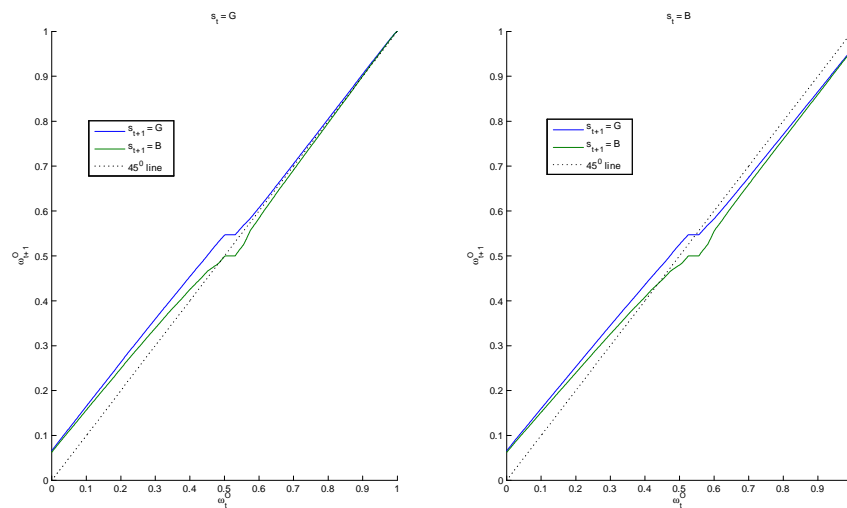Figure 6: Portfolio choice under regulated leverage
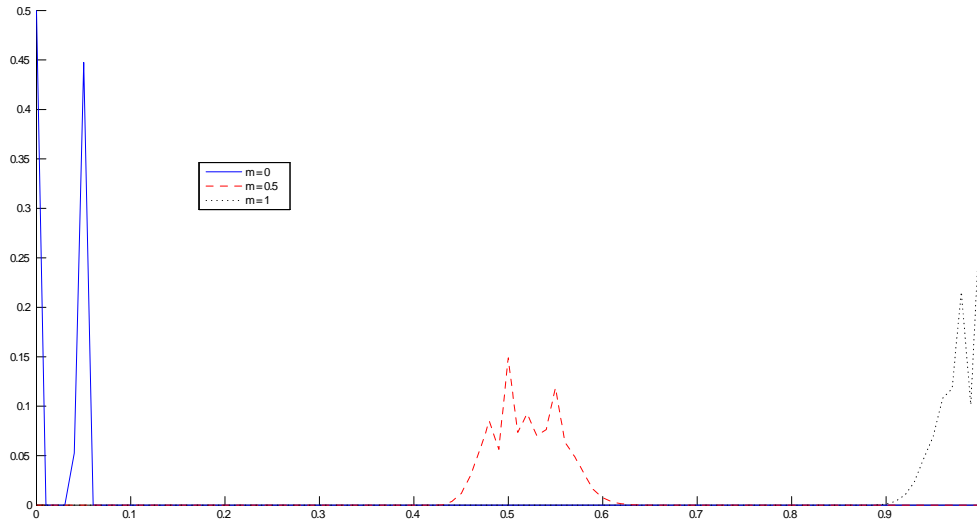

Figure 7: Wealth dynamics under regulated leverage

Figure 8: Stationary distributions of financial wealth under different market structures

Figure 9 shows that this increasing relationship between collateral requirement and long run asset return volatility holds for other intermediate values of m. This result is in contrast with the result obtained in Brumm, Grill, Kubler, and Schmedders (2011) (Figure 2) that collateral requirement decreases asset return volatility at high levels of margin requirement. This difference comes from the fact that, in Brumm, Grill, Kubler, and Schmedders (2011) agents trade due to their difference in risk-aversion (under Epstein-Zin recursive preferences) while in my paper, agents trade due to their difference in beliefs. Under belief heterogeneity, there is a new mechanism that does not exist in Brumm, Grill, Kubler, and Schmedders (2011): a higher margin requirement actually protects the agents with incorrect beliefs. Thus they are financially wealthier in the long run and their trading activity drives up asset price volatility.

5.2.4 Dynamic leverage cycles

Even though Subsubsections 5.2.1-5.2.3 capture some realistic behavior of asset price including the *debt deflation* channel, leverage defined in equation (3) is not consistent with what we observe in financial markets: high leverage in good times and low leverage in bad times, as documented in Geanakoplos (2009). In order to generate the procyclicality of leverage, I use the insight from Geanakoplos (2009) regarding aggregate uncertainty: bad news must generate more uncertainty and more disagreement in order to reduce equilibrium leverage significantly. The economy also constantly moves between low uncertainty and high uncertainty regimes. To formalize this idea, I assume that in the good state, $s = G$, next period dividend has low variance. However, when a bad shocks hits the economy, $s = GB$ or BB, the variance of next period dividend increases. In this dynamic setting, the formulation translates to a dividend process that depends not only on current exogenous shock but also on the last period exogenous shock. Therefore we need to use three exogenous shocks, instead
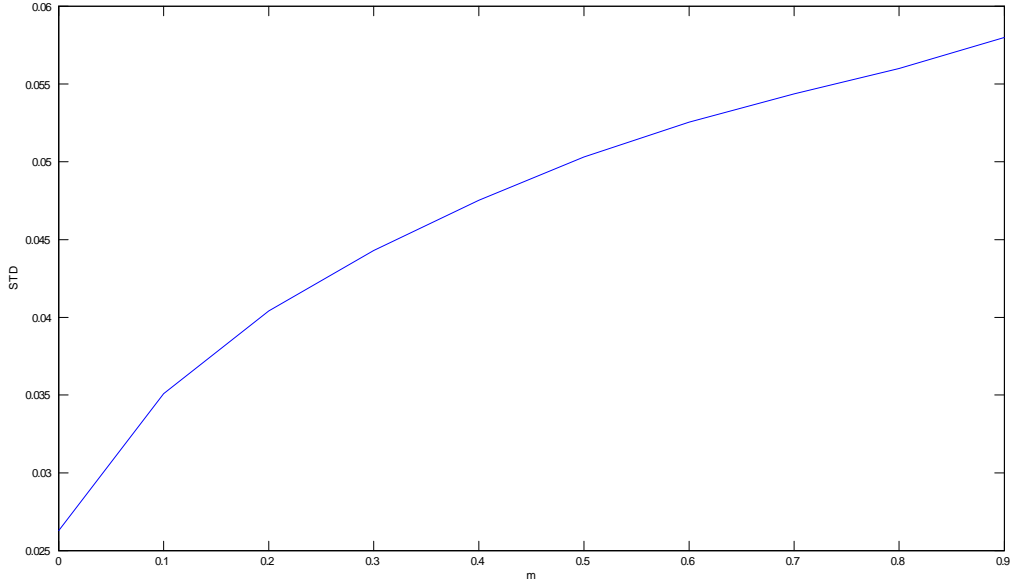
Figure 9: Asset price volatility and margin requirement

of the two exogenous shocks in the last Subsubsections:

$$s \in \{G, GB, BB\}.$$

Figure 10, left panel, shows that the good state, the variance of next period dividend is low, $d = 1$ or 0.8. However in bad states, the variance of next period dividend is higher, $d = 1$ or 0.2. The right panel of the figure shows the evolution through time of the exogenous states using Markov chain representation. Even though we have three exogenous states in this set-up, each state has only two immediate successors. So we can still use Proposition 5 to show that, in any history, there is only one leverage level in the economy. Moreover, in order to generate realistic levels of leverage observed in financial markets, i.e., around 20, we need to set $m = 0.05$ in the collateral constraint (23):

$$\phi_{t+1}^h + 0.95\theta_{t+1}^h \min_{s^{t+1}|s^t} \left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right)\right) \geq 0.$$

The uncertainty structure generates high leverage in good state, $s_t = G$ and low leverage in bad states $s_t = GB$ or BB. Figure 11 shows this pattern of leverage. The dashed line represents leverage level in good states $s = G$ as a function of the normalized wealth distribution. The two solid lines represent leverage level in bad states $s = GB$ or BB.

In addition to the fact that increased uncertainty significantly decreases leverage emphasized in Geanakoplos (2009), we also learn from Figure 11 that financial wealth distribution is another determinant of leverage. Figure 11 shows that leverage decreases dramatically from good states to bad states. However, in contrast to the static version in Geanakoplos (2009), we can quantify the relative contributions of the changes in wealth distribution and the changes in uncertainty to the changes in leverage over the business cycles. Figure 11
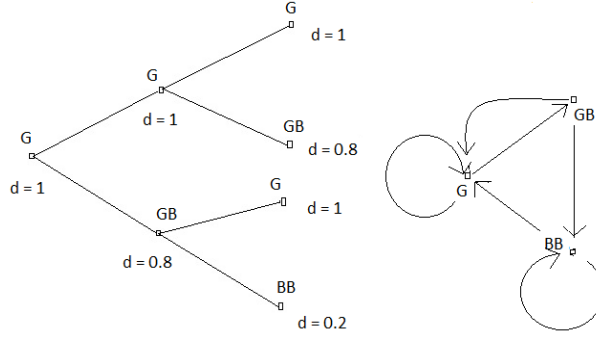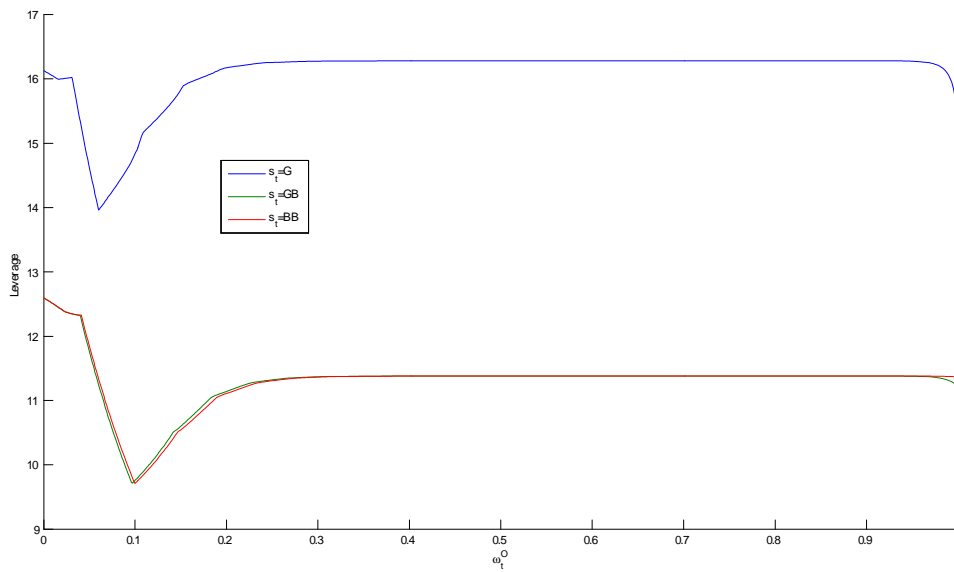
Figure 10: Evolution of the Aggregate States

Figure 11: Leverage Cycles

shows that changes in the wealth distribution contribute relatively little to the changes in leverage at higher levels of ω_t^O. At lower levels of ω_t^O, this version of dynamic leverage cycles generates a pattern of leverage build-up in good times. As shown in Figure 12, good shocks increase leverage as they increase the wealth of the optimists relative to the wealth of the pessimists and leverage is increasing in the wealth of the optimists when $\omega_t^O > 0.1$.

Notice, however, that even though leverage decreases significantly from 17 to 12 when a bad shock hits the economy, that leverage level is still too high compared to what observed during the last financial crisis. Gorton and Metrick (2010) document that leverage in some class of assets declined to almost 1. There are some other channels absent, such as counter-party risk, in this paper that might have caused the rapid decline in leverage. This rapid change in leverage is another important question for future research.
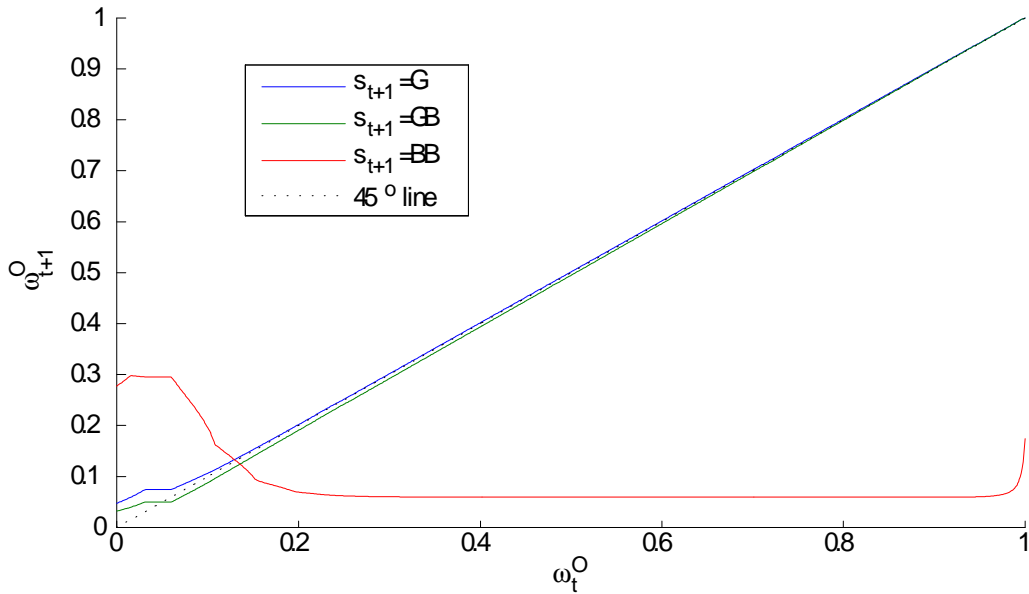
Figure 12: Wealth dynamics when $s_t = G$

6 Conclusion

In this paper I develop a dynamic general equilibrium model to examine the effects of belief heterogeneity on the survival of agents and on asset price dynamics and asset price volatility under different financial market structures. I show that, when financial markets are collateral constrained (endogenously incomplete), agents with incorrect beliefs survive in the long run. The survival of these agents leads to higher asset price volatility. This result contrasts with the frictionless complete markets case, in which agents holding incorrect beliefs are eventually driven out and as a result, asset price exhibits lower volatility. The survival of agents with incorrect beliefs also generates rich dynamics of asset price and wealth distribution.

In a companion paper, Cao (2010), I show the existence of stationary Markov equilibria in this framework with collateral constrained financial markets and with general production and capital accumulation technology. I also develop an algorithm for computing the equilibria. As a result, the framework can be readily used to investigate questions about asset pricing and about the interaction between financial markets and the macroeconomy. For instance, it would be interesting in future work to apply these methods in calibration exercises using more rigorous quantitative asset pricing techniques, such as in Alvarez and Jermann (2001). This could be done by allowing for uncertainty in the growth rate of dividends rather than uncertainty in the levels, as modeled in this paper, in order to match the rate of return on stock markets and the growth rate of aggregate consumption. Such a model would provide a set of moment conditions that could be used to estimate relevant parameters using GMM as in Chien and Lustig (2009). A challenge in such work, however, is that finding the Markov equilibria is computationally demanding. I start this exercise in Section 8 and further follow that path in Cao, Chen, and Scott (2011).

A second avenue for further research is to examine more normative questions in the

framework developed in this paper. My results suggest, for example, that financial regulation aimed at reducing asset price volatility should be state-depedent, as conjectured by Geanakoplos (2009). It would also be interesting to consider the effects of other intervention policies, such as bail-out or monetary policies.

7   Appendix I

Proof of Proposition 5.   To simplify the notations, I will suppress the subscripts t from the variables. Let $k_d = \max_{j \in \mathcal{J}, k_j \leq k^*} k_j$. Let p_d denote the price of the financial assets with collateral requirement k_d. Suppose that there is another financial asset in $\mathcal{J}$ that is actively traded and have collateral requirement $k \leq k^*$. Then by definition $k < k_d$. Let p_k denote the price of the financial asset. We have two cases:

Case 1) $\frac{1}{d} \geq k_d > k > \frac{1}{u}$: Consider the optimal portfolio choice of a seller of financial asset k. The pay-off from selling the asset is $(ku - 1, 0)$ and she has to pay $kq - p_k$ in cash: she buys k units of the real asset but she get p_k from selling the financial asset. So the return on the financial asset is $\frac{ku-1}{kq-p_k}$ (when good state realizes next period). Similarly, the return from selling financial asset k_d is $\frac{k_d u - 1}{k_d q - p_d}$. If there are sellers for financial asset k, it then implies that

$$\frac{ku - 1}{kq - p_k} \geq \frac{k_d u - 1}{k_d q - p_d}.$$

or equivalently

$$p_k \geq \frac{ku - 1}{k_d u - 1} p_d + \frac{k_d - k}{k_d u - 1} q, \tag{24}$$

otherwise, sellers will strictly prefer selling financial asset k_d to financial asset k. Now from the perspective of the buyers of financial assets, the pay-off of financial asset k is $(1, kd)$. We can write this pay-off as a portfolio of financial asset k_d and the real asset:

$$\begin{bmatrix} 1 \\ kd \end{bmatrix} = \frac{ku - 1}{k_d u - 1} \begin{bmatrix} 1 \\ k_d d \end{bmatrix} + \frac{k_d - k}{k_d u - 1} \begin{bmatrix} u \\ d \end{bmatrix}. \tag{25}$$

As a result, if there are buyers for financial asset k, we must have

$$p_k \leq \frac{ku - 1}{k_d u - 1} p_d + \frac{k_d - k}{k_d u - 1} q \tag{26}$$

otherwise the buyers will buy the portfolio (25) of financial asset k_d and the real asset instead. Thus we have both (24) and (26) happen with equality if financial asset k is actively traded. Armed with the equality, we can now prove the proposition. Consider each pair of seller and buyer of a unit of financial asset k: the buyer buys one unit, and the seller who sells one unit of financial asset k, at the same time is required to buy k units of the real asset. We alter the their portfolios in the following way: instead of buying one unit of financial asset k, the buyer buys $\frac{ku-1}{k_d u - 1}$ units of financial asset k_d from the seller and $\frac{k_d - k}{k_d u - 1}$ of the real asset. Given (25) and the equality (26), this changes in portfolio let the consumption and future wealth of the buyer unchanged. Now the seller instead of selling one unit of financial asset k, sells $\frac{ku-1}{k_d u - 1}$ units of financial asset k_d and holds $\frac{ku-1}{k_d u - 1} k_d$ units of the real asset as collateral. Similarly, due to the equality (24), this transaction costs the same and yields the same returns to the seller compared to selling one unit of financial asset k. So the consumption and the future wealth of the seller remains unchanged. Now we just need to verify that the total quantity of the real asset used remain unchanged. Indeed this is true because

$$\frac{k_d - k}{k_d u - 1} + \frac{ku - 1}{k_d u - 1} k_d = k.$$

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Case 2) $\frac{1}{u} \geq k$: Financial asset k's pay-off to the buyers is $k(u,d)$ and to the seller is 0. So the financial asset has exactly the same payoffs as the real asset. This implies, $p_k = kq$. The proposition follows immediately.

Now let $k_u = \min_{j \in \mathcal{J},\, k_j \geq k^*} k_j$. Let p_u denote the price of the financial asset k_u. The proof of the proposition is similar. First, we show that the price of any actively traded financial asset k with $k \geq k^*$ is $p_k = p_u$ and we can alter the portfolio of the buyers and sellers of financial asset k to transfer all the trade in the financial asset to financial asset k_u. ∎

8 Appendix II: Quantitative assessment

In this Section, I apply the numerical solution method to a more seriously calibrated setup used in Heaton and Lucas (1995). In order to do so, I need to modify the economy in Section 3 to allow for the possibility that aggregate endowment grows overtime. As in Heaton and Lucas (1995), I assume that the aggregate endowment $\overline{e}(s^t)$ evolves according to the process

$$\frac{\overline{e}(s^{t+1})}{\overline{e}(s^t)} = 1 + g(s^t).$$

There is only one Lucas tree that pays off the aggregate dividend income at time, D_t and

$$\delta(s^t) = \frac{D(s^t)}{\overline{e}(s^t)},$$

the remaining endowment is distributed among the consumers under the form of labor income

$$\sum_{h \in \mathcal{H}} e^h(s^t) = \overline{e}(s^t) - D(s^t).$$

Consumer h's labor income as a fraction of aggregate labor income is given by

$$\eta^h(s^t) = \frac{e^h(s^t)}{\sum_{h \in \mathcal{H}} e^h(s^t)}.$$

The following Proposition shows that, by working with the normalized variables, $\widehat{c}_t^h = \frac{c_t^h}{\overline{e}_t}$, $\widehat{e}_t^h = \frac{e_t^h}{\overline{e}_t}$, $\widehat{d}_t = \frac{d_t}{\overline{e}_t}$, and $\widehat{q}_t = \frac{q_t}{\overline{e}_t}$, we can find and compute collateral constrained equilibria in which these normalized variables depend only on the normalized financial wealth distribution.

Proposition 6 *Suppose that the per-period utility functions are CRRA $U_h(c) = \frac{c^{1-\sigma_h}}{1-\sigma_h}$, and (g, δ, η) depend only on the current state of the economy, there exists a collateral constrained equilibrium in this growth economy.*

Proof. To apply the existence proof and the numerical method used in Section 4, I use the following normalized variables

$$\widehat{c}_t^h = \frac{c_t^h}{\overline{e}_t}, \widehat{e}_t^h = \frac{e_t^h}{\overline{e}_t}, \widehat{d}_t = \frac{d_t}{\overline{e}_t},$$

and
$$\widehat{q}_t = \frac{q_t}{\overline{e}_t}.$$

Assuming CRRA for the agents, $U_h(c) = \frac{c^{1-\sigma_h}}{1-\sigma_h}$, the expected utility can also be re-written using the normalized variables

$$
\begin{aligned}
\mathbf{E}_t^h \left[\sum_{r=0}^{\infty} \beta^r U_h \left(c_{t+r}^h \right) \right] &= (\overline{e}_t)^{1-\sigma_h} \mathbf{E}_t^h \left[\sum_{r=0}^{\infty} \beta^r U_h \left(\widehat{c}_{t+r}^h \right) \left(\frac{\overline{e}_{t+r}}{\overline{e}_t} \right)^{1-\sigma_h} \right] \\
&= (\overline{e}_t)^{1-\sigma_h} \mathbf{E}_t^h \left[\sum_{r=0}^{\infty} \beta^r \prod_{r'=1}^{r} \left(1 + g \left(s^{t+r'} \right) \right)^{1-\sigma_h} U_h \left(\widehat{c}_{t+r}^h \right) \right]
\end{aligned}
$$

I focus on debt contracts by assuming that, in each node s^t, only financial contracts, i.e., $b_j \left(s^{t+1} \right) = 1$ for all $s^{t+1}|s^t$, are allowed to be traded. I consider the following collateral requirements

$$k_j \left(s^t \right) = \frac{1}{1-m} \max_{s^{t+1}|s^t} \frac{1}{q \left(s^{t+1} \right) + d \left(s^{t+1} \right)},$$

where $0 \leq m < 1$. These collateral constraints correspond to the borrowing constraints

$$\phi_{t+1}^h \leq - \left(1 - m \right) \theta_{t+1}^h \min_{s^{t+1}|s^t} \left\{ q \left(s^{t+1} \right) + d \left(s^{t+1} \right) \right\}.$$

When $m = 1$, no borrowing is possible, the agents are allowed to only trade on the real asset.[38] In these environment, I also use the normalized variables for the choice of debt holding $\widehat{\phi}_t^h = \frac{\phi_t^h}{\overline{e}_t}$.

We rewrite the optimization of the consumer as

$$\max_{\left\{ c_t^h, \theta_{t+1}^h, \phi_{t+1}^h \right\}} \mathbf{E}_0^h \left[\sum_{t=0}^{\infty} \beta_h^t U_h \left(\widehat{c}_t^h \right) \prod_{r=1}^{t} \left(1 + g \left(s^r \right) \right)^{1-\sigma_h} \right]$$

and in each history s^t, she is subject to the budget constraint

$$\widehat{c}_t^h + \widehat{q}_t \theta_{t+1}^h + \widehat{\phi}_{t+1}^h \leq \widehat{e}_t^h + \frac{1}{1+g\left(s^t\right)} \widehat{\phi}_t^h + \left(\widehat{q}_t + \widehat{d}_t \right) \theta_t^h,$$

the collateral constraints

$$\widehat{\phi}_{t+1}^h + \left(1 - m \right) \theta_{t+1}^h \min_{s^{t+1}|s^t} \left\{ \left(\widehat{q} \left(s^{t+1} \right) + \widehat{d} \left(s^{t+1} \right) \right) \left(1 + g \left(s^{t+1} \right) \right) \right\} \geq 0 \qquad (27)$$

---

[38] Alternatively, we can consider the collateral requirements

$$k_j \left(s^t \right) = \frac{1}{1-m} \max_{s^{t+1}|s^t} \frac{1}{q \left(s^{t+1} \right)},$$

which correspond to the borrowing constraints

$$\phi_{t+1}^h \leq - \left(1 - m \right) k_{t+1}^h \min_{s^{t+1}|s^t} q \left(s^{t+1} \right).$$

This constraint is similary to the one used in Kiyotaki and Moore (1997), when borrowers can only commit to deliver the part of the collateral asset without the current dividend. The quantiative implications of such requirements are very similar to the one used here.

and finally the no short-sale constraint in the real asset, $\theta_{t+1}^h \geq 0$, as before. We can just use the same analysis in Section 4 for this economy with the hat variables. ■

The following subsections show that, under reasonable calibrated parameters of the utility functions and exogenous shock processes, 1) the moments of asset prices such as equity premium behave differently under binding collateral constraints than under non binding collateral constraints; 2) due to risk-aversion, the equilibrium portfolio choice of the agents are such that the collateral constraints are not binding often in the stationary distribution; 3) consequently, the unconditional moments of asset prices are not very different from the unconditional moments when there are no collateral constraints. These findings hold both with and without belief heterogeneity. These findings are also consistent with the ones in Mendoza (2010), in which the author finds that "precautionary saving makes sudden stops low probability events nested within normal cycles, as observed in the data."

8.1   Homogeneous beliefs

Table 1 corresponds to Table in Heaton and Lucas (1995). The authors use the annual aggregate labor income and dividend data from NIPA and individual income from PSID to calibrate $g(.), \delta(.), \eta(.)$ and the transition matrix π. There are two representative agents in the economy and η corresponds to the endowment share of agent 1. In Panel A, the growth rate of the economy, g, fluctuates between 0.9904 and 1.0470, and the share of tradable income, δ, stays around 15% of the total endowment. Given the importance of income heterogeneity shown in column 4 of Panel A, I start the analysis with the benchmark in which both agents have the correct estimate of the transition matrix in Panel B. Despite the common belief, income heterogeneity gives the two agent strong incentives to trade with each other.

I use the equilibrium existence and computation procedure developed in Proposition 6 with the calibrated parameters in Table 1. Notice that, the economy in Heaton and Lucas (1995) is the same as in my economy except for the fact that the collateral constraint (27) is replaced by the exogenous borrowing constraint $\phi_t^h \geq -B^h$. Table 2 shows that collateral constraints do not alter significantly the quantitative result in Heaton and Lucas (1995). Despite the collateral constraints, the standard deviations of consumption and stock returns and bond returns are very similar in the economies with collateral constraints to the economy in Heaton and Lucas (1995).[39] The reason for this similarity is that the collateral constraints are not binding often, even though the behavior of the equilibrium variables are very different when the collateral constraints are binding from when they are not. Indeed, Figures 14 and 13 show the equity premium and portfolio choice of agent 1 as functions of her normalized financial wealth, ω_t^1. When the collateral constraint for this agent is binding, i.e. $\omega_t^1 < 0.1$, the equity premium is significantly higher than it is when the constraint is not binding. However, the stationary distribution of wealth shows that the constraints are binding in less than 5% in the distribution (See column 5, the second to last row of Table 3). In addition, the collateral value as a fraction of the price of the real asset is less 0.2% (See column 5, the last row of Table 3).

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[39]The last column in Table 2, $m = 1$, corresponds to the economy in which the agents can only trade in the real asset, subject to the no short selling. Thus there are no bonds traded.

A. Markov chain model for exogenous state variables							
State Number	States						
	g		δ		η		
1	0.9904		0.1402		0.3772		
2	1.0470		0.1437		0.3772		
3	0.9904		0.1561		0.3772		
4	1.0470		0.1599		0.3772		
5	0.9904		0.1402		0.6228		
6	1.0470		0.1437		0.6228		
7	0.9904		0.1561		0.6228		
8	1.0470		0.1599		0.6228		
B. Transition probability matrix $[\pi_{ij}]_{8x8}$							
0.3932	0.2245	0.0793	0.0453	0.1365	0.0799	0.0275	0.0157
0.3044	0.3470	0.0425	0.0484	0.1057	0.1205	0.0147	0.0168
0.0484	0.0425	0.3470	0.3044	0.0168	0.0147	0.1205	0.1057
0.0453	0.0793	0.2245	0.3932	0.0157	0.0275	0.0779	0.1365
0.1365	0.0779	0.0275	0.0157	0.3932	0.2245	0.0793	0.0453
0.1057	0.1205	0.0147	0.0168	0.3044	0.3470	0.0425	0.0484
0.0168	0.0147	0.1205	0.1057	0.0484	0.0425	0.3470	0.3044
0.0157	0.0275	0.0779	0.1365	0.0453	0.0793	0.2245	0.3932

Table 1: Summary Statistics for Baseline Case

8.2 Belief heterogeneity

Now, I introduce belief heterogeneity by assuming the first agent in Heaton and Lucas (1995) are more optimistic about the growth rate of the economy, while the second agents have the correct estimate of the transition matrix: $\pi^2 = \pi$ and

$$
\begin{aligned}
\pi^1(1,1) &= \pi(1,1) - p \\
\pi^1(1,2) &= \pi(1,2) + p \\
\pi^1(3,3) &= \pi(3,3) - p \\
\pi^1(3,4) &= \pi(3,4) + p \\
\pi^1(5,5) &= \pi(5,5) - p \\
\pi^1(5,6) &= \pi(5,6) + p \\
\pi^1(7,7) &= \pi(7,7) - p \\
\pi^1(7,8) &= \pi(7,8) + p,
\end{aligned}
$$

where $p = 0.3$.

Table 3 shows that the standard deviation of stock returns are still similar to when beliefs homogeneous. The reason is again that the collateral constraints are not binding often due to risk-aversion. Figures 15 and 16 plot the equity premium and portfolio choice of agent 1 as functions of agent 1's normalized financial wealth. Equity premium is very high when the collateral constraint is binding. The last two rows of Table 3 shows the small probabilities of

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Moments	Data	CM	HL	$m = 0$	$m = 0.9$	$m = 1$
Consumption Growth						
Average	0.020	0.018	0.018	0.019	0.019	0.019
Standard deviation	0.030	0.028	0.044	0.041	0.045	0.045
Bond return						
Average	0.008	0.080	0.077	0.078	0.077	
Standard deviation	0.026	0.009	0.012	0.012	0.013	
Stock return						
Average	0.089	0.082	0.079	0.079	0.080	0.080
Standard deviation	0.173	0.029	0.032	0.030	0.035	0.036

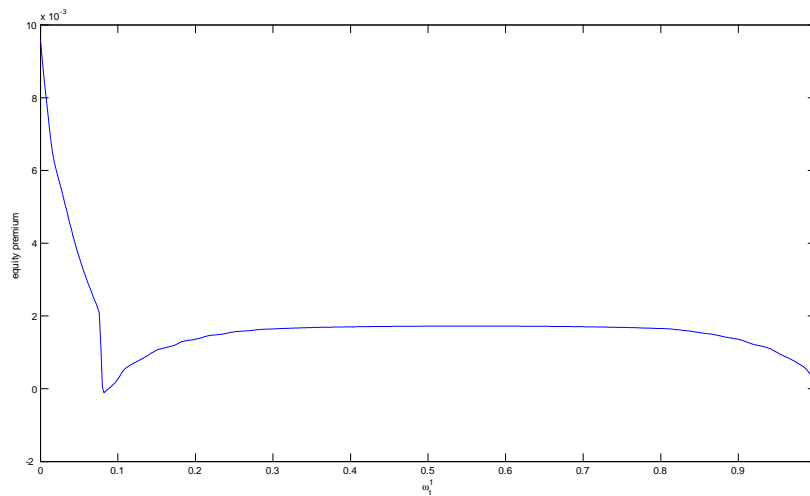Table 2: Summary Statistics for Baseline Case
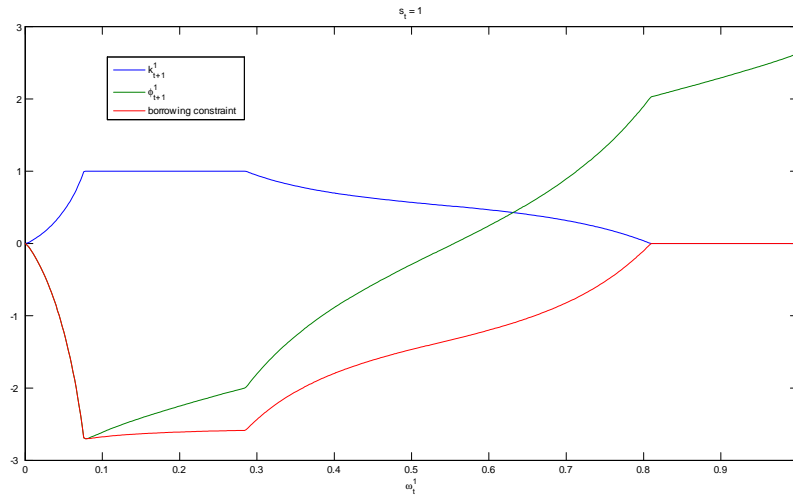
Figure 13: Portfolio choice

Figure 14: Equity premium

Moments	Data	CM	HL	$m=0$	$m=0, \pi^1$	$m=0, \pi^2$
Consumption Growth						
Average	0.020	0.018	0.018	0.019	0.027	0.020
Standard deviation	0.030	0.028	0.044	0.041	0.048	0.057
Bond return						
Average	0.008	0.080	0.077	0.078	0.084	0.080
Standard deviation	0.026	0.009	0.012	0.012	0.011	0.018
Stock return						
Average	0.089	0.082	0.079	0.079	0.086	0.076
Standard deviation	0.173	0.029	0.032	0.030	0.029	0.033
Collateral constraint						
Pr(binding constraint)				0.042	0.085	0.001
Collateral value				0.002	0.006	0.005

Table 3: Summary Statistics with Belief Heterogeneity

binding collateral constraints and the small collateral value as a fraction of asset price under either agent 1's belief or agent 2's belief.
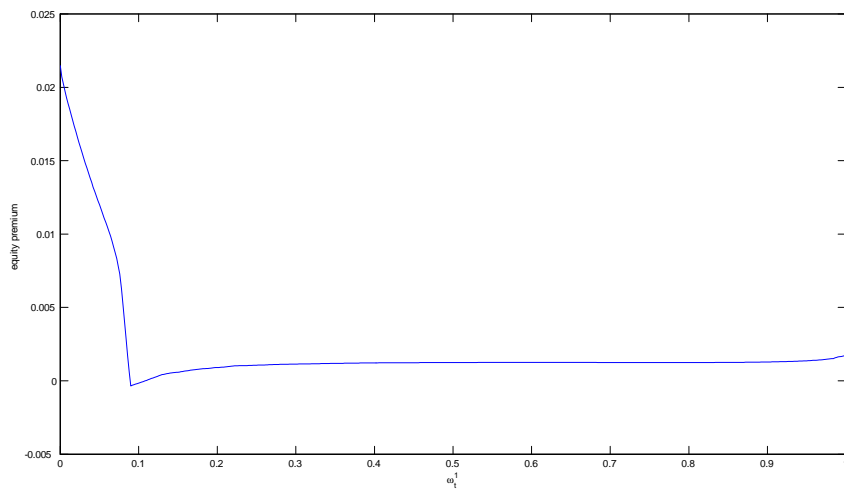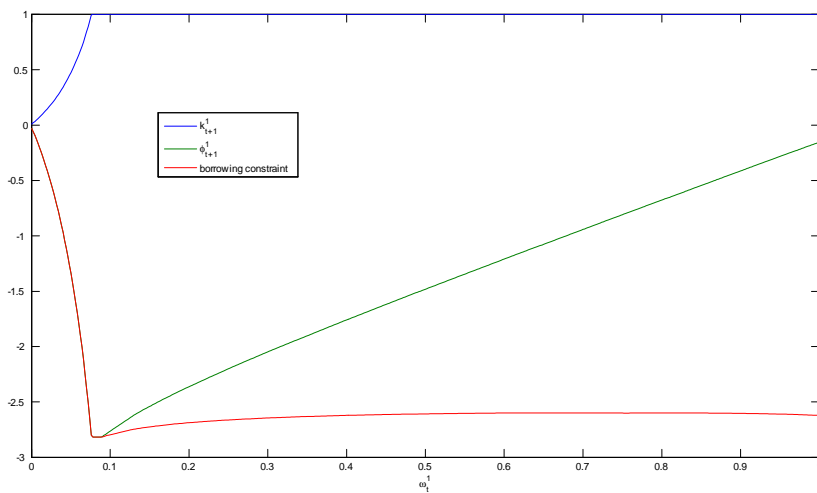
Figure 15: Equity premium

Figure 16: Portfolio choice

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