(30) 1. Consider the Lucas asset pricing model. Suppose that the tree has \( n \) possible dividend realizations \( y_1,\ldots,y_n \) each period and that dividends follow a Markov process.

(a) Suppose we add one-period Arrow securities to the Lucas model. Specifically, there are exactly \( n \) Arrow securities available for purchase each period. Arrow security \( i \) pays out 1 unit of consumption next period if the dividend of the tree next period equals \( y_i \) and pays out 0 in any other dividend realization.

What is the price function for Arrow security \( i \)? Explain the key steps in your reasoning.

(b) Suppose that at the beginning of each period the government imposes a 10 percent tax on the dividend of the tree. It is understood that the proceeds of the tax will be returned to the agent (or equally to all agents) in a lump-sum amount (i.e. independent of share holdings of the agent).

How does this tax and transfer arrangement affect the price function of the tree compared to the price function without taxes and transfers? Explain the key steps in your reasoning.

(25) 2. Consider a firm that maximizes discounted profit. There is a constant risk-free interest rate \( r > 0 \) that the firm uses to discount future profit. The profit in a period when the firm chooses labor input \( l \) is given by \( zF(l) - wl - \tau \max\{0, l^* - l\} \), where \( z \) is a Markovian productivity shock, \( F \) is the production function which is strictly concave, \( w \) is a constant wage and \( l^* \) is last period’s labor input choice. The last term in the profit function captures the cost the firm must pay to the government depending on the relation between current and last period’s labor choice. In this function \( \tau > 0 \) is a proportional tax rate.

(i) Provide a Bellman equation for this problem.

(ii) How does the value of the firm this period change as \( l^* \) (last period’s labor choice) increases, other things equal. Explain your reasoning.
3. Consider a consumption and savings problem of a consumer with preferences: \( \sum_{j=1}^{J} \psi_j u(c_j) \). In this problem, \( c_j \) is consumption in period \( j \) and \( \psi_j \) is the probability of surviving up to period \( j \). It is understood that \( \psi_j \) declines as \( j \) increases. The consumer gets to consume only while alive. Assume that the consumer has no labor income but has some amount of assets at age 1. The consumer can save (but not borrow) at a risk-free rate of \( r > 0 \).

(a) State the problem as a dynamic programming problem.

(b) Is the value function in period 1 differentiable in the asset level? If so, write down this derivative and indicate why this is so?

(c) Under what assumptions is it the case that the consumption path over time is to always reduce consumption?

4. Consider the problem of a planner who maximizes the ex-ante utility of a cohort of agents. Agents are identical in preferences but are heterogeneous in shocks received over the lifetime. Let \((s_1, \ldots, s_j) = s^j \in S^j\) denote an agent’s shock history up to period \( j \). It is understood that \( S^j \) is a finite set for each \( j \) and that \( P(s^j) \) is both the probability that an agent assigns to history \( s^j \) and the fraction of agents in the cohort that realize history \( s^j \). The only source of uncertainty is that agents receive an individual-specific skill shock \( s^j + 1 \) at the start of age \( j + 1 \) that alters their skill level \( h(s^j + 1) \) at the beginning of period \( j + 1 \). Each agent divides available time between work time \( L(s^j) \), learning time \( T(s^j) \) and leisure time \( l(s^j) \). Work time helps to produce a consumption good, whereas learning time helps to increase skill.

The planners problem is as follows:

\[
\max \sum_{j=1}^{J} \sum_{s^j \in S^j} \beta^{j-1} u(c(s^j), l(s^j)) P(s^j) \quad s.t.
\]

\[
(i) \sum_{s^j \in S^j} c(s^j) P(s^j) \leq \sum_{s^j \in S^j} h(s^j) L(s^j) P(s^j), \forall j
\]

\[
(ii) l(s^j) + L(s^j) + T(s^j) = 1, \forall j, s^j
\]

\[
(iii) h(s^j + 1) = H(h(s^j), T(s^j), s_{j+1}), \forall j, s^j
\]

Derive a necessary condition that the intratemporal marginal rate of substitution between \( c(s^j) \) and \( l(s^j) \) must satisfy in a solution to the planners problem. Be clear on exactly how you derive this condition.
Answers:

1(a). Arrow security price function: \( p_i(y) = \frac{\beta u'(y_i)\pi(y_i|y)}{u'(y)} \)

Method:

\[
u'(c(z, y))p_i(y) = \beta u'(c(z', y_i))\pi(y_i|y).
\]

\[
u'(y)p_i(y) = \beta u'(y_i)\pi(y_i|y).
\]

\[
p_i(y) = \frac{\beta u'(y_i)\pi(y_i|y)}{u'(y)}
\]

First equation is a FOC from Bellman’s equation. Second equation imposes equilibrium condition. Third rearranges the second.

1(b) The first equation below we can derive from the usual analysis of the Lucas model but with the tax rate factored in. The leftmost equality in the second equation is a specific solution to the price equation. We know that there is only 1 solution from Lucas’ paper. The other equalities hold by rearrangement. The result is that the price function with a tax rate of \( \tau > 0 \) simply shifts downward the price function for the \( \tau = 0 \) case by the factor \((1 - \tau)\).

\[
p(y; \tau) = E[\frac{\beta u'(y')}{u'(y)} (p(y'; \tau) + y'(1 - \tau)|y]
\]

\[
p(y; \tau) = E[\sum_{t=1}^{\infty} \frac{\beta u'(y_t)}{u'(y)} y_t(1 - \tau)|y] = (1 - \tau)E[\sum_{t=1}^{\infty} \frac{\beta u'(y_t)}{u'(y)} y_t|y] = (1 - \tau)p(y; 0)
\]

2.

(i) Bellman’s equation:

\[
v(z, l^*) = \max_{l \geq 0} [zF(l) - wl - \tau \max\{0, l^* - l\}] + \frac{1}{1 + r} E[v(z', l)|z]
\]

(ii) One could try to answer the second part by arguing that \( v(z, l^*) \) is differentiable in its second component. Then one could try to take a derivative. The problem with this is that for states for which the labor choice is set equal to last period’s labor choice, then the value function will not be differentiable. Thus, one might try something different.

Denote \( g(z, l^*, l) \) the return function in Bellman’s equation above. Also let \( \hat{l} \) be best choice from state \((z, l^* + \delta)\). The inequalities below then hold for \( \delta \geq 0 \).

This implies that \( v(z, l^*) \) weakly decreases as \( l^* \) increases.

\[
v(z, l^*) \geq g(z, l^*, \hat{l}) + \frac{1}{1 + r} E[v(z', \hat{l})|z] \geq g(z, l^* + \delta, \hat{l}) + \frac{1}{1 + r} E[v(z', \hat{l})|z] = v(z, l^* + \delta)
\]
Reasoning: The equality holds by definition. The rightmost inequality holds since \( g(z, l^*, l) \) weakly decreases as \( l^* \) increases. The leftmost inequality holds as \( \hat{l} \) is feasible but may not be the best choice from state \( l^* \).

3. (a) 

\[ v_j(a) = \max \psi_j u(c) + v_{j+1}(a') \quad s.t. \quad c + a' \leq a(1 + r), \quad c, a' \geq 0 \]

(b) Claim : \( v'_j(a) = \psi_j u'(c_j(a))(1 + r) \)

Proof: Set \( w(a) = \psi_j u(a(1 + r) - a^*) + v_{j+1}(a^*) \), where \( a^* \) is the asset choice solving Bellman’s equation from asset level \( a \). Provided that \( u \) is concave and differentiable, it is clear that \( w \) is concave and differentiable in \( a \). Moreover, \( w(a) = v_j(a) \) by construction. Thus, when \( v_j \) is concave, then we can apply Benveniste and Scheinkman’s Thm to get that \( v_j \) is differentiable and see that \( v'_j(a) = w'(a) = \psi_j u'(c_j(a))(1 + r) \).

(c) The first equation is a FOC directly from BE. The second equation applies BS to take the derivative of the value function. The third equation reorganizes the second and expresses consumption using different notation.

\[ \psi_j u'(c) = v'_{j+1}(a') \]
\[ \psi_j u'(c_j(a)) = \psi_{j+1} u'(c_{j+1}(a_j(a))(1 + r) \]
\[ u'(c_j) = \frac{\psi_{j+1}}{\psi_j} u'(c_{j+1})(1 + r) \]

Claim: Assume that \( u \) is strictly increasing concave and differentiable. If \( \frac{\psi_{j+1}}{\psi_j}(1 + r) \leq 1 \), then the last equation above implies that \( c_j \geq c_{j+1} \). If the first inequality is strict, then so is the second inequality.

4. The students might think of two ways of going.

Method 1: Argue that a solution to the planners problem has the feature that there is no feasible perturbation of the solution that improves the objective function.

One useful perturbation is to in state \( s^j \) vary leisure \( l(s^j) \) time by \( \delta \), increase work time \( L(s^j) \) by \( \delta \) and then increase consumption by \( \delta h(s^j) \). Such a perturbation is feasible if consumption, leisure and labor are non-zero. This perturbation only changes the objective function at age \( j \) in state \( s^j \). If there is no gain or loss to a marginal change in \( \delta \), then \( u_1(c(s^j), l(s^j))h(s^j) - u_2(c(s^j), l(s^j)) = 0 \)

\[ \frac{u_2(c(s^j), l(s^j))}{u_1(c(s^j), l(s^j))} = h(s^j) \]

Method 2: Set up a Lagrangian w/ multipliers on all constraints. Produce FOC from the Lagrangian. Reorganize to get the same result.