Aggregate Production Function

\[ Y_t = A_t F(K_t, L_t) \]

- \( Y_t \) - output at time \( t \)
- \( A_t \) - technology level at time \( t \)
- \( (K_t, L_t) \) - capital and labor at time \( t \)
Properties: constant returns to scale

A Production function \( F(K, L) \) has constant returns to scale (CRS) provided that whenever all inputs are scaled up or down by a given factor \( \lambda \), then output is scaled by exactly the same factor.

\[
\lambda Y = F(\lambda K, \lambda L), \forall \lambda > 0
\]

Key Implication: No advantage to being a large country. Only capital-labor ratio matters - set \( \lambda = 1/L \).

\[
\frac{Y}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = F\left(\frac{K}{L}, 1\right)
\]
Properties: constant returns to scale

Geometry- a constant returns to scale production function has some useful geometry. ALL the isoquants are "radial blowups" of a single isoquant.
Properties: diminishing marginal products

The marginal product of a factor of production is the extra output caused by increasing the input by one unit, other things equal. We will assume that marginal products are diminishing or falling as the input is increased. Marginal products are partial derivatives.

Notation:

\[ F_K(K, L) \] - marginal product of capital

\[ F_L(K, L) \] - marginal product of labor
Properties: profit maximization

\[
Profit = F(K, L) - wL - RK
\]

Necessary Conditions for Profit Maximization:

\[F_L(K, L) = W\]

\[F_K(K, L) = R\]

The theory asserts that factors are paid their marginal products when the producer takes input prices \((W, R)\) as given. This is the assumption of competitive input markets.
Factor Prices and Marginal Products

\[ F_L(K, L) \]
Cobb-Douglas Production Function: \( Y = AK^\beta L^{1-\beta} \)

\[
F_L(K, L) = (1 - \beta)AK^\beta L^{-\beta} = (1 - \beta)A\left(\frac{K}{L}\right)^\beta
\]

\[
F_K(K, L) = \beta AK^{\beta-1}L^{1-\beta} = \beta A\left(\frac{L}{K}\right)^{1-\beta}
\]

\[ Y = F(K, L) = F_L(K, L)L + F_K(K, L)K \text{ - CRS} \]

Important Property: Constant Factor Shares

*Labor's Share* = \( \frac{F_L(K, L)L}{Y} = (1 - \beta) \)

*Capital's Share* = \( \frac{F_K(K, L)K}{Y} = \beta \)
Technological Change: key reason why people live vastly better today than a hundred years ago.

Embodied Technological Change - you must build or purchase a new piece of capital to take advantage of the new technology (e.g. new computer chip, new car, new cell phone, new stove, new airplane, ...).

Disembodied Technological Change - you can take advantage of the new technology (largely) by using existing inputs (e.g. pin factory, assembly line, double-entry book keeping, just-in-time inventory methods, computer programs, ...).

Theory: $Y_t = A_t F(K_t, L_t)$
Aggregate Production Function $Y_t = A_t F(K_t, L_t)$

We follow Solow in using the abstraction that there is a technological relationship between aggregate measures of capital and labor $(K, L)$ and aggregate output $Y$. This greatly simplifies the analysis of growth theory issues.

When is this justified from micro principles?

One obvious justification: all firms have the same CRS production function and face the same input prices. Then the whole economy behaves as if there is one competitive firm with a CRS production function.