Speculation and Financial Wealth Distribution under Belief Heterogeneity

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Abstract

Under limited commitment that prevents agents from pledging their future non-financial wealth, agents with incorrect beliefs always survive by holding on to their non-financial wealth. Friedman (1953)’s market selection hypothesis suggests that their financial wealth trends towards zero in the long run. However, in this paper, I present a dynamic general equilibrium model in which over-optimistic agents not only survive but also prosper by speculation. The numerical method developed to solve for the equilibrium asset prices and allocations in the model can be used for many other heterogeneous agent models with incomplete markets and portfolio constraints in the presence of non-financial endowments.

*I am grateful to Daron Acemoglu and Ivan Werning for their infinite support and guidance during my time at MIT. I wish to thank Guido Lorenzoni and Robert Townsend for their advice since the beginning of this project, and Markus Brunnermeier, Jinhui Bai, Turan Bali, Pablo Beker, Tim Cogley, Behzad Diba, John Geanakoplos, Pedro Gete, Mark Huggett, Felix Kubler, Dirk Krueger, Per Krusell, Viktor Tsyrennikov, and participants at many seminars and conferences for helpful comments and discussions.
†The previous version of this paper was circulated as Cao (2010) - “Collateral shortages, asset price, and investment volatility with heterogeneous beliefs” - which can be found at http://www9.georgetown.edu/faculty/dc448/papers.html
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1 Introduction

The events leading to the financial crisis of 2007-2008 have highlighted the importance of belief heterogeneity and how financial markets create opportunities for agents with different beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds profited from the securities by short-selling them.

One reason why economic theory has paid relatively little attention to the heterogeneity of beliefs and how it interacts with financial markets is the market selection hypothesis. The hypothesis, originally formulated by Friedman (1953), claims that in the long run, there should be limited differences in beliefs because agents with incorrect beliefs will be taken advantage of and eventually driven out of the markets by those with the correct beliefs. Therefore, agents with incorrect beliefs will have no influence on economic activity in the long run. This hypothesis has been formalized and extended in recent work by Sandroni (2000) and Blume and Easley (2006). However these papers assume that financial markets are complete, an assumption that plays a central role in allowing agents to pledge all their wealth, including financial and non-financial wealth.¹

In this paper, I present a dynamic general equilibrium framework in which agents differ in their beliefs but markets are endogenously incomplete because of collateral constraints, equivalently portfolio constraints or margin constraints. Collateral constraints limit the extent to which agents can pledge their future non-financial wealth and ensure that agents with incorrect beliefs never lose so much as to be driven out of the market. Consequently, all agents, regardless of their beliefs, survive in the long run and continue to trade on the basis of their heterogeneous beliefs.

In this environment, it is natural to ask the question what happens to the financial wealth of the agents with incorrect beliefs. The market selection hypothesis suggests that these agents will lose most of their financial wealth in the long run, leaving them only their non-financial wealth. The answer to the question is not simple. The long run distribution of financial wealth between agents depends on the exact structure of incomplete financial markets. For example, when agents are allowed to trade in only one real asset, over-optimistic agents (agents with incorrect beliefs) prosper by holding an increasingly

¹In this paper, it is important to differentiate financial wealth, i.e., the value of financial asset holdings, from non-financial wealth, i.e., non-financial endowments such as wages. The literature on survival focuses on total wealth distribution, i.e., the total financial and non-financial wealth, while the focus of this paper is on financial wealth distribution. See Lustig and Van Nieuweburgh (2005) for a similar distinction between housing wealth and human wealth.
larger share of the real asset and by driving up the price of the asset: they *prosper by speculation*. In the same example, when these agents can use the real asset as collateral to borrow, they end up with low financial wealth, that is their share in the real asset net of their borrowing in the long run. The simultaneous determination in equilibrium of financial wealth distribution and asset price dynamics is essential for these results.

The study of financial wealth distribution thus requires the solution of the endogenous asset price dynamics. The infinite-horizon, dynamic general equilibrium approach adopted in this paper provides a transparent mapping between the financial wealth distribution and economic variables such as asset prices and portfolio choices. It also allows for a characterization of the effects of financial regulation on asset price volatility.\(^2\)

More specifically, I study an economy in dynamic general equilibrium with both aggregate shocks and idiosyncratic shocks and heterogeneous, infinitely-lived agents. The shocks follow a Markov process. Consumers differ in their beliefs on the transition matrix of the Markov process. For simplicity, these belief differences are never updated because there is no learning; in other words agents in this economy agree to disagree.\(^3\) There is a unique final consumption good, one real asset and potentially one bond. The real asset, modelled as a *Lucas tree* as in Lucas (1978), is in fixed supply. I assume that agents cannot short sell the real asset. Endogenously incomplete (financial) markets are introduced by assuming that borrowing, i.e. selling bonds, has to be collateralized by the real asset. I refer to the equilibria of the economy with these assets as *collateral constrained equilibria*.\(^4,5\)

Households (consumers) can differ in many aspects, such as risk-aversion and endowments. Most importantly, they differ in their beliefs concerning the transition matrix governing the transitions across the exogenous states of the economy. Given the con-

\(^2\)In the earlier version of this paper - Cao (2010), attached to this submission - I show that the dynamic stochastic general equilibrium model with endogenously incomplete markets presented here also includes well-known models as special cases, including recent models, such as those in Fostel and Geanakoplos (2008) and Geanakoplos (2010), as well as more classic models including those in Kiyotaki and Moore (1997) and Krusell and Smith (1998). For instance, a direct generalization of the current model allows for capital accumulation with adjustment costs in the same model in Krusell and Smith (1998) and shows the existence of a recursive equilibrium. The generality is useful in making this framework eventually applicable to a range of questions on the interaction between financial markets, heterogeneity, aggregate capital accumulation and aggregate activity.

\(^3\)Alternatively, one could assume that even though agents differ with respect to their initial beliefs, they partially update them. In this case, similar results would apply provided that the learning process is sufficiently slow (as will be the case when agents start with relatively firm priors).

\(^4\)In Appendix C, I show that, under some restriction, this setup is equivalent to the one in which there are many bonds with different levels of collateral requirement as in Geanakoplos and Zame (2002)

\(^5\)The liquidity constrained equilibrium in Kehoe and Levine (2001) corresponds to a special case of collateral constrained equilibrium when the margin on collateralized borrowing is set to 1. The numerical solution in this paper completely characterizes the equilibrium which Kehoe and Levine (2001) conjecture that the dynamics can be very complicated. In the paper, the authors also show that the dynamics in liquidity constrained equilibrium is more complicated than the dynamics in debt constrained equilibrium.
sumers’ subjective expectations, they choose their consumption and real asset and bond holdings to maximize their intertemporal expected utility.

Before analyzing the dynamic of financial wealth distribution, I show that in any collateral constrained equilibrium, every agent survives in the standard definition a la Sandroni (2000) and Blume and Easley (2006) because of the constraints. When agents lose their bets, they can simply walk away from their debt at only the cost of losing collateral and keep their current and future (non-financial, non-pledgeable) endowments. They can return and trade again in the financial markets in the same period. They cannot walk away from their debt under complete markets because they can commit to delivering all their future endowments. Using this simple survival result, I establish the existence of collateral constrained equilibria with a stationary structure - Markov equilibria - in which equilibrium prices and quantities depend only on the distribution of financial wealth distribution. In addition, I also develop an algorithm to compute these equilibria.

The numerical solutions of these equilibria allow me to answer the questions on the dynamics of financial wealth distribution, as well as asset prices. I use the algorithm to solve for collateral constrained equilibria and present these dynamics in a numerical example. In this example, I assume that there are two types of agents: pessimists who have correct beliefs and optimists who are over-optimistic about the return of the real asset.

First of all, to answer the question asked at the beginning of the introduction on the financial wealth of agents with incorrect beliefs, I start the numerical analysis by studying a collateral constrained economy in which agents are allowed to trade only in the real asset subject to the no-short-selling constraint. This economy corresponds to the liquidity constrained economy in Kehoe and Levine (2001), but allows for heterogeneous beliefs. In this economy, not only do agents with incorrect beliefs (the optimists) survive but also prosper by postponing consumption to invest in the real asset. The speculative activities of the optimists - combined with their increasing financial wealth - constantly pushes up the price of the real asset which is essential for their increasing financial wealth. I call this phenomenon prosper by speculation. However when I allow the agents to borrow using the real asset as collateral, the optimists lose their financial wealth and end up with low

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6 The collateral constraints are a special case of limited commitment because there will be no need for collateral if agents can fully commit to their promises.

7 This model is also a generalization of Harrison and Kreps (1978) to allow for risk-averse agents.

8 Appendix B shows that the existence of non-financial wealth of the optimists is crucial for this result.

9 Interestingly, the increasing price dynamics are such that the pessimists do not always want to short-sell the asset, i.e. the short-selling constraint does not always bind for the pessimists. They attempt to short-sell the asset only when the price of the asset is too high, at which point, their short-selling constraint binds strictly.
financial wealth in the long run. Thus the structure of the financial markets really matters for the (long run) financial wealth of agents with incorrect beliefs through the price of the real asset.

One implication of this result is that - also in the numerical example above - simple and extreme forms of financial regulations such as shutting down collateralized borrowing or uniformly restricting leverage surprisingly increase the long run financial wealth of the optimists and long run asset price volatility. The intuition for greater volatility under such regulations is similar to the intuition for why long run asset price volatility is higher under collateral constrained economies than under complete markets economies. Financial regulations act as further constraints protecting the agents with incorrect beliefs. Thus, in the long run these agents hold most of the assets that they believe, incorrectly, to have high rates of return. The shocks to the rates of return on these assets then create large movements in the marginal utilities of the agents and, hence, generate large volatility of the prices of the assets.

The endogenous price of the real asset which plays an important role in the distribution of financial wealth between agents with different beliefs also exhibits interesting dynamics. For example, in the last economy with collateralized borrowing, the dynamic general equilibrium captures the debt-deflation channel as in Mendoza (2010), which models a small open economy. The economy in this example also follows two different dynamics in different times, normal business cycles and debt-deflation cycles, depending on whether the collateral constraints are binding for any of the agents. In a debt-deflation cycle, some collateral constraint binds. When a bad shock hits the economy, the constrained agents are forced to liquidate their real asset holdings. This fire sale of the real asset reduces the price of the asset and tightens the constraints further, starting a vicious cycle of falling asset prices. This example shows that the debt-deflation channel still operates when we are in a closed-economy with an endogenous interest rate, as opposed to exogenous interest rates as in Mendoza (2010). The nonlinear dynamics are also emphasized in a recent paper by Brunnermeier and Sannikov (2014).

The numerical algorithm developed in this paper (presented in Appendix A) is general and can be used for models with incomplete markets, portfolio constraints (potentially different across agents), and with heterogeneity in beliefs, discount rates, preferences, and endowments. The algorithm extends the one in Kubler and Schmedders (2003) to allow for belief heterogeneity, endogenous production and investment. The idea of the

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10This version of the paper focuses on endowment economy, but in the earlier version, Cao (2010), I allow for investment and capital accumulation. In Cao and Nie (2014), we allow for endogenous labor supply and housing consumption. It is also easy to adapt the algorithm to models with recursive Epstein-Zin preferences.
algorithm is to solve for the equilibrium mapping from the financial wealth distribution at time \( t \) to asset prices and the allocation of consumption and asset holdings at time \( t \) (and the multipliers on borrowing constraints), as well as the future financial wealth distributions at time \( t + 1 \), assuming that the equilibrium mapping at time \( t + 1 \) has already been calculated. In finite horizon economy, the algorithm starts from the last period, in which the equilibrium mapping is immediately given, and iterates backward until the first period. As we increase the horizon of the economy, the equilibrium mapping for the first period becomes independent of the horizon and converges to the equilibrium mapping for the infinite horizon economy. This convergence is shown rigorously in Appendix A. In a recent, important paper, Dumas and Lyasoff (2012) suggests a similar procedure that uses stochastic discount factors, i.e. the distributions of marginal utilities as the endogenous state variables, instead of the financial wealth distribution. While this approach works beautifully in their setting, it does not apply when there are portfolio constraints, such as collateral constraints, because the Euler’s equations involve the multipliers on the constraints.\(^{11}\)

The rest of the paper proceeds as follows. The next section reviews the related literature. In Section 3, I present the general model of an endowment economy and an analysis of the survival of agents with incorrect beliefs and asset price volatility under collateral constraints. In this section, I also define collateral constrained equilibrium as well as Markov equilibrium. Section 4 focuses on a numerical example with two agents to present the equilibrium dynamics of financial wealth distribution and asset prices. Section 5 concludes with potential applications of the framework in this paper. Appendix A shows the existence of Markov equilibrium and derives a numerical algorithm to compute the equilibrium. Appendix B shows the importance of non-financial wealth for the \textit{prosper by speculation} mechanism. Appendix C shows the equivalence between collateral constrained equilibrium in this paper and the collateral equilibrium in Geanakoplos and Zame (2002) and Appendix D presents a continuous time version of the model in Section 3, with and without non-financial wealth. In Online Appendix E, I apply the model to the U.S. economy using the parameters estimated in Heaton and Lucas (1995). Finally, in Online Appendix F, I make changes to the structure of the shocks in Section 4 in order to match the patterns of leverage over the business cycles.

\(^{11}\)Equation (11) in their paper will involve the multipliers. Thus, we cannot derive the mapping from the distribution of marginal utilities to financial wealth distribution, i.e. equation (12) in their paper.
2 Related Literature

This paper is related to the growing literature studying collateral constraints in dynamic general equilibrium, started by early papers including Kiyotaki and Moore (1997) and Geanakoplos and Zame (2002). The dynamic analysis of collateral constrained equilibria is related to Kubler and Schmedders (2003). They pioneer the introduction of financial markets with collateral constraints into a dynamic general equilibrium model with aggregate shocks and heterogeneous agents. The technical contribution of this paper relative to Kubler and Schmedders (2003) is to introduce heterogeneous beliefs using the rational expectations equilibrium concept in Radner (1972): even though agents assign different probabilities to both aggregate and idiosyncratic shocks, they agree on the equilibrium outcomes, including prices and quantities, once a shock is realized. This rational expectation concept differs from the standard rational expectation concept, such as the one used in Lucas and Prescott (1971), in which subjective probabilities should coincide with the true conditional probabilities given all available information.

Related to the literature on the survival of agents with incorrect beliefs such as Blume and Easley (2006) and Sandroni (2000) under complete markets, and Beker and Chattopadhyay (2010), and Cogley, Sargent, and Tsyrennikov (2014) under incomplete markets, my paper focuses on the dynamics of the financial wealth distribution among agents - taking non-financial wealth as exogenously given - while the existing literature investigates total wealth.

While the focus of my paper is on financial wealth, the survival (in terms of consumption) result is easily obtained because of the collateral constraints. As mentioned in footnote 6, collateral constraints are a special case of limited commitment. However, this special case of limited commitment is different from an alternative limited commitment in the literature in which agents are assumed to be banned from trading in financial markets after their defaults such as in Kehoe and Levine (1993) and Alvarez and Jermann (2000). In my paper, agents can always return to the financial markets and trade using their non-financial endowment after defaulting and losing all their financial wealth. Given this outside option, the financial constraints are more stringent than they are in other papers. Beker and Espino (2015) and Tsyrennikov (2012) use a similar survival mechanism based on the limited commitment framework in Alvarez and Jermann (2000). Under

\[\text{In Appendix C I show a mapping between the equilibria in the two papers.}\]
\[\text{However, solving for the equilibrium asset prices and the dynamics of financial wealth distribution is not easy.}\]
\[\text{Beker and Espino (2015) apply the framework to U.S. data to explain the magnitude of short-term momentum and long-term reversal in the excess returns of U.S. stock market.}\]
complete markets, Kogan et al. (2011) shows the importance of preferences (in the family of DARA utility functions), beside endowments and beliefs, in determining the survival (and price impact) of agents with incorrect beliefs. Relatedly, also under complete markets, Borovicka (2015) shows that agents with incorrect beliefs can survive if all agents have Epstein-Zin preferences and risk aversion is larger than the inverse of the intertemporal elasticity of substitution. In my paper, agents have standard separable CRRA preferences, but agents with incorrect survive because of market incompleteness coming from collateral constraints, and non-financial endowments. The survival of irrational traders is also studied in Long et al. (1990) and Long et al. (1991) but they do not have a fully dynamic framework to study the long run survival of the traders.

In a recent paper, Cogley, Sargent, and Tsyrennikov (2014) study an incomplete markets economy with belief heterogeneity in which agents can only trade in state-incontingent bonds (but they can pledge their future non-financial wealth). They show numerically that, due to precautionary saving, agents with incorrect beliefs not only survive but also prosper and drive agents with correct beliefs out of the market in the long run. They call this phenomenon survival by precautionary saving.\textsuperscript{15} The prosper by speculation mechanism mentioned in my introduction offers another way in which agents with incorrect beliefs can survive (and prosper). Because the real asset is long-lived in my paper, its price dynamics is important for my mechanism, in contrast to Cogley, Sargent, and Tsyrennikov (2014) in which financial assets are short-lived (either bonds or Arrow securities) and the prices of the financial assets have little effect on the survival results in their paper. With long-lived assets, I also show that differences in beliefs alone do not suffice for prospering in my paper, we also need the existence of non-financial wealth of the agents.

My paper is also related to the literature on the effect of heterogeneous beliefs on asset prices such as Cogley and Sargent (2008) and Chen, Joslin, and Tran (2012). The authors, however, consider only complete markets.

In Harrison and Kreps (1978), the authors show that beliefs heterogeneity can lead to asset price bubbles under short-selling constraints and linear utility functions. My paper includes this set-up as a special case and allows for risk-aversion. I show that only when asset prices deviate too far away from their fundamental values, rational agents try to short-sell the real asset but are constrained by the short-selling constraint. Simsek (2013) also studies the effects of belief heterogeneity on asset prices under short-selling and collateral constraints, but in a static setting. He assumes exogenous wealth distributions to investigate whether heterogeneous beliefs affect asset prices. In contrast, I study the

\textsuperscript{15}Blume and Easley (2006) - Section 5 - offers a similar example with incomplete markets and short-lived assets in which agents with incorrect beliefs dominates by saving more than agents with correct beliefs do.
effects of the endogenous wealth distribution on asset prices.

The channel through which asset prices deviate from their fundamental values in my paper is related to the limited arbitrage mechanism in Shleifer and Vishny (1997). When asset prices deviate too much from the fundamental value according to the agents with correct beliefs, i.e. the pessimists, they want to to short-sell the asset but they are constrained by the no-short selling constraint. However, the limits to arbitrage channel in Shleifer and Vishny (1997) does not always apply in this model. The pessimists, are not constrained by the short-selling constraint all the time, while the price of the asset always exceeds the pessimists’ valuation. The dynamics of asset prices is such that the pessimists are happy to hold the asset as well. Only when asset prices are sufficiently high, or equivalently when the asset is sufficiently over-valued, the pessimists attempt to short-sell the asset and hit the no-short-selling constraint. In Shleifer and Vishny (1997), the deviation arises because agents with correct beliefs hit their financial constraints before being able to arbitrage away the price anomalies. In this paper, agents with incorrect beliefs hit their financial constraint more often and are protected by the constraint.

On the normative question of how financial regulation affects asset price volatility, in recent papers, Rytchkov (2014) and Brumm et al. (2015) show that at high levels of margin requirement, i.e. restricting leverage, decreases asset price volatility. This result is different from the normative result in my paper. In their papers, agents trade due to their difference in risk-aversion, while in my paper, agents trade due to the differences in beliefs. Under belief heterogeneity, there is a new mechanism that does not exist in Rytchkov (2014) and Brumm et al. (2015): a higher margin requirement actually protects the agents with incorrect beliefs. Thus they are financially wealthier in the long run and their trading activity drives up asset price volatility.

In an earlier version of this paper, Cao (2010), attached to this submission, I introduce capital accumulation to the current framework. The model presented there is a generalization of Krusell and Smith (1998) with financial markets and adjustment costs. In particular, the Existence Theorem 1 in Appendix A shows that a recursive equilibrium in Krusell and Smith (1998) (with finitely many number of agents) exists. Krusell and Smith (1998) numerically solve for such an equilibrium, but they do not formally show its existence. Cao (2010) is also related to Kiyotaki and Moore (1997). I provide a microfoun-
dation for the borrowing constraint in Kiyotaki and Moore (1997) using the endogeneity of the set of actively traded financial assets.

The economy with heterogeneous agents subject to portfolio constraints in this paper is similar to the ones in many papers in the continuous time finance literature such as De-temple and Murthy (1997), Gallmeyer and Hollifield (2008), and more recently Rytchkov (2014). These papers assume that there is no non-financial wealth and take advantage of the resulting homogeneity in the constrained optimization problem of the agents to solve for the equilibrium. I show in Appendix D that this homogeneity breaks down in the presence of non-financial wealth. However, one can still solve for the equilibrium by looking for an equilibrium mapping from the financial wealth distribution to the equilibrium allocations (of consumptions and asset holdings), asset prices, and the multipliers on the portfolio constraints as done in this paper.

3 General Model

In this general model, there are heterogeneous agents who differ in their beliefs about the future streams of dividends and the individual endowments. In order to study the effects of belief heterogeneity on asset prices, I allow for only one real asset in fixed supply. After presenting the model and defining collateral constrained equilibrium in Subsection 3.1, I show some general properties of the equilibrium in Subsection 3.2 and a stationary form of collateral constrained equilibrium in Subsection 3.3.

3.1 The Endowment Economy

Consider an endowment, a single consumption (final) good economy in infinite horizon with infinitely-lived agents (consumers). Time runs from $t = 0$ to $\infty$. There are $H$ types of consumers

$$ h \in \mathcal{H} = \{1, 2, \ldots, H\} $$

in the economy with a continuum of measure 1 of identical consumers in each type. These consumers might differ in many dimensions including their per period utility function $U_h(c)$ (i.e., risk-aversion), and their endowment of final good $e_h$. The consumers might also differ in their initial endowment of a real asset, Lucas’ tree, that pays off real dividend in terms of the consumption good. However, the most important dimension of heterogeneity is the heterogeneity in belief over the evolution of the exogenous state of
the economy. There are $S$ possible exogenous states (or equivalently exogenous shocks)

$$s \in \mathcal{S} = \{1, 2, \ldots, S\}.$$  

The states capture both idiosyncratic uncertainty (uncertain individual endowments), and aggregate uncertainty (uncertain aggregate dividends).  

The evolution of the economy is captured by the realizations of the shocks over time: $s^t = (s_0, s_1, \ldots, s_t)$ is the history of realizations of shocks up to time $t$. I assume that the shocks follow a Markov process with the transition probabilities $\pi(s, s')$. In order to rule out transient states, I make the following assumption:

**Assumption 1.** $\mathcal{S}$ is ergodic.

In contrast to the standard rational expectation literature, I assume that the agents do not have the perfect estimate of the transition matrix $\pi$. Each of them has their own estimate of the matrix, $\tilde{\pi}$. However, these estimates are not very far from the truth.

**Assumption 2.** $\pi(s, s') = 0$ if and only if $\tilde{\pi}(s, s') = 0$.  

This condition implies that every agent believes that $\mathcal{S}$ is ergodic.

**Real Asset:** There is one real asset that pays off state-dependent dividend $d(s)$ in the final good. The asset can be both purchased and used as collateral to borrow. The ex-dividend price of each unit of the asset in history $s^t$ is denoted by $q(s^t)$. I assume that agents cannot short-sell the real asset. The total supply 1 of the asset is given at the beginning of the economy, under the form of asset endowments to the consumers.

**Collateralized Bond:** In addition to the real asset, the agents in this economy can also borrow subject collateral constraints. The agents borrow by selling one-period-ahead bonds but bonds have to be collateralized by the real asset. In history $s^t$, a bond that pays off one unit of consumption good next period is sold at price $p(s^t)$.

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17 A state $s$ can be a vector $s = (A, \epsilon_1, \ldots, \epsilon_H)$ where $A$ consists of aggregate shocks and $\epsilon_h$ are idiosyncratic shocks.

18 Learning can be incorporated into this framework by allowing additional state variables which are the current beliefs of agents in the economy. In Blume and Easley (2006) and Sandroni (2000), agents who learn slower will disappear under complete markets. However they all survive under collateral constraints. The dynamics of asset prices described here corresponds to the short-run behavior of asset prices in the economy with learning.

19 This general specification allows for time-varying belief heterogeneity as in He and Xiong (2012). In particular, agents might share the same beliefs in good states, $\tilde{\pi}(s, .) = \tilde{\pi}'(s, .)$, but their beliefs can start diverging in bad states, $\tilde{\pi}(s, .) \neq \tilde{\pi}'(s, .)$. Simsek (2013) shows, in a static model, that only the divergence in beliefs about bad states matters for asset prices.

20 I can relax this assumption by allowing for limited short-selling.
When agents borrow by selling bonds, they have to simultaneously purchase the real asset to use it as collateral for the bonds. This transaction is equivalent to buying the real asset using leverage. The difference between how much they pay for the real asset and how much can borrow against it is called margin in the literature.

**Consumers:** Consumers are the most important actors in this economy. They can be hedge fund managers or banks’ traders in financial markets. In each state $s_t$, each consumer is endowed with a potentially state dependent (non-financial) endowment $e^h_t = e^h(s_t)$ units of the consumption good. I suppose that there is a strictly positive lower bound on these endowments. This lower bound guarantees a lower bound on consumption if a consumer decides to default on all her debt.\textsuperscript{21}

**Assumption 3.** $\min_{h,s} e^h(s) > \varepsilon > 0$.

Consumers maximize their intertemporal expected utility with the per period utility functions $U^h(.) : \mathbb{R}^+ \rightarrow \mathbb{R}$ that satisfy the following assumption.

**Assumption 4.** $U^h(.)$ is concave and strictly increasing.\textsuperscript{22}

I also assume that consumers share the same discount factor $\beta$.\textsuperscript{23} Consumer $h$ takes the sequences of prices $\{q_t, p_t\}$ as given and solves

$$\begin{align*}
\max_{\{c^h_t, \theta^h_{t+1}, \phi^h_{t+1}\}} \mathbb{E}_0^h & \left[ \sum_{t=0}^{\infty} \beta^t U^h(c^h_t) \right] \\
\text{subject to the budget constraint} & \quad c^h_t + q_t \theta^h_{t+1} + p_t \phi^h_{t+1} \leq e^h_t + \phi^0_t + (q_t + d_t) \theta^h_t, \\
\text{no-short sale constraint on the real asset} & \quad \theta^h_{t+1} \geq 0, \\
\text{and the collateral constraint, or equivalently leverage or margin constraint,} & \quad \phi^h_{t+1} + (1 - m) \theta^h_{t+1} \min_{s^{t+1}|s^t} (q_{t+1} + d_{t+1}) \geq 0,
\end{align*}$$

\textsuperscript{21}I also introduce the disutility of labor in the general existence proof in Cao (2010) in order to study employment in this environment. When we have strictly positive labor endowments, $l^h$, we can relax Assumption 3 on final-good endowments, $e^h$.

\textsuperscript{22}Notice that I do not require $U^h(.)$ to be strictly concave. This assumption allows for linear utility functions in Geanakoplos (2010) and Harrison and Kreps (1978).

\textsuperscript{23}The general formulation and solution method in Cao (2010) allow for heterogeneity in the discount rates. In this paper I assume homogeneous discount factor to focus on beliefs heterogeneity.
where \(0 \leq m \leq 1\) is the margin that determines how much an agent can borrow per unit of the real asset holding.\(^{24}\)

I impose the collateral constraint as it is widely used in financial markets. It is also relevant to the issue of survival and prosper under beliefs heterogeneity. Indeed, when \(m = 0\), collateral constraint (4) allows for the possibility of some agent to lose all her financial wealth, leaving alone her non-financial wealth. An increase in \(m\) corresponds to a tighter collateral constraint, or some financial regulation. We will investigate the effects of changing \(m\) in Subsection 4.4. However, the exact form of the constraint does not affect the conclusions in this paper as long as the agents cannot pledge all her current and future non-financial wealth.\(^{25,26}\)

The most important feature of the objective function is the superscript \(h\) in the expectation operator \(E^h[.].\) which represents the subjective beliefs when an agent calculates her future expected utility. Entering period \(t\), agent \(h\) holds \(\theta^h_t\) old units of real asset and \(\phi^h_t\) units of collateralized bonds. She can trade old units of real asset at price \(q\), and buy new units of real asset \(\theta^h_{t+1}\) for time \(t+1\) at the same price. She can also buy and sell bonds \(\phi^h_{t+1}\) at price \(p\). If she sells bonds she is subject to collateral constraint (4). The collateral constraint (4) has the usual property of financial constraints that higher (future) asset prices should enable more borrowing. When \(m = 1\), agents are not allowed to borrow, so they can only trade in the real asset.

In this environment, I define an equilibrium as follows:

\(^{24}\)One implicit condition from the assumption on utility functions is that consumption is positive, i.e., \(c^h_t \geq 0\).

\(^{25}\)We can also consider the alternative collateral constraint

\[
\phi^h_{t+1} + (1 - m) \theta^h_{t+1} \min_{s^t+1} q(s^t+1) \geq 0,
\]

which is the constraint used in Kiyotaki and Moore (1997), or

\[
\phi^h_{t+1} + (1 - m) \theta^h_{t+1} q_t \geq 0,
\]

as in Mendoza (2010). The quantitative implications of such constraint are very similar to the ones in this paper. In Cao and Nie (2014), we carry out a complete set of comparisons of the effects of the different forms of constraints in a model without beliefs heterogeneity (and with housing and endogenous labor supply) and find little differences in the quantitative effects of the different constraints. In the continuous time limit presented in Appendix D, all these constraints are identical. In Cao and Nie (2014), we also allow for exogenous borrowing limits instead of the endogenous borrowing limits arising from collateral constraints as in this current paper.

\(^{26}\)See Miller (1977) for an early discussion of the importance of trading constraints such as no-short sale constraints, or collateral constraints, and beliefs heterogeneity on stock returns. Diether et al. (2002) finds empirical evidence for Miller’s hypothesis using dispersion in analysts’ forecasts as a proxy for differences in opinion. Similarly, Bali et al. (2011) show that lottery investors (or over-optimist investors) generate demand for stocks with high probabilities of large short-term up moves in the stock price, pushing the prices of such stocks up.
Definition 1. A collateral constrained equilibrium for an economy with initial asset holdings
\[ \left\{ \theta^h_0, \phi^h_0 \right\}_{h \in H} \]
and initial shock \( s_0 \) is a collection of consumption, real asset, and bond holdings and prices in each history \( s^t \),
\[
\left( \left\{ c^h_t (s^t), \theta^h_{t+1} (s^t), \phi^h_{t+1} (s^t) \right\}_{h \in H}, q_t (s^t), p_t (s^t) \right)
\]
satisfying the following conditions:

i) The markets for final good, real asset, and bond in each period clear:
\[
\sum_{h \in H} \theta^h_{t+1} (s^t) = 1 \\
\sum_{h \in H} \phi^h_{t+1} (s^t) = 0 \\
\sum_{h \in H} c^h_t (s^t) = \sum_{h \in H} e^h (s_t) + d (s_t)
\]

ii) For each consumer \( h \), \( \left\{ c^h_t (s^t), \theta^h_{t+1} (s^t), \phi^h_{t+1} (s^t) \right\} \) solves the individual maximization problem subject to the budget constraint (2), and the no-short sale and collateral constraints, (3) and (4).

The collateral constrained equilibrium is based on the exogenous collateral constraint (4) which is often assumed in the literature. In Appendix C, I present a micro-foundation for this collateral constraint based on the limited commitment that requires agents to hold collateral when they borrow.

Proposition 1. Assume that each aggregate state \( s^t \) has only two possible future successor states \( s^{t+1} \), each collateral constrained equilibrium has an equivalent collateral equilibrium a la Geanakoplos and Zame (2002), i.e., the two equilibria have the same prices (of bonds and real asset) and consumption allocation (but the equilibria might differ in the portfolio holdings of the agents in the economy).

Appendix C formalizes and proves this proposition.

3.2 Survival in Collateral Constrained Equilibrium

In this subsection, I show that all agents (including the ones with incorrect beliefs) survive in a collateral constrained equilibrium.
Given the endowment economy, we can easily show that the total supply of final good in each period is bounded by a constant \( \bar{e} \). Indeed, in each period, the total supply of final good is bounded by

\[
\bar{e} = \max_{s \in S} \left( \sum_{h \in \mathcal{H}} e^h (s) + d(s) \right). \tag{5}
\]

The first term on the right hand side is the total endowment of all consumers. The second term is the dividend from the real asset. In collateral constrained equilibrium, the market clearing condition for the final good implies that the total consumption of all consumers is bounded from above by \( \bar{e} \).

We can show that in any collateral constrained equilibrium, the consumption of each consumer is bounded from below by a strictly positive constant \( c \). Two assumptions are important for this result. First, the no-default-penalty assumption allows consumers, at any moment in time, to walk away from their past debt and only lose their collateral. After defaulting, they can always keep their non-financial wealth - inequality (7) below. Second, increasingly large speculation by postponing current consumption is not an optimal plan in equilibrium, because total consumption is bounded by \( \bar{e} \), in inequality (8).\(^{27}\)

This assumption prevents agents from constantly postponing their consumption to speculate in the real asset and represents the main difference with the survival channel in Alvarez and Jermann (2000) which is used by Beker and Espino (2015) and Tsyrennikov (2012) for heterogeneous beliefs. Formally, we arrive at:

**Proposition 2.** Suppose that there exists a \( c \) such that

\[
U_h (c) < \frac{1}{1 - \beta} U_h (\bar{e}) - \frac{\beta}{1 - \beta} U_h (\bar{e}), \forall h \in \mathcal{H}, \tag{6}
\]

where \( \bar{e} \) is defined in (5). Then in a collateral constrained equilibrium, the consumption of each consumer in each history always exceeds \( c \).

**Proof.** This result is shown in an environment with homogenous beliefs (see Lemma 3.1 in Duffie et al. (1994)). It can be done in the same way under heterogenous beliefs. I replicate the proof in order to provide the economic intuition in this environment.

By the market clearing condition in the market for the final good, the consumption of each consumer in each future period is bounded by future aggregate endowment. In each period, a feasible strategy of consumer \( h \) is to default on all of her past debt at only the

\(^{27}\)Even though an atomistic consumer may have unbounded consumption, in equilibrium, prices will adjust such that a consumption plan in which consumption sometime exceeds \( \bar{e} \) will not be optimal.
cost of losing all her collateral. However, she can still at least consume her endowment from the current period onwards. Therefore

$$U_h(c_{h,t}) + \mathbb{E}_t^h \left[ \sum_{r=1}^{\infty} \beta^r U_h(c_{h,t+r}) \right] \geq \frac{1}{1-\beta} U_h(\bar{c}). \quad (7)$$

Notice that in equilibrium, $\sum_{t} c_{h,t+r} \leq \bar{c}$ and so $c_{h,t+r} \leq \bar{c}$. Hence

$$U_h(c_{h}) + \frac{\beta}{1-\beta} U_h(\bar{c}) \geq \frac{1}{1-\beta} U_h(\bar{c}). \quad (8)$$

This implies

$$U_h(c_{h}) \geq \frac{1}{1-\beta} U_h(\bar{c}) - \frac{\beta}{1-\beta} U_h(\bar{c}) > U_h(\bar{c}).$$

Thus, $c_{h} \geq \xi$. \hfill \Box

Condition (6) is satisfied immediately if $\lim_{c \to 0} U_h(c) = -\infty$, for example, with log utility or with CRRA utility with the CRRA coefficient exceeding 1.

One immediate corollary of Proposition 2 is that every consumer survives in equilibrium. Sandroni (2000) shows that in complete markets equilibrium, almost surely the consumption of agents with incorrect beliefs converges to 0 at infinity. Therefore, collateral constrained equilibrium differs from complete markets equilibrium when consumers strictly differ in their beliefs.

The survival mechanism in collateral constrained equilibrium is similar to the one in Alvarez and Jermann (2000), Beker and Espino (2015) and Tsyrennikov (2012). In particular, the first term on the right hand side of (6) captures the fact that the agents always have the option to default and go to autarky. In which case, they only consume their endowment which exceeds $\xi$ in each period and which is the lower bound for consumption in Alvarez and Jermann (2000). However, the two survival mechanisms also differ because, in this paper, agents can always default on their promises and lose all their real asset holdings. Yet they can always return to financial markets to trade right after defaulting. The second term in the right hand side of (6) shows that this possibility might actually hurt the agents if they have incorrect beliefs. The prospect of higher reward for speculation, i.e. high $\bar{c}$, will induce these agents to constantly postpone consumption to speculate. As a result, their consumption level might fall well below $\xi$. Indeed, the lower bound of consumption $\xi$ is decreasing in $\bar{c}$. The more there is of the total final good, the more profitable speculative activities are and the more incentives consumers have to defer current consumption to engage in these activities.
This survival mechanism is also different from the limited arbitrage mechanism in Shleifer and Vishny (1997), in which asset prices differ from their fundamental values because agents with correct beliefs hit their financial constraints (or short-selling constraints) before they can arbitrage away the difference between assets’ fundamental value and their market price. In this paper, agents with incorrect beliefs hit their financial constraint more often than the agents with correct beliefs do and are protected by the constraint. In the equilibria computed in Section 4, agents with the correct belief (the pessimists) sometime do not hit their borrowing constraint (or short-selling constraints).

The next subsection is devoted to showing the existence of these equilibria with a stationary structure and Appendix A presents an algorithm to compute the equilibria.

3.3 Markov Equilibrium

Proposition 2 is established under the presumption that collateral constrained equilibria exist. In Appendix A, I show that under weak conditions on endowments and utility functions, a collateral constrained equilibrium exists and has the following stationary structure.

Define the financial wealth share of each agent as

\[
\omega_t^h = \frac{(q_t + d_t) \theta_t^h + \phi_t^h}{q_t + d_t}.
\]  

Let \( \omega \left( s^t \right) = (\omega^1 \left( s^t \right), ..., \omega^H \left( s^t \right)) \) denote the financial wealth distribution. Then in equilibrium \( \omega \left( s^t \right) \) always lies in the (H-1)-dimensional simplex \( \Omega \), i.e., \( \omega^h \geq 0 \) and \( \sum_{h=1}^{H} \omega^h = 1 \). \( \omega^h \)'s are non-negative because of the collateral constraint (4) that requires the value of each agent’s asset holding to exceed the liabilities from their past borrowings. The sum of \( \omega^h \) equals 1 because of the real asset market clearing and bond market clearing conditions. Using the definition of financial wealth distribution, I define a Markov equilibrium as follows.

**Definition 2.** A Markov equilibrium is a collateral constrained equilibrium in which the prices of real asset and bond and the allocation of consumption, real asset and bond in each history depend only on the exogenous shock \( s_t \) and the endogenous financial wealth share distribution \( \omega \left( s^t \right) \).

This Markov equilibrium definition features the endogenous state variable \( \omega_t \) that depends on asset price \( q_t \) (which by itself depends on the state variable). This equilibrium was first studied in Duffie et al. (1994). Brunnermeier and Sannikov (2014) and He and
Krishnamurthy (2013) use the same type of equilibrium definition in their continuous time models, with endogenous financial wealth distribution as a sufficient endogenous state variable. In Appendix D, I build a continuous time version of my model to show explicitly the similarity between the Markov equilibrium concept used in this paper and the equilibrium concepts in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013). In the appendix, I also show that there is one-to-one mapping between the Markov equilibrium in this paper and the recursive equilibrium concept used in Rytchkov (2014) in which endogenous consumption share is the sufficient state variable.\footnote{One important difference is that the construction and solution method for Markov equilibrium in this paper allow for non-financial endowments, which are absent from these papers. I discuss this issue further in Appendix D.}

In Appendix A, I show the conditions under which a Markov equilibrium exists, and I develop a numerical method to compute Markov equilibria. Markov equilibria inherit all the properties of collateral constrained equilibria. In particular, in a Markov equilibrium, every consumer survives (Proposition 2). Regarding asset prices, the construction of Markov equilibria shows that asset prices can be history-dependent in the long run through the evolution of the financial wealth share distribution, defined in (9).

This result is in contrast with the one in Sandroni (2000) in which - under complete markets - asset prices depend on the wealth distribution that converges in the long run. So, in the long run, asset prices only depend on the current exogenous state $s_t$. In a Markov equilibrium, the financial wealth share distribution constructed in (9) constantly moves over time, even in the long run. For example, if an agent $h$ with incorrect belief loses all her real asset holding due to leverage, next period she can always use her endowment to speculate in the real asset again. In this case, $\omega^h$ will jump from 0 to a strictly positive number. Asset prices therefore depend on the past realizations of the exogenous shocks, which determine the evolution of the financial wealth distribution $\omega$. Consequently, we have the following result.

**Proposition 3.** When the aggregate endowment is constant across states $s \in S$, and shocks are I.I.D., long run asset price volatility is higher in Markov equilibria than it is in complete markets equilibria.

**Proof.** As shown in Sandroni (2000), in the long run, under complete markets, the economy converges to an economy with homogenous beliefs because agents with incorrect beliefs will eventually be driven out of the markets and the real asset price $q(s_t)$ converges to a price independent of time and state. Hence, due to I.I.D. shocks and constant aggregate endowment, under complete markets, asset price volatility converges to zero in
In Markov equilibrium, asset price volatility remains above zero as the exogenous shocks constantly change the financial wealth distribution that, in turn, changes real asset price.\textsuperscript{29} 

There are two components of asset price volatility. The first and standard component comes from the volatility in the dividend process and the aggregate endowment. The second component comes from the financial wealth distribution when agents strictly differ in their beliefs. In general, it depends on the correlation of the two components, that we might have asset price volatility higher or lower under collateral constraints versus under complete markets. However, the second component disappears under complete markets because only agents with the correct belief survive in the long run. In contrast, under collateral constraints, this component persists. As a result, when we shut down the first component, asset price is more volatile under collateral constraints than it is under complete markets in the long run. In general, whether the same comparison holds depends on the long-run correlation between the first and the second volatility components.

\section{The Dynamics of Financial Wealth Distribution and Asset Prices}

In this section, I focus on a special case of the general framework analyzed in section 3. I restrict myself to an economy with only two types of agents. In such an economy, the financial wealth distribution can be summarized by only one number, which is the fraction of the financial wealth share held by one of the two types of agents. Under this simplification, I can compute collateral constrained equilibria. In the following Subsections 4.2 through 4.4, I compute Markov equilibria in a simple economy to show the complex joint dynamics of financial wealth distribution and asset prices, including the prosper by speculation mechanism that I described in the introduction. In Online Appendix E, I allow for richer and more realistic structures of shocks as well as of the endowments of the agents.

\subsection{Two-Agent Economy}

Consider a special case of the general model presented in Section 3. There are two exogenous states $S = \{G, B\}$ and one real asset of which the dividend depends on the

\textsuperscript{29}This result holds except in knife-edge cases in which, even in collateral constrained equilibrium, asset price is independent of the financial wealth distribution and the exogenous shocks, or when financial wealth distribution does not move over time. These cases never appear in numerical solutions.
exogenous state:

\[ d(G) > d(B). \]

The exogenous state follows an I.I.D. process, with the probability of high dividend, \( \pi \), unknown to agents in this economy.

There are two types of consumers (measure one in each type), the optimists, \( O \), and the pessimists, \( P \), who differ in their beliefs. They have different estimates of the probability of high dividend \( \pi^h, h \in \{O, P\} \). I suppose that \( \pi^O > \pi^P \), i.e., optimists always think that good states are more likely than the pessimists believe. Again, due to the collateral constraint, in equilibrium, \( \omega^h_t \) must always be positive and

\[ \omega^O_t + \omega^P_t = 1. \]

Therefore, we just need to keep track of the wealth share of the optimists, \( \omega^O_t \). The pay-off relevant state space

\[ \left\{ \left( \omega^O_t, s_t \right) : \omega^O_t \in [0, 1] \text{ and } s_t \in \{G, B\} \right\} \]

is compact.\(^{30}\) Appendix A shows the existence of collateral constrained equilibria under the form of Markov equilibria in which prices and allocations depend solely on that state defined above. Appendix A also provides an algorithm to compute such equilibria. As explained in Subsection 3.3, the equilibrium asset prices depend not only on the exogenous state but also on the financial wealth share share \( \omega^O_t \).

I choose suggestive parameters to illustrate the dynamics of financial wealth distribution and asset prices. Imagine that the optimists correspond to the investment banking sector that is bullish about the profitability of the mortgage-backed securities market (the real asset) and the pessimists correspond to the rest of the economy. The parameters are chosen such that the income size of the investment banking sector is about 4% (\( e^O \)) of the U.S. economy, and the size of the mortgage-backed security market in the U.S. is about

\(^{30}\)Given that the optimists prefer holding the real asset, i.e., \( \theta^O_t > 0 \), \( \omega^O_t \) corresponds to the fraction of the asset owned by the optimists.
15% of the U.S. annual GDP. In particular, let

\[
\beta = 0.95 \\
d(G) = 1 > d(B) = 0.5 \\
U(c) = \log(c).
\]

While the assumption of log utility is important in the absence of non-financial endowments because consumption choice will be just a constant fraction \((1 - \beta)\) of financial wealth (as used in Appendix B and shown in classic papers such as Merton (1969)), with non-financial endowments, log utility assumption does not simplify the calculation of Markov equilibrium. \(^{32}\) The same numerical procedure and numerical findings in this section hold for utility functions other than log utility. \(^{33}\)

The beliefs are \(\pi^O = 0.6 > \pi^P = 0.5\).

I fix the endowments of the pessimists and the optimists at

\[
e^P = \begin{bmatrix} 100 & 100.5 \end{bmatrix} \\
e^O = \begin{bmatrix} 4 & 4 \end{bmatrix}.
\]

The endowment of the pessimists is chosen as slightly counter-cyclical so that the aggregate endowment is kept constant at 105. This assumption of constant aggregate endowment makes it easy to compare asset prices in Markov equilibrium to asset prices under complete markets (in the long run) in which the price of the real asset is just the present discounted value of dividend evaluated at constant stochastic discount factor (Proposition 3). But given the very high level of endowment of the pessimists, if their endowment were constant (or slightly procyclical) instead, the numerical results below would not change quantitatively.

In the next subsections, I study the properties of equilibria under different market structures (different margin requirement \(m's\)) using the numerical method developed in Appendix A.

\(^{31}\)The distribution of endowments are similar to the one in Fostel and Geanakoplos (2008), but asset returns and beliefs are different from theirs, and this economy is in infinite horizon as opposed to 2- or 3-period economies in their paper.

\(^{32}\)See further discussions on non-homogeneity in the presence of non-financial wealth in Appendix D.

\(^{33}\)In Cao (2010), I compare the collateral constrained equilibria in this section to the complete markets equilibrium. In this exercise, the assumption of log utility facilitates the solution of the complete markets equilibrium.
4.2 Liquidity Constrained Equilibrium

As a benchmark, I assume that \( m = 1 \). In this case, a collateral constrained economy is equivalent to an economy in which agents can only trade in the real asset

\[
e^h_t + q_t \theta^h_{t+1} \leq e^h_t + (q_t + d_t) \theta^h_t
\]  

and \( \theta^h_{t+1} \geq 0 \). In the absence of bond holdings, \( \omega^h_t = \theta^h_t \). This economy corresponds to the liquidity constrained economy studied in Kehoe and Levine (2001) with belief heterogeneity.

The upper panel in Figure 1 shows the relationship between the price of the asset and the current fraction of the real asset \( \theta^O_t \) held by the optimists, when the current exogenous state \( s_t = G \) (bold red line) and \( s_t = B \) (thin blue line) respectively. The two black dotted bands show the prices of the real asset evaluated using the belief of the optimists (dotted upper band) and the belief of the pessimist (dotted lower band).

To understand the patterns of asset prices, Figure 2 shows the future fraction of the real asset held by the optimists, \( \theta^O_{t+1} \), as a function of the current holding \( \theta^O_t \) (bold red line for \( s_t = G \) and blue line for \( s_t = B \), and dashed line for \( 45^o \) line). We can see that when \( \theta^O_t \) is far from 1, \( \theta^O_{t+1} \) lies strictly between 0 and 1. Thus, both sets of agents are marginal buyers of the real asset. Given that the consumption of the pessimists changes relatively little due to their high level of non-tradable endowment, the price of the asset is close to

\[ p^h = \frac{\beta}{1-\beta} \left( \pi^h(G) d(G) + \left(1 - \pi^h(G) \right) d(B) \right) \]
Figure 2: Asset Holdings Without Leverage

Figure 3: Stationary Distributions of the Optimists’ Financial Wealth Share under different Market Structures
the pessimists’ estimate (lower band). This is true if the non-short selling constraint of the pessimists is not binding or going to bind in the near future, i.e., \( \theta_t^O \), sufficiently far away from 1. When \( \theta_t^O \) is close to 1, \( \theta_{t+1}^O \) is equal to 1, thus \( \theta_{t+1}^P = 0 \), i.e., the optimists are the only marginal buyers of the real asset, thus its price is close to the optimists’ valuation (upper band). Notice also that when \( s_t = G \) and \( \theta_t^O \) is close to 1, \( \theta_{t+1}^O = 1 \), and the price of the asset exceeds the valuation of any of the agents in the economy. This is because the marginal utility of the optimists is very high, relative to their long run level of average income, given their large share in the real asset and dividend is high. As they are the only marginal buyers of the real asset (\( \theta_{t+1}^O = 1 \) and \( \theta_{t+1}^P = 0 \)), this high level of marginal utility drives up the price of the real asset.\(^{35,36}\) At this point, the pessimists would like to short sell the real asset. However they face the no-short selling constraint, the limits to arbitrage channel in Shleifer and Vishny (1997) kicks in.

Nevertheless, the limits to arbitrage channel in Shleifer and Vishny (1997) does not always apply in this model. The agents with correct beliefs, i.e., the pessimists, are not constrained by the short-selling constraint all the time (\( \theta_{t+1}^P > 0 \) when \( \theta_t^O \) is not too close to 1), while the price of the real asset always exceeds the pessimists’ valuation. The dynamics of asset price is such that the pessimists are happy to hold the asset as well.\(^{37}\) Only when \( \theta_t^O \) is sufficiently high, or equivalently when the asset is sufficiently over-valued, the pessimists attempt to short-sell the asset and hit the no-short-selling constraint.

I define the volatility of asset price as one period ahead volatility\(^{38}\)

\[
v_t \left( s^t \right) = std_{s^{t+1} \mid s^t} \left( q_{t+1} \left( s^{t+1} \right) \right) . \tag{11}
\]

The lower panel in Figure 1 shows asset price volatility as functions of \( \theta_t^O \) (bold red line for \( s_t = G \) and blue line for \( s_t = B \)). Asset price volatility is the highest when \( s_t = G \) and \( \theta_t^O \) is close two 1. The budget constraint (10) and the policy function \( \theta_{t+1}^O \) in the right

\(^{35}\)Another way to put it, because there is no bond in this model, the real asset also serves as a saving device and derives additional value from this function. Indeed, when \( s = G \) and \( \theta_t^O \) is close to 1 (but not too close), \( \theta_{t+1}^O > \theta_t^O \) because dividend is high so the optimists save by purchasing more when \( s = B \) and \( \theta_t^O \) is close to 1, \( \theta_{t+1}^O < \theta_t^O \) because dividend is low so the optimists dissave.

\(^{36}\)This channel for why asset price exceeds all agents’ valuation comes from risk-aversion. It is different from the channel in Harrison and Kreps (1978), in which agents expect to be able to sell their assets to other agents in the economy because their degrees of optimism and pessimism change over time. In my model, the optimists and pessimists remain being so so over time.

\(^{37}\)The pessimists do not necessarily want to short-sell the asset because of the possibility that asset price can increase by a lot when a good shock hits next period and drives up the wealth of the optimists together with the price of the real asset.

\(^{38}\)This definition corresponds to the instantaneous volatility in continuous time asset pricing models. See Xiong and Yan (2010) for the most recent use of the definition in the context of belief heterogeneity and complete markets.
panel of Figure 1 show that the consumption of the optimists \( c^O_t \) is close to \( c^O_t + d_t \theta^O_t \). As \( \theta^O_t \) is close to 1, the optimists’ consumption changes more with the changes in \( d_t \), which translates into higher asset price volatility because the optimists are always the marginal buyer of the real asset. The same logic explains why asset price volatility is increasing in \( \theta^O_t \).

Figure 2 also tells us about the dynamics of the economy over time, given that \( \theta^O_t \) is the only endogenous state variable in this economy. The bold red line (blue line) corresponds to the asset holding of the optimists next period as a function of their asset holding this period when \( s = G \) (\( s = B \)). The dashed-line is the 45° line. The two policy functions lie above the 45° lines for most of the values of current asset holding of the optimists. This suggests that the optimists tend to hold a larger fraction of the real asset relative to the pessimists’ holdings. Indeed, the crossed blue line in Figure 3 shows the long run distribution of fraction of the real asset held by the optimists (the other two lines correspond to the long run distributions of the optimists’ financial wealth in cases where they can take on leverage, which we will discuss in the next subsections). In the long run, the optimists end up holding most of the real asset. The optimists therefore are not driven out of the markets as suggested in Sandroni (2000) but in fact prosper. They survive by forgoing current consumption, even if the real asset pays off low dividend, to hold on to an ever-increasing share of the asset. This survival by “speculation” channel is complementary to the survival by precautionary saving channel suggested by Cogley et al. (2014). However, due to the fact that the real asset is long-lived in this framework but not in Cogley et al. (2014), the possibility to speculate in this asset allows the optimists to not only survive but also to prosper. In Appendix B, I show that in addition to belief heterogeneity, the optimists need non-financial wealth to prosper.

The stationary distribution combined with the fact that volatility is increasing in \( \theta^O_t \) (see the lower panel in Figure 1), shows that the price of the real asset tend to be very volatile in the long run, as the optimists hold most of the real asset. We will return to this result when we study the effect of financial regulation on volatility in Subsection 4.4.

### 4.3 Collateral Constrained Equilibrium

Now let \( m = 0 \). The agents can borrow up to the minimum future value of their real asset holdings. This economy is thus a mixture of the liquidity constrained and debt constrained economies studied in Kehoe and Levine (2001).
I rewrite the budget constraint of the optimists using financial wealth share, \( \omega^O_t \),

\[
e^O_t + q_t \theta^O_{t+1} + p_t \phi^O_{t+1} \leq e^O_t + (q_t + d_t) \omega^O_t.
\]

In Figure 4, the upper panel plots the price of the real asset as a function of the optimists’ financial wealth share \( \omega^O_t \) for the good state \( s_t = G \) (bold red line) and for the bad state \( s_t = B \) (blue line). The black dotted band corresponds to the valuations of the optimists. Interest rate \( r \) is also endogenously determined in this economy. Most of the time it hovers around the common discount factors of the two agents, i.e. \( r (s^t) \approx \frac{1}{p} - 1 \) because the pessimists have relatively large endowments and they are the marginal buyers of the collateralized bonds.

To understand the shape of the price functions in Figure 4, it is useful to look at the equilibrium portfolio choice of the agents in Figure 5. The bold red lines on the left and right panels are the real asset holdings of the optimists, \( \theta^O_{t+1} \), as functions of the financial wealth share \( \omega^O_t \) when the state \( s_t \) is \( G \) or \( B \) respectively. Similarly, the blue lines correspond to the bond holdings of the optimists, \( \phi^O_{t+1} \) and the black dashed lines correspond to the collateral constraint on borrowing: 

\[
-\theta^O_{t+1} \min_{s^t+1 | s^t} (q (s^t+1) + d (s^t+1)).
\]

Negative \( \phi^O_{t+1} \) means that the optimists borrow from the pessimists to invest into the real asset and the blue and black lines overlap when the collateral constraint is binding. Leverage induces the optimists to hold the total supply of the real asset. The pessimists are no longer the marginal buyers of the real asset as in Subsection 4.2, so the price of the real asset is very close to the valuation of the optimists.

Figure 5 also shows that the collateral constraint (4) is binding when \( \omega^O_t \) is close to 0. Due to this binding constraint, the optimists are restricted in their ability to smooth consumption, so their consumption becomes relatively low. Low consumption drives up marginal utility and consequently lowers the price of the real asset. Thus the slope of the price function in financial wealth share is higher when \( \omega^O_t \) is small relative to when \( \omega^O_t \) is higher (but not too close to 1). As prices and portfolio choices are all endogenous in equilibrium, we can re-interpret the high slope under the light of the “debt-deflation” channel: In the region in which the borrowing constraint binds - or is going to bind in the near future - when a bad shock hits the economy, that is \( s_{t+1} = B \), the optimists are forced to liquidate their real asset holdings. This fire sale of the real asset reduces its price and tightens the constraints further, thus setting off a vicious cycle of falling asset price and binding collateral constraint. These dynamics of asset price under borrowing constraint are called the debt-deflation channel in a small-open economy in Mendoza (2010). This example shows that the channel still operates when we are in a closed-economy with an
endogenous interest rate, $r \left( s^t \right)$ as opposed to exogenous interest rates in Mendoza (2010) or in Kocherlakota (2000).

On the right side of the upper panel in Figure 4, when $\omega_t^O$ is close to 1, the debt-deflation channel is not present as the collateral constraint is not binding or going to be binding in the near future. High asset price elasticity with respect to the financial wealth share of the optimists is due to their high exposure to the asset. As the optimists own most of the real asset without significant borrowing from the pessimists (Figure 5), the dividend from the real asset directly impacts the consumption of the optimists. Higher dividend implies higher consumption, lower marginal utility and higher asset price. The behavior of asset price when $\omega_t^O$ is close to 1 is thus similar to the case without leverage in Subsection 4.2.

The lower panel in Figure 4 plots asset price volatility, $v_t$ in (11) as functions of the financial wealth share of the optimists, the bold red line for $s_t = G$ and the blue line for $s_t = B$. Comparing the lower panels in Figure 4 to Figure 1, for higher $\omega_t^O$ and $s_t = B$, asset price volatility is also lower in this economy than it is in the no leverage economy in Subsection 4.2 because the ability to borrow allows the optimists to mitigate the drop in consumption when the bad shock with low dividend realizes.

In order to study the dynamics of asset price, we need to combine asset price as a function of the financial wealth share shown in Figure 4 with the evolution of the exogenous state and the evolution of the financial wealth share distribution, $\omega_t^O$. Figure 6 shows the
evolution of $\omega_t^O$. The left and right panels represent the transition of the financial wealth share when the current state is $G$ and $B$ respectively. The bold red lines represent next period financial wealth share of the optimists as a function of the current financial wealth share if the good shock realizes next period. The blue lines represent the same function when the bad shock realizes next period. I also plot the 45 degree lines for comparison (the dash-dotted black lines). This figure shows that, in general, good shocks tend to increase - and bad shocks tend to decrease - the financial wealth share of the optimists. This is because the optimists bet more on the likelihood that the good state realizes next period (by borrowing collateralized and investing into the real asset).\(^{39}\)

We can also think of financial wealth share, $\omega_t^O$, as the fraction of the real asset that the optimists owns. When the current state is good, and the fraction is high, the optimists will receive a high dividend from their real asset holding. Due to consumption smoothing, they will not consume all the dividend, but will use some part of the dividend to buy more real asset. Thus as we see from the left panel of Figure 6, the real asset holding of the optimists normally increases at high $\omega_t^O$. Similarly, when the current state is bad and the fraction is high, the optimists will sell off some of their real asset holding to smooth consumption. As a result, we see on the right panel that the real asset holding of the optimists tends to decrease at high $\omega_t^O$.

\(^{39}\)A similar evolution of the wealth distribution holds for complete markets, see Cao (2010).
Figure 6: Wealth Dynamics under Full Leverage

The wealth dynamics in Figure 6 imply that, as opposed to the case without leverage in Subsection 4.2, in the stationary distribution the optimists end up with low financial wealth share (Figure 3, bold black line). As we see from Figure 5, at low levels of $\omega_t^O$, the optimists always leverage up to the collateral constraint to buy the real asset, and when the bad shock hits they will lose all their asset holdings (they have to sell off their asset holdings to pay off their debt), in which case, their financial wealth reverts to zero. However, they can always use their non-financial endowment to return to the financial markets by investing in the real asset again (with leverage). Sometimes they are lucky - that is, when the asset pays high dividend and its price appreciates - their financial wealth increases rapidly. But due to the fact that the optimists have an incorrect belief, they are unlucky more often than they think, such that most of the time they end up with low financial wealth. The ability to leverage actually hurts these optimists because they can suffer larger financial losses. So leverage brings the equilibrium close to the complete markets equilibrium in Sandroni (2000), where the optimists’ total wealth goes to zero over time.40

Lastly, by combining Figures 4 and 6, we can choose sequences of shock realizations to generate booms and busts documented in Burnside et al. (2014). In that paper, to generate the booms and busts in asset prices, the authors rely on the slow propagation of beliefs

40See Cogley et al. (2014) for a similar point when Arrow securities are introduced in addition to bonds.
but shut down the effect of wealth distribution by assuming risk-neutrality.

### 4.4 Regulating Leverage and Asset Price Volatility

Proposition 3 in Section 3 suggests that the variations in the wealth distribution drive up asset price volatility relative to the long run complete markets benchmark. It is thus tempting to conclude that by restricting leverage, we can reduce the variation of wealth of the optimists, therefore reducing asset price volatility.\(^{41}\) However, this simple intuition is not always true. Similar to collateral constraints, financial regulation acts as another device to protect the agents with incorrect beliefs from making wrong bets and from disappearing. Worse yet, regulation may help these agents prosper by speculation as in Subsection 4.2. The higher wealth of the agents with incorrect beliefs increases their impact on real asset price, thus making real asset price more volatile.

To show this result, I consider a financial regulation that chooses a margin requirement \(m \in [0, 1]\). When \(m = 0\), the equilibrium is the unregulated one in Subsection 4.3. When \(m = 1\), collateral requirement is infinite, so there are no bonds traded. The equilibrium is that agents only trade in the real asset subject to no short-selling in Subsection 4.2. Figure 7 shows that the long run asset return volatility is increasing in collateral requirement, \(m\).\(^{42}\)

To explain this increasing relationship, I study in detail an economy with an intermediate value of \(m\), \(m = 0.5\). Figures 8, 9, and 10 are the counterparts of Figure 4, 5, and 6 when \(m = 0.5\). The portfolio choice in Figure 9, shows that due to restricted leverage the optimists are able to purchase all the supply of the real asset only when they have sufficiently large financial wealth, i.e., \(\omega_{t}^{O} > 0.5\). Below that threshold, the pessimists are also the marginal buyers of the asset, so the price of the asset is closer to the pessimists’ valuation, as depicted in Figure 8 (the lower dotted band in the upper panel). The same figure also shows that asset price is the most volatile around \(\omega_{t}^{O} = 0.5\), at that point, when the bad shock hits, the optimists have to give up some of the real asset. Asset price falls as the pessimists become the marginal buyers. In addition, the debt deflation channel kicks in because of the collateral constraint on the optimists. Debt deflation further depresses the price of the real asset. The wealth dynamics are described by Figure 10. Below \(\omega_{t}^{O} = 0.5\), the optimists are protected by the regulation, as they cannot borrow as much as they

---

\(^{41}\)Welfare analysis is delicate in environments with belief heterogeneity, as shown in Brunnermeier, Simsek, and Xiong (2014), so I only focus on volatility in this discussion. The equilibrium effects of margin requirements and leverage restrictions on asset price volatility are of independent interest to economists and policy practitioners, as discussed in Rytchkov (2014) and many other related papers.

\(^{42}\)The long run return volatility is computed using the long run stationary distribution of financial wealth share of the optimists.
want, most of their investment is in the real asset, so they accumulate wealth through speculation as in Subsection 4.2. However, above $\omega t = 0.5$, their borrowing constraint is not binding, so they tend to borrow too much, relative to when they have correct belief, and lose wealth over time as in Subsection 4.3.

Figure 3 shows the stationary distributions of the financial wealth share of the optimists under three different financial structures, $m = 0$ (bold black line), 0.5 (red circled line) and 1 (blue crossed line). The stationary mean level of financial wealth of the optimists is increasing in $m$. As suggested by the analysis of volatility above, stricter financial regulation, which corresponds to higher collateral requirement, i.e., higher margin, higher $m$, leads to higher long run financial wealth of the optimists. Consequently, asset price volatility is higher due to the optimists’ greater influence on asset price.

This increasing relationship between collateral requirement and asset returns volatility is in contrast with the result obtained in Rytchkov (2014) and Brumm et al. (2015) that collateral requirement decreases asset return volatility at high levels of margin requirement. This difference comes from the fact that Rytchkov (2014) and Brumm et al. (2015) assume that agents trade due to their difference in risk-aversion, while in my paper agents trade due to their difference in beliefs. Under belief heterogeneity, there is a new mechanism that does not exist in Rytchkov (2014) and Brumm et al. (2015): a higher margin requirement actually protects the agents with incorrect beliefs. Thus they are financially wealthier in the long run and their trading activity drives up asset price volatility.
Figure 8: Asset Price and Asset Price Volatility under Regulated Leverage

Figure 9: Portfolio Choice under Regulated Leverage
5 Conclusion

In this paper I develop a dynamic general equilibrium model to examine the effects of belief heterogeneity on the financial wealth of agents and on asset price dynamics and asset price volatility under different financial market structures. I show that when financial markets are collateral constrained (endogenously incomplete markets) agents with incorrect beliefs sometime prosper in the long run. The prosperity of these agents leads to higher asset price volatility. This result contrasts with the frictionless complete markets case in which agents holding incorrect beliefs are eventually driven out and, as a result, asset price exhibits lower volatility. The prosperity of agents with incorrect beliefs also generates rich dynamics of asset prices and financial wealth distribution.

In the earlier version of this paper - Cao (2010) - I show the existence of stationary Markov equilibria in this framework with collateral constrained financial markets and with general production and capital accumulation technology. I also develop an algorithm for computing the equilibria. As a result, the framework can be readily used to investigate questions about asset pricing and about the interaction between financial markets and the macroeconomy. For instance, it would be interesting in future work to apply these methods in calibration exercises using more rigorous quantitative asset pricing techniques, such as in Alvarez and Jermann (2001).
uncertainty in the growth rate of dividends rather than uncertainty in the levels, as in Online Appendix E, in order to match the rate of return on stock markets and the growth rate of aggregate consumption.

A second avenue for further research is to examine more normative questions in the framework developed in this paper. My normative results suggest, for example, that financial regulation aimed at reducing asset price volatility should be state-dependent, as conjectured by Geanakoplos (2010).
A Markov Equilibrium

In this appendix, I argue that collateral constrained equilibrium exists with a stationary structure. The equilibrium prices and allocation depend on the exogenous state of the economy and a measure of the financial wealth distribution. The detailed proofs are in Cao (2010). I also show a numerical algorithm to compute the equilibrium in Subsection A.3.

A.1 The State Space and Definition

Let $\omega_t$ denote the financial wealth distribution and $\Omega$ denote the $(H-1)$-dimensional simplex in Subsection 3.3. I show that, under the conditions detailed in Subsection A.2 below, there exists a Markov equilibrium over the compact space $S \times \Omega$ defined in Subsection 3.3.

In particular, for each $(s_t, \omega_t) \in S \times \Omega$, we need to find a vector of prices and allocations

$$\nu_t \in \hat{\mathcal{V}} = \mathbb{R}_+^H \times \mathbb{R}_+^H \times \mathbb{R}_+^H \times \mathbb{R}_+ \times \mathbb{R}_+$$

that consists of the consumers’ decisions: consumption of each consumer $\left( c^h (\sigma) \right)_{h \in \mathcal{H}} \in \mathbb{R}_+^H$, real and bond holdings $\left( \theta^h (\sigma), \phi^h (\sigma) \right)_{h \in \mathcal{H}} \in \mathbb{R}_+^H \times \mathbb{R}_+^H$, the price of the real asset $q(\sigma) \in \mathbb{R}_+$ and the price of bonds $p(\sigma) \in \mathbb{R}_+$ must satisfy the market clearing conditions and the budget constraint of consumers bind. Moreover, for each future state $s_{t+1} \in S$ succeeding $s_t$, we need to find a corresponding financial wealth distribution $\omega_{t+1}^s$ and equilibrium allocation and prices $\nu_{t+1}^s \in \hat{\mathcal{V}}$ such that for each household $h \in \mathcal{H}$ the following conditions hold:

a) For each $s^+ \in S$ succeeding $s$

$$\omega_{s^+}^h = \frac{\theta^h (q_{s^+}^h + d_{s^+}^h) + \phi^h}{q_{s^+}^h + d_{s^+}^h},$$

where $q_{s^+}^h, d_{s^+}^h$ are the price of the real asset and dividend at state $s^+$.

b) There exist multipliers $\mu^h, \eta^h \in \mathbb{R}_+$ corresponding to collateral constraints and no-
short selling constraints such that
\[ 0 = -q U'_h \left( c^h \right) + \eta^h q + \mu^h (1 - m) \theta^h \min (q^+ + d^+) + \beta E^h \left\{ (q^+ + d^+) U'_h \left( c^{h+} \right) \right\} \]  
(14a)
\[ 0 = -p U'_h \left( c^h \right) + \beta E^h \left\{ U'_h \left( c^{h+} \right) \right\} \]  
(14b)
\[ 0 = \mu^h \left( \phi^h + (1 - m) \theta^h \min (q^+ + d^+) \right) \]  
(14c)
\[ 0 = \eta^h \theta^h \]  
(14d)
\[ 0 \leq \phi^h + (1 - m) \theta^h \min (q^+ + d^+) \]  
(14e)
\[ 0 \leq \theta^h . \]  
(14f)

Condition a) guarantees that the future financial wealth distributions are consistent with the current equilibrium decision of the consumers. Conditions b) are the first-order conditions and the complimentary-slackness conditions with respect to real asset and bond holdings from the maximization problem (1) of the consumers. Given that the maximization problem is convex, the first-order conditions are sufficient for a maximum.

Before continue, let me briefly discuss the prices of real asset and bond in a Markov equilibrium. We can rewrite the first-order conditions with respect to real asset holding (14a) as
\[ q U'_h \left( c^h \right) = \eta^h q + \mu^h (1 - m) \theta^h \min (q^+ + d^+) + \beta E^h \left\{ (q^+ + d^+) U'_h \left( c^{h+} \right) \right\} \]
\[ \geq \beta E^h \left\{ (q^+ + d^+) U'_h \left( c^{h+} \right) \right\} . \]

By re-iterating this inequality we obtain
\[ q_t \geq E^h_i \left\{ \sum_{r=1}^\infty \beta^r d_{t+r} \frac{U'_h \left( c^h_{t+r} \right)}{U'_h \left( c^h_t \right)} \right\} . \]

We have a strict inequality if there is a strict inequality \( \mu^h_{t+r} > 0 \) or \( \eta^h_{t+r} > 0 \) in the future. So the asset price is higher than the present discounted value of the stream of its dividend because in future it can be sold to other agents, as in Harrison and Kreps (1978) or it can be used as collateral to borrow as in Fostel and Geanakoplos (2008). In Cao (2010), I show that under belief heterogeneity collateral constraints will eventually be binding for every agent when they strictly differ in their belief. As a result, the price of the real asset is strictly higher than the present discounted value of its dividend.\(^{43}\)

\(^{43}\)We can also derive a formula for the equity premium that depends on the multipliers \( \mu \) similar to the
Equation (14a) also shows that the real asset will have collateral value when some
\( \mu^h > 0 \), in addition to the asset’s traditional pay-off value weighted at the appropriate
discount factors. Unlike in Alvarez and Jermann (2000), attempts to find a pricing kernel
which prices assets using their pay-off value might prove fruitless because assets with the
same payoffs but different collateral values will have different prices. This point is also
emphasized in Geanakoplos’ papers. Following Fostel and Geanakoplos (2008), I define
the collateral value of the real asset (as a proportion of the total value of the asset) by the
cross-sectional expected value

\[
CV_h = E^h \left[ \frac{\mu^h (1-m) \varrho^h \min (q^+ + d^+)}{q U^f_h (c^h)} \right]
\]

in the long run stationary distribution. Online Appendix E offers an estimate of this value
in a calibration for the U.S. economy.

A.2 Existence and Properties of Markov Equilibrium

The existence proof is similar to the ones in Kubler and Schmedders (2003) and Magill
and Quinzii (1994). We approximate the Markov equilibrium by a sequence of equilibria
in finite horizon. There are three steps in the proof. First, using Kakutani fixed point
theorem to prove the existence proof of the truncated T-period economy. Second, show
that all endogenous variables are bounded. And lastly, show that the limit as T goes to
infinity is the equilibrium of the infinite horizon economy.

For the second step, we need the following assumption:

**Assumption 5.** There exist \( \bar{c}, \bar{\zeta} > 0 \) such that

\[
U_h (\bar{c}) + \max \left\{ \frac{\beta}{1 - \beta} U_h (\bar{\zeta}) , 0 \right\} \\
\leq \min \left\{ \frac{1}{1 - \beta} \min_{s \in S} U_h (\bar{c}) , \min_{s \in S} U_h (\bar{\zeta}) \right\} \forall h \in \mathcal{H}.
\]

and

\[
U_h (\bar{c}) + \min \left\{ \frac{\beta}{1 - \beta} U_h (\bar{c}) , 0 \right\} \\
\geq \max \left\{ \frac{1}{1 - \beta} U_h (\bar{c}) , U_h (\bar{\zeta}) \right\} \forall h \in \mathcal{H}.
\]

\[
\text{equity premium formula in Mendoza (2010)}
\]
The intuition for the first inequality is detailed in the existence proof in Cao (2010); it ensures a lower bound for consumption. The second inequality ensures that the price of the real asset is bounded from above. Both inequalities are obviously satisfied by CRRA functions with CRRA coefficients exceeding 1.

**Lemma 1.** Suppose that Assumption 5 is satisfied. Then there is a compact set that contains the equilibrium endogenous variables constructed in Cao (2010) for every $T$ and every initial condition lying inside the set.

*Proof.* Cao (2010) 

**Theorem 1.** Under the same conditions, a Markov equilibrium exists.

*Proof.* In Cao (2010), I show the existence of Markov equilibria for a general model also with capital accumulation. As in Kubler and Schmedders (2003), we extract a limit from the T-finite horizon equilibria. Lemma 1 guarantees that equilibrium prices and quantities are bounded as $T$ goes to infinity. The proof follows Kubler and Schmedders (2003) but uses an alternative definition of attainable sets in order to accommodate production.

### A.3 Numerical Method

The construction of Markov equilibria in the last section also suggests an algorithm to compute them. The following algorithm is based on the construction of Markov equilibrium in Subsection 3.3.

As in the existence proof, we look for the following correspondence

$$
\rho : S \times \Omega \rightarrow \hat{V} \times \Omega^2 \times \mathcal{L}
$$

$$(s, \omega) \mapsto (\hat{v}, \omega^+, s, \mu, \eta)$$

$\hat{v} \in \hat{V}$ is the set of endogenous variables excluding the financial wealth distribution as defined in (12). $(\omega^+_s)_{s \in S}$ are financial wealth distributions in the $S$ future states and $(\mu, \eta) \in \mathcal{L}$ are Lagrange multipliers as defined in subsection A.1.

From a given continuous initial mapping $\rho^0 = (\rho^0_1, \rho^0_2, \ldots, \rho^0_S)$, we construct the sequence of mappings $\{\rho^n = (\rho^n_1, \rho^n_2, \ldots, \rho^n_S)\}_{n=0}^{\infty}$ by induction. Suppose we have obtained $\rho^n$, for each state variable $(s, \omega)$, we look for

$$
\rho^{n+1}_s(\omega) = (\hat{v}_{n+1}, \omega^+_s, \mu_{n+1}, \eta_{n+1})
$$

that solves the first-order conditions (14), market clearing conditions, and the consistency of the future financial wealth distribution (13).
We construct the sequence \( \{\rho^n\}_{n=0}^\infty \) on a finite discretization of \( S \times \Omega \). Given the equilibrium mapping \( \rho^n \), we solve for the equilibrium mapping \( \rho^{n+1} \). Fixing a precision \( \delta \), the algorithm stops when \( \|\rho^{n+1} - \rho^n\| < \delta \).

To solve for the equilibrium mapping \( \rho^{n+1} \), given the equilibrium mapping \( \rho^n \) as following. At each point on the grid for \( \omega \) and exogenous state \( s \), we solve for a system of \( 5H + 2S \) unknowns (assuming that the exogenous state \( s \) has \( S \) successors) and \( 5H + 2S \) equations. The unknowns include the allocations and asset prices \( \delta_{n+1} = \{(c^h_{n+1}, \theta^h_{n+1}, \phi^h_{n+1}) \mid h \in H, q_{n+1}, p_{n+1} \} \), \( 2H \) multipliers \( \{(\mu^h_{n+1}, \eta^h_{n+1}) \mid h \in H' \} \) and \( S \) future financial wealth distribution \( \omega^{+}_{s,n+1} \). The equations include \( 4H \) first order conditions (14a-14d), \( H \) budget constraints of \( H \) agents, \( 2 \) market clearing conditions for the real asset and for bonds, and \( S \) consistency conditions, (13), for financial wealth distributions in \( S \) future exogenous states. Equations (14a-14d), require the future consumptions \( c^h_+ \) and real asset price \( q_+ \). To obtain these values, we interpolate the equilibrium mapping \( \rho_n \) over the grid to obtain the values at each \( (s_+, \omega^{+}_{s,n+1}) \). We use MATLAB’s solve (or a more powerful version, ktrlink by Knitro) to solve for the system of \( 5H + 2S \) equations.

There are two important details in implementing this algorithm: First, in order to calculate the \((n+1)\)-th mapping \( \rho^{n+1} \) from the \(n\)-th mapping, we need to only keep track of the consumption decisions \( c^h \) and asset prices \( q \) and \( p \). Even though other asset holding decisions and Lagrange multipliers might not be differentiable functions of the financial wealth distribution, the consumption decisions and asset prices normally are. Relatedly, when there are redundant assets, there might be multiple asset holdings that implement the same consumption policies and asset prices. Second, if we choose the initial mapping \( \rho^0 \) as an equilibrium of the 1-period economy as in Subsection A.2, then \( \rho^n \) corresponds to an equilibrium of the \((n+1)\)-period economy. I follow this choice in computing an equilibrium of the two agent economy presented in Section 4.

**B No Non-financial Wealth**

In this section, I show the importance of non-financial wealth for the dynamics of financial wealth distribution and asset prices by studying a model in Section 4 in the absence of non-financial wealth. Specifically, the budget constraints of the agents become

\[
c^h_i + q_i \theta^h_{i+1} + p_i \phi^h_{i+1} \leq (q_i + d_i) \theta^h_i + \phi^h_i
\]
and under the collateral constraint (4). When the margin \( m = 1 \), agents trade only in the real asset. So the budget constraints of the agents become

\[
 c_t^h + q_t \theta_{t+1}^h \leq (q_t + d_t) \theta_t^h.
\]

**Proposition 4.** Suppose that agents have log utility and agents trade only in the real asset, i.e. \( m = 1 \). Regardless of the beliefs of the agents, the equilibrium price of the real asset is

\[
 q_t = \frac{\beta}{1 - \beta} d_t.
\]

and the asset holdings of the agents do not change over time, i.e. \( \theta_{t+1}^h = \theta_t^h \) for all \( t, s^t \), and \( h \in \mathcal{H} \).

**Proof.** Given log utility

\[
 c_t^h = (1 - \beta) (q_t + d_t) \theta_t^h.
\]

Therefore the market clearing condition in the consumption-market at time \( t \) implies that

\[
 d_t = \sum_{h \in \mathcal{H}} c_t^h = (1 - \beta) (q_t + d_t) \sum_{h \in \mathcal{H}} \theta_t^h = (1 - \beta) (q_t + d_t).
\]

Thus we have the expression (16) for the price of the real asset. By plugging this price into the budget constraint of the agents and using the solution for consumption (17), we obtain the constant asset holdings of the agents over time.

This proposition shows that the *prosper by speculation* mechanism in Subsection 4.2 is driven by the non-financial wealth of the agents in the economy.

However when agents are allowed to trade also in bonds then the standard disappearance result applies.

**Proposition 5.** Suppose that agents are allowed use the real asset as collateral to borrow, i.e. \( m = 0 \) and in each history \( s^t \), there are only two possible future exogenous states \( s_{t+1} \). Then only the agents with correct beliefs survive, i.e. almost surely:

\[
 \lim_{t \to \infty} c_t^h = 0.
\]

and

\[
 \lim_{t \to \infty} \omega_t^h = 0.
\]
for all $h \in H$ and $P^h \neq P$, where $P$ is the correct belief and some agent has the correct belief.

Proof. Financial markets are dynamically complete so this proposition is a direct application of Sandroni (2000).

\[\square\]

C Collateral Constrained Equilibrium and Collateral Equilibrium

In this appendix, I show that the collateral constrained equilibrium defined in Section 3.1 is the same as the collateral equilibrium in Geanakoplos and Zame (2002) under the condition that in each history there are only two possible exogenous future states. As in Geanakoplos and Zame (2002), I assume that agents have access to a wide range of financial assets instead of only one collateralized bond in Section 3.1.

Financial Assets: In each history $s^t$, there are also (collateralized) financial assets, $j \in J_t(s^t)$. $J_t(s^t)$ have a finite number of elements, $J_t(s^t)$. Each financial asset $j$ (or financial security) is characterized by a vector and a scalar, $(b_j, k_j)$, of promised payoffs and collateral requirement. Promises are a standard feature of financial assets similar to Arrow’s securities, i.e., asset $j$ traded in history $s^t$ promises next-period pay-off $b_j(s_{t+1}) > 0$ in terms of final good at the successor history $s^{t+1} = (s^t, s_{t+1})$. The non-standard feature is the collateral requirement. Agents can only sell the financial asset $j$ if they hold shares of the real asset as collateral. If an agent sells one unit of security $j$, she is required to hold $k_j$ units of the real asset as collateral.\(^{44}\)

There are no penalties for default except for the seizure of collateral. Thus, a seller of the financial asset can default in history $s^{t+1}$ whenever the total value of collateral falls below the promise at that state at the cost of losing the collateral. By individual rationality, the effective pay-off of security $j$ in history $s^t$ is always given by

\[
f_j(s^t, s_{t+1}) = \min \left\{ b_j(s_{t+1}), \left( q(s^{t+1}) + d(s_{t+1}) \right) k_j \right\}.
\]

Let $p_{j,t}(s^t)$ denote the price of security $j$ in history $s^t$.

Assumption 6. Each financial asset requires a strictly positive level of collateral

\[
\min_{j \in J_t(s^t)} k_j > 0
\]

\(^{44}\)Notice that, there are only one-period ahead financial assets. See He and Xiong (2012) for a motivation why longer term collateralized financial assets are not used in equilibrium.

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If a financial asset \( j \) requires no collateral then its effective pay-off, determined by (18), will be zero. Hence it will be easy to show that in equilibrium its price, \( p_{j,t} \), will be zero as well. We can thus ignore these financial assets.

Remark. Selling one unit of financial asset \( j \) is equivalent to purchasing \( k_j \) units of the real asset and at the same time pledging these units as collateral to borrow \( p_{j,t} \). It is shown in Cao (2010) that

\[ k_j q_t - p_{j,t} > 0, \]

which is to say that the seller of the financial asset always has to pay some margin. So the decision to sell financial asset \( j \) using the real asset as collateral corresponds to the desire to invest in the real asset at margin rather than the simple desire to borrow.

Remark. We can then define the leverage ratio associated with the transaction as

\[ L_{j,t} = \frac{k_j q_t}{k_j q_t - p_{j,t}} = \frac{q_t}{q_t - \frac{p_{j,t}}{k_j}}. \]  

(19)

Even though there are many financial assets available, in equilibrium only some financial asset will be actively traded, which in turn determines which leverage levels prevail in the economy in equilibrium. In this sense, both asset price and leverage are simultaneously determined in equilibrium, as emphasized in Geanakoplos (2010). Online Appendix F uses this definition of leverage to investigate how leverage varies over the business cycles.

Consumer \( h \) takes the sequences of prices \( \{ q_t, p_{j,t} \} \) as given and solves

\[
\max_{\{c_t, \phi_t, \theta_t \}} E^h_0 \left[ \sum_{t=0}^{\infty} \beta^t U_h(c_t^h) \right]
\]

subject to the budget constraint

\[
c_t^h + q_t \phi_{t+1}^h + \sum_{j \in J_t} p_{j,t} \phi_{j,t+1}^h \leq c_t^h + \sum_{j \in J_{t-1}} f_{j,t} \phi_{j,t}^h + (q_t + d_t) \theta_t^h
\]

(21)

and the collateral constraint

\[
\theta_{t+1}^h + \sum_{j \in J_t: \phi_{j,t} < 0} \phi_{j,t+1}^h k_j \geq 0
\]

(22)

The collateral constraint (22) implies the no-short sale constraint. I define a collateral equilibrium as in Geanakoplos and Zame (2002):
Definition 3. A collateral equilibrium for an economy with initial asset holdings
\[ \{ \theta^h, \phi^h \}_{h \in \mathcal{H}} \]
and initial shock \( s_0 \) is a collection of consumption, real and financial asset holdings and prices in each history \( s^t \),

\[
\left( \left\{ c^h_t \left( s^t \right), \theta^h_{t+1} \left( s^t \right), \phi^h_{j,t+1} \left( s^t \right) \right\} \right)_{h \in \mathcal{H}} \\
q_t \left( s^t \right), \left\{ p_{j,t} \left( s^t \right) \right\}_{j \in \mathcal{J}_t(s^t)}
\]
satisfying the following conditions:

i) The markets for final good, real and financial assets in each period clear:
\[
\sum_{h \in \mathcal{H}} \theta^h_{t+1} \left( s^t \right) = 1 \\
\sum_{h \in \mathcal{H}} \phi^h_{j,t+1} \left( s^t \right) = 0 \quad \forall j \in \mathcal{J}_t \left( s^t \right) \\
\sum_{h \in \mathcal{H}} c^h_t \left( s^t \right) = \sum_{h \in \mathcal{H}} e^h \left( s_t \right) + d \left( s_t \right)
\]

ii) For each consumer \( h \), \( \left\{ c^h_t \left( s^t \right), \theta^h_{t+1} \left( s^t \right), \phi^h_{j,t+1} \left( s^t \right) \right\}_{j \in \mathcal{J}_t(s^t)} \) solves the individual maximization problem subject to the budget constraint (2), and the collateral constraint (4).

Now, to study the standard debt contracts, I consider the sets \( \mathcal{J}_t \) of financial assets which promise state-independent payoffs in the next period. I normalize these promises to \( b_j = 1 \). Asset \( j \) also requires \( k_j \) units of the real asset as collateral. The effective pay-off is therefore
\[
f_{j,t+1} \left( s^{t+1} \right) = \min \left\{ 1, k_j \left( q \left( s^{t+1} \right) + d \left( s^{t+1} \right) \right) \right\}.
\] (23)

Due to the finite supply of the real asset, in equilibrium only a subset of the financial assets in \( \mathcal{J}_t \) are traded. It turns out that in some special cases we can determine exactly which financial assets are traded. For example, Fostel and Geanakoplos (2008) argue that if we allow for the set \( \mathcal{J}_t \) to be dense enough, then in equilibrium the only financial asset traded in equilibrium is the one with the minimum collateral level to avoid default. This statement also applies to the infinite-horizon set-up in this paper under the condition that in any history, there are only two succeeding future exogenous states. I formalize the result in Proposition 6 below. The proposition uses the following definition
Definition 4. Two collateral equilibria are equivalent if they have the same allocation of consumption to the consumers and the same prices of real and financial assets. The equilibria might differ in the consumers’ portfolios of real and financial assets.

Proposition 6. Consider a collateral equilibrium and suppose in a history \( s^t \), there are only two possible future exogenous states \( s^{t+1} \). Let

\[
    u_t = \max_{s^{t+1}|s^t} \left( q \left( s^{t+1} \right) + d \left( s^{t+1} \right) \right)
\]

\[
    d_t = \min_{s^{t+1}|s^t} \left( q \left( s^{t+1} \right) + d \left( s^{t+1} \right) \right)
\]

and

\[
    k^*_t = \frac{1}{d_t}.
\]

We can find an equivalent collateral equilibrium such that only the financial assets with the collateral requirements

\[
    \max_{j \in J, k_j \leq k^*_t} k_j \text{ and } \min_{j \in J, k_j \geq k^*_t} k_j
\]

are actively traded. In particular, when \( k^*_t \in J^t \), we can always find an equivalent equilibrium in which only the financial assets with the collateral requirement exactly equal to \( k^*_t \) are traded.

Proof. Intuitively, given only two future states, using two assets - a financial asset and the real asset - we can effectively replicate the pay-off of all other financial assets. But we need to make sure that the collateral constraints, including the no short-selling constraint, are satisfied in the replications. To simplify the notations, I will suppress the subscripts \( t \) from the variables. Let \( k_d = \max_{j \in J, k_j \leq k^*_t} k_j \). Let \( p_d \) denote the price of the financial assets with collateral requirement \( k_d \). Suppose that there is another financial asset in \( J \) that is actively traded and have collateral requirement \( k \leq k^* \). Then by definition \( k < k_d \). Let \( p_k \) denote the price of the financial asset. We have two cases:

Case 1) \( \frac{1}{d} \geq k_d > k > \frac{1}{u} \): Consider the optimal portfolio choice of a seller of financial asset \( k \). The pay-off from selling the asset is \( (ku - 1, 0) \) and she has to pay \( kq - p_k \) in cash: she buys \( k \) units of the real asset but she get \( p_k \) from selling the financial asset. So the return on the financial asset is \( \frac{ku - 1}{kq - p_k} \) (when the good state realizes next period). Similarly, the return from selling financial asset \( k_d \) is \( \frac{k_d u - 1}{k_d q - p_d} \). If there are sellers for financial asset \( k \), it then implies that

\[
    \frac{ku - 1}{kq - p_k} \geq \frac{k_d u - 1}{k_d q - p_d}.
\]
or equivalently
\[ p_k \geq \frac{ku - 1}{kd - 1} P_d + \frac{k_d - k}{kd - 1} q, \]  
(24)

otherwise, sellers will strictly prefer selling financial asset \( k_d \) to financial asset \( k \). Now from the perspective of the buyers of financial assets, the pay-off of financial asset \( k \) is \((1, kd)\). We can write this pay-off as a portfolio of financial asset \( k_d \) and the real asset:
\[ \begin{bmatrix} 1 \\ kd \end{bmatrix} = \frac{ku - 1}{kd - 1} \begin{bmatrix} 1 \\ kd \\ d \end{bmatrix} + \frac{k_d - k}{kd - 1} \begin{bmatrix} u \\ d \end{bmatrix}. \]  
(25)

As a result, if there are buyers for financial asset \( k \), we must have
\[ p_k \leq \frac{ku - 1}{kd - 1} P_d + \frac{k_d - k}{kd - 1} q, \]  
(26)

otherwise the buyers will buy the portfolio (25) of financial asset \( k_d \) and the real asset instead. Thus, both (24) and (26) happen with equality if financial asset \( k \) is actively traded. Armed with the equality, we can now prove the proposition. Consider each pair of seller and buyer of a unit of financial asset \( k \): the buyer buys one unit, and the seller who sells one unit of financial asset \( k \). The seller, at the same time, is required to buy \( k \) units of the real asset. We alter their portfolios in the following way: instead of buying one unit of financial asset \( k \), the buyer buys \( \frac{ku - 1}{kd - 1} \) units of financial asset \( k_d \) from the seller and \( \frac{k_d - k}{kd - 1} \) of the real asset. Given (25) and the equality (26), these changes in portfolio leave the consumption and future wealth of the buyer unchanged. Now the seller instead of selling one unit of financial asset \( k \), sells \( \frac{ku - 1}{kd - 1} \) units of financial asset \( k_d \) and holds \( \frac{ku - 1}{kd - 1} k_d \) units of the real asset as collateral. Similarly, due to the equality (24), this transaction costs the same and yields the same returns to the seller compared to selling one unit of financial asset \( k \). So the consumption and the future wealth of the seller remains unchanged. Now we just need to verify that the total quantity of the real asset used remain unchanged. Indeed this is true because
\[ \frac{k_d - k}{kd - 1} + \frac{ku - 1}{kd - 1} k_d = k. \]

Case 2) \( \frac{1}{u} \geq k \): Financial asset \( k \)'s pay-off to the buyers is \( k (u, d) \) and to the seller is 0. So the financial asset has exactly the same payoffs as the real asset. This implies, \( p_k = kq \). The proposition follows immediately.

Now let \( k_u = \min_{j \in J, k_j \geq k} k_j \). Let \( p_u \) denote the price of the financial asset \( k_u \). The proof of the proposition is similar. First, we show that the price of any actively traded financial
asset \( k \) with \( k \geq k^* \) is \( p_k = p_u \) and we can alter the portfolio of the buyers and sellers of financial asset \( k \) to transfer all the trade in the financial asset to financial asset \( k_u \). \( \Box \)

Unless specified otherwise, I assume that the set of financial assets, \( \mathcal{J}_t \), includes all collateral requirements \( k \in \mathbb{R}^+, k > 0 \). Proposition 6 implies that, for any collateral equilibrium, we can find an equivalent collateral equilibrium in which the only financial asset with the collateral requirement exactly equal to \( k^* (s^t) \) is traded in equilibrium. Therefore in such an equilibrium the only actively traded financial asset is riskless to its buyers. Let \( p(s^t) \) denote the price of this financial asset. The endogenous interest rate is therefore \( r(s^t) = \frac{1}{p(s^t)} - 1 \).\(^{46,47}\)

We can also establish the following corollary which corresponds to Proposition 1:

**Corollary.** [Formalization of Proposition 1] When \( \mathcal{J}_t \) includes all collateral requirements \( k \in \mathbb{R}^+, k > 0 \), a collateral equilibrium is equivalent to a collateral constrained equilibrium in which agents can borrow \( \phi^h_{t+1} \) but are subject to the collateral constraint (4) with \( m = 0 \):

\[
\phi^h_{t+1} + \theta^h_{t+1} \min_{s^t+1 \mid s^t} \left( q(s^t+1) + d(s_{t+1}) \right) \geq 0. \tag{27}
\]

**Proof.** Proposition 6 shows that we can find an equivalent collateral equilibrium in which only financial assets with collateral level \( k^t_n \) is traded. The collateral constraint (22) at this collateral level can be re-written as the collateral constraint (27). \( \Box \)

By assuming that lenders can seize only the real asset but not the current dividend in (18), we can also show that - under the same conditions as in Proposition 6 - the collateral equilibrium is equivalent to a collateral constrained economy in which the agents face the collateral constraint

\[
\phi^h_{t+1} + \theta^h_{t+1} \min_{s^t+1 \mid s^t} q(s^t+1) \geq 0.
\]

This is the constraint used in Kiyotaki and Moore (1997). The authors microfound this condition by assuming that human capital is inalienable. This paper thus provides an alternative microfoundation using the endogeneity of the set of actively traded financial assets.\(^{48}\)

\(^{45}\)To apply the existence theorem 1 I need \( \mathcal{J} \) to be finite. But we can think of \( \mathcal{J} \) as a fine enough grid.

\(^{46}\)The uniqueness of actively traded financial assets established in Geanakoplos (2010) and He and Xiong (2012) is in the “equivalent” sense in Definition 4.

\(^{47}\)In a two-period economy, Araujo et al. (2012) show that if there are \( S \) future states, for any collateral equilibrium, we can find an equivalent collateral equilibrium in which only \( S - 1 \) collateralized bonds are actively traded. This Proposition can be extended for the infinite-horizon economy in this paper as in Proposition 6.

\(^{48}\)Mendoza (2010) and Kocherlakota (2000) use a similar collateral constraint but
By restricting the set $J_t$, we can also find a collateral equilibrium equivalent to a collateral constrained equilibrium in which the margin $m \in (0, 1)$.

## D Continuous Time Model

In this appendix, I present a continuous time version of my model. Using the method developed in Rytchkov (2014), in the absence of non-financial endowment, I show that the Markov equilibrium concept translates to a continuous time equilibrium with wealth share as a sufficient state variable. However, this method does not apply when there is non-financial endowment, because the constrained optimization problems of the agents cease to be homogenous. Thus the value function does not admit a simple power form in financial wealth.

The environment is exactly the same as in the main paper, except that time is continuous. The aggregate dividend $D_t$ follows a geometric Brownian motion

$$
\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t,
$$

where $B_t$ is a standard Brownian motion. However, agents in this economy might have an imprecise estimate of the growth rate of dividend, as modelled in Gallmeyer and Hollifield (2008) and Yan (2008).\footnote{See Chabakauri (2014) for an application of Rytchkov (2014)’s method to this environment with heterogeneous beliefs.} From agent $h$’s point of view, the dividend follows the process

$$
\frac{dD_t}{D_t} = \mu^h_D dt + \sigma_D dB^h_t,
$$

where

$$
dB^h_t = dB_t - \frac{\mu^h_D - \mu_D}{\sigma_D} dt.
$$

By Girsanov’s theorem, $B^h_t$ is a Brownian motion in agent $h$’s filter.

The total return on stock is determined by

$$
\frac{dq_t + D_t dt}{q_t} = \mu_t dt + \sigma_t dB_t,
$$

where $q_t$ is the equilibrium price of the real asset and $\mu_t$ and $\sigma_t$ are endogenously determined by

$$
\min_{x_{t+1} \mid x_t} \left( q(s^{t+1}) + d(s_{t+1}) \right)
$$

replaced by $q(s^t)$. I have not seen a microfoundation for such a collateral constraint but the quantitative implications of the constraint should be similar if asset prices do not vary too much across immediate future histories.
mined in equilibrium. From agent $h$’s perspective

$$\frac{dq_t + D_t dt}{q_t} = \mu^h_t dt + \sigma_t dB^h_t,$$

where, by (28),

$$\mu^h_t = \mu_t + \frac{\mu_D^h - \mu_D}{\sigma_D} \sigma_t.$$

Given the process of stock price $q_t$, interest rate $r_t$, agent $h$ maximizes his lifetime utility function given by

$$\mathbb{E}_0^h \left[ \int_0^\infty e^{-\rho t} \left( \frac{c^h_t}{1 - \sigma_h} \right) \right],$$

where $\mathbb{E}_0^h [.]$ is the expectation operator, $\rho > 0$ is the discount rate, $c^h_t$ is consumption at time $t$. The agents can trade in the market for stocks as well as in a market for state non-contingent bonds that yield instantaneous rate of return $r_t$. Agents also receive a flow of non-financial endowment, $e_t$. To simplify the analysis we assume that $e_t$ is proportional to dividend:

$$e^h_t = e^h D_t.$$

It is straightforward to allow $e_t$’s to have their own processes.

Let $\theta^h_t, \phi^h_t$ denote the holdings of stock and state-incontingent bond of the agent. The agent is subject to the following constraint on the dynamics of their financial wealth $W^h_t$:

$$dW^h_t = \theta^h_t q_t \left( \mu^h_t dt + \sigma_t dB^h_t \right) + r_t \phi^h_t dt - c^h_t dt + e^h D_t dt$$

$$W^h_t = q_t \theta^h_t + \phi^h_t,$$

and portfolio constraints

$$\theta^h_t \geq 0$$

$$\phi^h_t + (1 - m) q_t \theta^h_t \geq 0.$$  \hspace{1cm} (33)

Let $s^h_t$ denote the share of investment of agent $h$ in stock,

$$\theta^h_t q_t = s^h_t W^h_t$$

then $1 - s^h_t$ is the share of investment in bonds,

$$\phi^h_t = (1 - s^h_t) W^h_t.$$
Using \( s^h_t \), we rewrite (32) as

\[
dW^h_t = s^h_tW^h_t \left( \mu^h_t dt + \sigma^h_t dB^h_t \right) + r_t(1 - s^h_t)W^h_t dt - c^h_t dt + e^h_t D_t dt.
\]

The portfolio constraint becomes

\[
0 \leq s^h_t \leq \frac{1}{m}.
\]

The definition of the sequential competitive equilibrium for this economy is standard and is a continuous version of the competitive equilibrium in the main paper.

**Definition 5.** A competitive equilibrium consists of sequences of prices \( \{q_t, r_t\}_{t=0}^{\infty} \) and allocations \( \{c^h_t, \theta^h_t, \phi^h_t\}_{h \in H} \) such that (i) \( q_t \) follows the dynamics (29), (ii) the allocation \( \{c^h_t, \theta^h_t, \phi^h_t\} \) maximizes agent \( h \)'s (31) subject to the dynamic net worth constraint (32) and portfolio constraints (33) given \( \{q_t, r_t\} \) and initial asset holdings \( \{\theta^h_0, \phi^h_0\} \); (iii) land, consumption, stock and bond markets clear: \( \sum_{h \in H} \theta^h_t = 1, \sum_{h \in H} \phi^h_t = 0, \sum_{h \in H} c^h_t = (1 + \sum_{h \in H} e^h_t)D_t \) in every instant \( t \).

Let \( \omega^h_t \) denote the financial wealth share of agent \( h \):

\[
\omega^h_t = \frac{W^h_t}{q_t}.
\]

By the stock and bond market clearing conditions, we have

\[
\sum_{h \in H} \omega^h_t = 1,
\]

in any competitive equilibrium. Therefore in order to keep track of the financial wealth distribution between the agents, \( (\omega^h_t)_{h \in H} \) in equilibrium, we only need to keep track of \( H - 1 \) wealth shares: \( \omega_t = (\omega^h_t)_{h=1,\ldots,H-1} \).

From now on, we make the following assumption.

**Assumption 7.** Assume that \( H = 2 \) and there are no non-financial endowments, i.e. \( e^h = 0 \).

Following the main paper, I look for a Markov equilibrium in which the wealth share of one of the two agents, for example, \( \omega_t = \omega^1_t \), is the sufficient state variable. I show below that this is the only state variable needed to characterize the equilibrium in the economy.

Indeed, we conjecture that, the state variable \( \omega_t \) evolves according to:

\[
d\omega_t = \mu_{\omega}(\omega_t) dt + \sigma_{\omega}(\omega_t) dB_t,
\]

(34)
where $\mu_\omega, \sigma_\omega$ are endogenous functions to be determined.

From agent $h$’s perspective:

$$d\omega_t = \mu_h^\omega(\omega_t)dt + \sigma_\omega(\omega_t)dB_t^h,$$

where by (28),

$$\mu_h^\omega(\omega_t) = \mu(\omega_t) + \frac{\mu_D^h - \mu_D}{\sigma_D}\sigma_\omega(\omega_t).$$

Furthermore, all the equilibrium variables depend only on this state variable $\omega_t$. For example, the return and volatility of assets price are $\mu_t = \mu(\omega_t)$ and $\sigma_t = \sigma(\omega_t)$.

As in Rytchkov (2014), the solution to the portfolio’s problem of agent $h$ is determined in the following lemma.

**Lemma 2.** Given the perceived processes for the rate of return on stock, (30) and interest rate, $r_t$, as well as the dynamics of the state $\omega_t$, (34), the value function of agent $h$ is given by

$$J^h(\omega, W, t) = \frac{1}{1 - \sigma_h} W^{1 - \sigma_h} \exp(K^h(\omega)) \exp(-\rho t).$$

(35)

where $K^h(\omega)$ satisfies the second order differential equation:

$$\sigma_h \exp \left( -\frac{K^h(\omega)}{\sigma_h} \right) + (1 - \sigma_h) s^h(\omega) \mu^h(\omega) + (1 - \sigma_h) (1 - s^h(\omega)) r(\omega)$$
$$- \frac{\sigma_h}{2} (\sigma(\omega))^2 \left( s^h(\omega) \right)^2 + \frac{1}{2} (\sigma_\omega(\omega))^2 \left( K^{hh}(\omega) + \left( K^{hh}(\omega) \right)^2 \right)$$
$$+ (1 - \sigma_h) K^{hh}(\omega) s^h(\omega) \sigma(\omega) \sigma_\omega(\omega) - \rho = 0.$$  

(36)

where

$$s^h(\omega) = \max(\min(\frac{1}{m}, s^h_*(\omega)), 0),$$

given that

$$s^h_*(\omega) = \frac{\mu^h(\omega) - r(\omega) + K^{hh}(\omega) \sigma(\omega) \sigma_\omega(\omega)}{\sigma_h \sigma(\omega))^2}.$$

Lastly,

$$c^h(\omega) = W \exp \left( -\frac{K^h(\omega)}{\sigma_h} \right).$$

**Proof.** The Bellman equation for the value function of agent $h$ is:

$$\max_{c,s} \left[ e^{-\rho t} \frac{c^{1 - \sigma_h}}{1 - \sigma_h} + DJ^h \right] = 0,$$

(37)
where

\[ DJ^h = J^h_W(sW\mu^h + (1 - s)Wr - c) + \frac{1}{2}J^h_{WW}W^2\sigma^2s^2 + \frac{1}{2}\sigma^2_{\omega}J^h_{\omega}\omega + J_t. \]

From the conjecture (35) of the value function, the maximization over \( c \) in equation (37) yields

\[ e^{-\rho t}c_h = J^h_W, \]

or

\[ c_h = W\exp\left(-\frac{K^h(\omega)}{\sigma_h}\right). \]

The constrained maximization over \( s \) reduces to

\[ s^h = \arg\max_{0 \leq s \leq 1/m} \left( (J^h_W(\mu^h - r) + J^h_{W\omega}\sigma\sigma_\omega) s + \frac{1}{2}J^h_{WW}W\sigma^2s^2 \right). \]

Consider the unconstrained problem:

\[ s^*_h = \arg\max_s \left( (J^h_W(\mu^h - r) + J^h_{W\omega}\sigma\sigma_\omega) s + \frac{1}{2}J^h_{WW}W\sigma^2s^2 \right). \]

From the functional form for \( f^h \):

\[ s^*_h = \arg\max_s \left( \frac{W^{-\sigma_h} \exp(K^h(\omega))(\mu^h - r) + W^{-\sigma_h} \exp(K^h(\omega))K^{ht}(\omega)\sigma\sigma_\omega) s}{\frac{1}{2}\sigma_h W^{-\sigma_h} \exp(K^h(\omega))\sigma^2s^2} \right) = \frac{\mu^h - r + K^{ht}(\omega)\sigma\sigma_\omega}{\sigma_h\sigma^2}. \]

Therefore

\[ s^h = \max(\min(\frac{1}{m}, s^*_h), 0). \]

Plugging the solution for \( c \) and \( s \) back to the Bellman’s equation 37, we obtain the ODE equation on \( H, (36). \)

Given the solution to the optimization problem of the agents, we can go back to find the equations that characterize the dynamics of the economy. First, we solve for the price-to-dividend ratio as a function of the state of the economy:

\[ f(\omega_t) = \frac{q_t}{D_t}. \]
Using the Ito’s lemma, from the fact that \( q_t = D_t f(\omega_t) \), we obtain

\[
 dq_t = dD_t f(\omega_t) + D_t df(\omega_t) + dD_t f(\omega_t) \\
= D_t (\mu_D dt + \sigma_D dB_t) f(\omega_t) + D_t f'(\omega_t) (\mu_\omega(\omega_t) dt + \sigma_\omega(\omega) dB_t) + \frac{1}{2} D_t f''(\omega_t) (\sigma_\omega(\omega))^2 dt \\
+ f'(\omega_t) D_t \sigma_D \sigma_\omega(\omega_t) dt.
\]

Dividing both sides by \( q_t \):

\[
 \frac{dq_t}{q_t} = \left( \mu_D + \frac{f'(\omega_t)}{f(\omega)} (\mu_\omega(\omega_t) + \sigma_D \sigma_\omega(\omega_t)) + \frac{(\sigma_\omega(\omega))^2}{2} \frac{f''(\omega)}{f(\omega)} \right) dt \\
+ \left( \sigma_D + \frac{f'(\omega_t)}{f(\omega_t)} \right) dB_t.
\]

Therefore

\[
 \mu(\omega) = \mu_D + \frac{f'(\omega)}{f(\omega)} (\mu_\omega(\omega) + \sigma_D \sigma_\omega(\omega)) + \frac{(\sigma_\omega(\omega))^2}{2} \frac{f''(\omega)}{f(\omega)} + \frac{1}{f(\omega)} 
\]

and

\[
 \sigma(\omega) = \sigma_D + \frac{f'(\omega)}{f(\omega)}.
\]

From the definition of \( \omega_t \) and the wealth dynamics of agent 1:

\[
 d\omega_t = d \left( \frac{W_t^1}{q_t} \right) \\
= \frac{dW_t^1}{q_t} - \frac{W_t^1 dq_t}{q_t^2} + \frac{W_t^1 (dq_t)^2}{q_t^3} - \frac{1}{q_t^2} dW_t^1 dq_t \\
= s^1(\omega_t) \omega_t (\mu(\omega_t) dt + \sigma(\omega_t) dB_t) + r(\omega_t) (1 - s^1(\omega_t)) \omega_t dt - \omega_t \exp \left( - \frac{K^1(\omega_t)}{\sigma_1} \right) dt \\
- \omega_t \left( \mu(\omega_t) dt + \sigma(\omega_t) dB_t - \frac{1}{f(\omega_t)} dt \right) + \omega_t (\sigma(\omega_t))^2 dt - s^1(\omega_t) (\sigma(\omega_t))^2 \omega_t dt.
\]

This equation enables us to solve for the functions \( \mu(\omega) \) and \( \sigma(\omega) \) as functions of \( s^1(\omega), r(\omega), \mu(\omega), \sigma(\omega) \)

\[
 \mu(\omega) = s^1(\omega) \omega \mu(\omega) + r(\omega) (1 - s^1(\omega)) \omega - \omega \exp \left( - \frac{K^1(\omega)}{\sigma_1} \right) \\
- \omega \mu \left( (\omega) - \frac{1}{f(\omega)} \right) + \omega (\sigma(\omega))^2 - s^1(\omega) (\sigma(\omega))^2 \omega_t,
\]

\[52\]
and
\[ \sigma_\omega(\omega) = \omega \sigma(\omega)(s^1(\omega) - 1). \] (42)

From the bond or stock market clearing conditions:
\[ \left(1 - s^1(\omega)\right) \omega + \left(1 - s^2(\omega)\right)(1 - \omega) = 0. \] (43)

Lastly, the market clearing condition in the consumption good markets yields
\[ c^1_t + c^2_t = D_t \]
or
\[ \omega \exp \left(- \frac{K^1(\omega)}{\sigma_1}\right) + (1 - \omega) \exp \left(- \frac{K^1(\omega)}{\sigma_1}\right) = \frac{1}{f(\omega)}, \] (44)

We sum up the derivations above in the following proposition.

**Proposition 7.** A equilibrium of the economy can be characterized by 8 unknown functions 
\[ \mu(\cdot), \sigma(\cdot), r(\cdot), f(\cdot), \mu_\omega(\cdot), \sigma_\omega(\cdot), K^1(\cdot), K^2(\cdot) \]
that satisfy the system of 8 functional equations, equation (36) for agent 1 and agent 2, (38), (39), (41), (42), (43), (44). The boundary conditions at \( \omega = 0 \) and \( \omega = 1 \) are determined by the economies in with only one agent, agent 1 (\( \omega = 1 \)) or agent 2 (\( \omega = 0 \)).

If the system of functional equations admits a solution, the solution corresponds to an equilibrium.\(^{50}\) This equilibrium is also a continuous version of Markov equilibrium in the main paper and similar to the equilibrium concepts used in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013). There is a one-to-one mapping from this equilibrium to the equilibrium in Rytchkov (2014), in which consumption share is the sufficient state variable. Indeed,
\[
\frac{c^1_t}{c^2_t} = \frac{W^1_t \exp \left(- \frac{K^1(\omega_t)}{\sigma_1}\right)}{W^2_t \exp \left(- \frac{K^2(\omega_t)}{\sigma_2}\right)} = \frac{\omega_t \exp \left(- \frac{K^1(\omega_t)}{\sigma_1}\right)}{1 - \omega_t \exp \left(- \frac{K^2(\omega_t)}{\sigma_2}\right)},
\]
so we can inverse wealth share \( \omega_t \) as a function of consumption share, assuming that the last expression is monotone in \( \omega_t \). Thus consumption share is also a sufficient state variable.

However, the method presented in this appendix does not apply when non-financial

\(^{50}\)The system of functional equations can be solved numerically as in Rytchkov (2014).
endowments $e^h$ is different from 0. To see why, we use the change of variable $\tilde{c}_i^h = c_i^h - e^h D_t$. Using these variables, the budget constraint (32) transforms back to the one without non-financial wealth

$$dW_i^h = \theta_i^h q_t \left( \mu_i^h dt + \sigma_i dB_i^h \right) + r_t \phi_i^h dt - \tilde{c}_i^h dt.$$ 

However, the objective function of the agent $h$ becomes

$$\mathbb{E}_0^h \left[ \int_0^\infty e^{-\rho t} \left( \tilde{c}_i^h + e^h D_t \right)^{1-\sigma_h} \right],$$

which is no longer homogeneous in $\{\tilde{c}_i^h\}_{t=0}^\infty$. Therefore Lemma 2 on the functional form of the value function no longer applies.

The construction of Markov equilibrium and solution method in the discrete time suggests that it might still be possible to solve for the equilibrium in this economy by looking for an equilibrium mapping from the financial wealth distribution to the allocations, asset prices, and the multipliers on the portfolio constraints.
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Speculation and Financial Wealth Distribution under Belief Heterogeneity: Online Appendix

Abstract

In Appendix E, I apply the model in the main paper to the U.S. economy using the parameters estimated in Heaton and Lucas (1995). I find that 1) the equity premium is higher under binding collateral constraints than under non-binding collateral constraints; 2) due to risk-aversion, the equilibrium portfolio choice of the agents is such that the collateral constraints are not often binding in the stationary distribution; 3) consequently, the unconditional moments of asset prices are not very different from the unconditional moments when there are no collateral constraints. These findings are consistent with Mendoza (2010). In Appendix F, I make changes to the structure of the shocks in Section 4 in order to match the patterns of leverage over the business cycles.
1 Online Appendix E: A Quantitative Assessment

In this Appendix, I apply the numerical solution method to a more seriously calibrated setup used in Heaton and Lucas (1995). In order to do so, I need to modify the economy in Section 3 to allow for the possibility that aggregate endowment grows overtime. As in Heaton and Lucas (1995), I assume that the aggregate endowment \( \bar{e}(s^t) \) evolves according to the process

\[
\frac{\bar{e}(s^{t+1})}{\bar{e}(s^t)} = 1 + g(s^t).
\]

There is only one Lucas tree that pays off the aggregate dividend income at time, \( D_t \)

\[
\delta(s^t) = \frac{D(s^t)}{\bar{e}(s^t)},
\]

the remaining endowment is distributed among the consumers under the form of labor income

\[
\sum_{h \in H} e^h(s^t) = \bar{e}(s^t) - D(s^t).
\]

Consumer \( h \)’s labor income as a fraction of aggregate labor income is given by

\[
\eta^h(s^t) = \frac{e^h(s^t)}{\sum_{h \in H} e^h(s^t)}.
\]

The following Proposition shows that, by working with the normalized variables, \( \hat{c}^h_i = \frac{c^h_i}{\bar{e}_t}, \hat{z}^h_l = \frac{z^h_l}{\bar{e}_t}, \hat{d}_t = \frac{d_t}{\bar{e}_t}, \) and \( \hat{q}_t = \frac{q_t}{\bar{e}_t}, \) we can find and compute collateral constrained equilibria in which these normalized variables depend only on the financial wealth share distribution.

**Proposition.** Suppose that the per-period utility functions are CRRA \( U_h(c) = \frac{c^{1-\sigma_h}}{1-\sigma_h} \), and \( (g, \delta, \eta) \) depend only on the current state of the economy. There exists a collateral constrained equilibrium in this growth economy.

**Proof.** To apply the existence proof and the numerical method used in Subsection 3.3, I
use the following normalized variables
\[
\hat{c}_t = \frac{c_t}{e_t}, \hat{d}_t = \frac{d_t}{e_t}, \hat{q}_t = \frac{q_t}{e_t},
\]
and
\[
\hat{\theta}_t = \frac{\theta_t}{e_t}.
\]
Assuming CRRA for the agents, \( U_h(c) = \frac{c^{1-\gamma_h}}{1-\gamma_h} \), the expected utility can also be re-written using the normalized variables
\[
E_h^t \left[ \sum_{r=0}^{\infty} \beta^r U_h \left( \frac{\hat{c}^h_{t+r}}{e_t} \right) \right] = (\frac{e_t}{\hat{e}_t})^{1-\gamma_h} E_h^t \left[ \sum_{r=0}^{\infty} \beta^r U_h \left( \frac{\hat{c}^h_{t+r}}{e_t} \right) \left( \frac{\hat{e}_t}{e_t} \right)^{1-\gamma_h} \right] = (\frac{e_t}{\hat{e}_t})^{1-\gamma_h} E_h^t \left[ \sum_{r=0}^{\infty} \beta^r \prod_{r'=1}^{r} (1 + g(s^{t+r}))^{1-\gamma_h} U_h \left( \frac{\hat{c}^h_{t+r}}{e_t} \right) \right]
\]
These collateral constraints is
\[
\phi^h_{t+1} + (1-m) \theta^h_{t+1} \min_{s^{t+1}|s^t} \left\{ q \left( s^{t+1} \right) + d \left( s^{t+1} \right) \right\} \geq 0,
\]
where \( m \in [0,1] \) and when \( m = 1 \), no borrowing is possible, the agents are allowed to only trade on the real asset. In this environment, I also use the normalized variables for the choice of debt holding \( \hat{\phi}^h_t = \frac{\phi^h_t}{\hat{e}_t} \).
We rewrite the optimization of the consumer as
\[
\max_{\{c^h_t, \theta^h_{t+1}, \phi^h_{t+1}\}} E_0^h \left[ \sum_{t=0}^{\infty} \beta_t^h U_h \left( \frac{\hat{c}^h_t}{e_t} \right) \prod_{r=1}^{t} (1 + g(s^r))^{1-\gamma_h} \right]
\]
and in each history \( s^t \), she is subject to the budget constraint
\[
\hat{c}^h_t + \hat{q}_t \theta^h_{t+1} + \hat{\phi}^h_{t+1} \leq \hat{e}^h_t + \frac{1}{1 + g(s^t)} \hat{\phi}^h_t + \left( \hat{q}_t + \hat{\phi}^h_t \right) \theta^h_t,
\]
the collateral constraints

\[ \hat{\phi}^{h}_{t+1} + (1 - m) \theta^{h}_{t+1} \min_{s^{t+1}|s^t} \left\{ \left( \hat{q} \left( s^{t+1} \right) + \hat{d} \left( s^{t+1} \right) \right) \left( 1 + g \left( s^{t+1} \right) \right) \right\} \geq 0 \]  

and finally the no short-sale constraint in the real asset, \( \theta^{h}_{t+1} \geq 0 \), as before. We can just use the same analysis in Subsection 3.3 for this economy with the hat variables.

The following subsections show that, under reasonable calibrated parameters of the utility functions and exogenous shock processes, 1) the moments of asset prices such as equity premium behave differently under binding collateral constraints than under non binding collateral constraints; 2) due to risk-aversion, the equilibrium portfolio choice of the agents are such that the collateral constraints are often not binding in the stationary distribution; 3) consequently, the unconditional moments of asset prices are not very different from the unconditional moments when there are no collateral constraints. These findings hold both with and without belief heterogeneity. These findings are also consistent with the ones in Mendoza (2010), in which the author finds that “precautionary saving makes sudden stops low probability events nested within normal cycles, as observed in the data.”

1.1 Homogeneous Beliefs

Table 1 corresponds to Table 1 in Heaton and Lucas (1995). The authors use the annual aggregate labor income and dividend data from NIPA and individual income from PSID to calibrate \( g (.) \), \( \delta (.) \), \( \eta (.) \) and the transition matrix \( \pi \). There are two representative agents in the economy and \( \eta \) corresponds to the endowment share of agent 1. In Panel A, the growth rate of the economy, \( g \), fluctuates between 0.9904 and 1.0470, and the share of tradable income, \( \delta \), stays around 15% of the total endowment. Given the importance of income heterogeneity shown in column 4 of Panel A, I start the analysis with the benchmark in which both agents have the correct estimate of the transition matrix in Panel B.
Despite the common belief, income heterogeneity gives the two agents strong incentives to trade with each other.

I use the equilibrium existence and computation procedure developed in Proposition 6 with the calibrated parameters in Table 1. Notice that, the economy in Heaton and Lucas (1995) is the same as in my economy except for the fact that the collateral constraint (29) is replaced by the exogenous borrowing constraint $\phi^h_t \geq -B^h$. Table 2 shows that collateral constraints do not alter significantly the quantitative result in Heaton and Lucas (1995). Despite the collateral constraints, the standard deviations of consumption, stock returns and bond returns are very similar in the economies with collateral constraints to the economy in Heaton and Lucas (1995). The reason for this similarity is that the collateral constraints are often not binding, even though the behavior of the equilibrium variables are very different when the collateral constraints are binding from when they are not. Indeed, Figures 2 and 1 show the equity premium and portfolio choice of agent 1 as functions of her financial wealth share, $\omega^1_t$. When the collateral constraint for this agent is binding, i.e. $\omega^1_t < 0.1$, the equity premium is significantly higher than it is when the constraint is not binding. However, the stationary distribution of wealth share shows that the constraints are binding in less than 5% of the distribution (See column 5, the second to last row of Table 3). In addition, the collateral value as a fraction of the price of the real asset is less than 0.2% (See column 5, the last row of Table 3).

1.2 Heterogeneous Beliefs

Now, I introduce belief heterogeneity by assuming that the first agent in Heaton and Lucas (1995) is more optimistic about the growth rate of the economy, while the second

---

1The last column in Table 2, $m = 1$, corresponds to the economy in which the agents can only trade in the real asset, subject to the no short selling. Thus there are no bonds traded.
A. Markov chain model for exogenous state variables

State Number | States | \( g \) | \( \delta \) | \( \eta \)
--- | --- | --- | --- | ---
1 | 0.9904 | 0.1402 | 0.3772
2 | 1.0470 | 0.1437 | 0.3772
3 | 0.9904 | 0.1561 | 0.3772
4 | 1.0470 | 0.1599 | 0.3772
5 | 0.9904 | 0.1402 | 0.6228
6 | 1.0470 | 0.1437 | 0.6228
7 | 0.9904 | 0.1561 | 0.6228
8 | 1.0470 | 0.1599 | 0.6228

B. Transition probability matrix \( [\pi_{ij}]_{8 \times 8} \)

\[
\begin{array}{cccccccc}
0.3932 & 0.2245 & 0.0793 & 0.0453 & 0.1365 & 0.0799 & 0.0275 & 0.0157 \\
0.3044 & 0.3470 & 0.0425 & 0.0484 & 0.1057 & 0.1205 & 0.0147 & 0.0168 \\
0.0484 & 0.0425 & 0.3470 & 0.3044 & 0.0168 & 0.0147 & 0.1205 & 0.1057 \\
0.0453 & 0.0793 & 0.2245 & 0.3932 & 0.0157 & 0.0275 & 0.0799 & 0.1365 \\
0.1365 & 0.0779 & 0.0275 & 0.0157 & 0.3932 & 0.2245 & 0.0793 & 0.0453 \\
0.1057 & 0.1205 & 0.0147 & 0.0168 & 0.3044 & 0.3470 & 0.0425 & 0.0484 \\
0.0168 & 0.0147 & 0.1205 & 0.1057 & 0.0484 & 0.0425 & 0.3470 & 0.3044 \\
0.0157 & 0.0275 & 0.0779 & 0.1365 & 0.0453 & 0.0793 & 0.2245 & 0.3932 \\
\end{array}
\]

Table 1: Summary Statistics for Baseline Case

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>CM</th>
<th>HL</th>
<th>( m = 0 )</th>
<th>( m = 0.9 )</th>
<th>( m = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Consumption Growth</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>0.020</td>
<td>0.018</td>
<td>0.018</td>
<td>0.019</td>
<td>0.019</td>
<td>0.019</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.030</td>
<td>0.028</td>
<td>0.044</td>
<td>0.041</td>
<td>0.045</td>
<td>0.045</td>
</tr>
<tr>
<td><strong>Bond return</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>0.008</td>
<td>0.080</td>
<td>0.077</td>
<td>0.078</td>
<td>0.077</td>
<td></td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.026</td>
<td>0.009</td>
<td>0.012</td>
<td>0.012</td>
<td>0.013</td>
<td></td>
</tr>
<tr>
<td><strong>Stock return</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>0.089</td>
<td>0.082</td>
<td>0.079</td>
<td>0.079</td>
<td>0.080</td>
<td>0.080</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.173</td>
<td>0.029</td>
<td>0.032</td>
<td>0.030</td>
<td>0.035</td>
<td>0.036</td>
</tr>
</tbody>
</table>

Table 2: Summary Statistics for Baseline Case
Figure 1: Portfolio Choice
Figure 2: Equity Premium
agent have the correct estimate of the transition matrix: \( \pi^2 = \pi \) and

\[
\begin{align*}
\pi^1(1,1) & = \pi(1,1) - p \\
\pi^1(1,2) & = \pi(1,2) + p \\
\pi^1(3,3) & = \pi(3,3) - p \\
\pi^1(3,4) & = \pi(3,4) + p \\
\pi^1(5,5) & = \pi(5,5) - p \\
\pi^1(5,6) & = \pi(5,6) + p \\
\pi^1(7,7) & = \pi(7,7) - p \\
\pi^1(7,8) & = \pi(7,8) + p,
\end{align*}
\]

where \( p = 0.3 \).

Table 3 shows that the standard deviation of stock returns are still similar to when beliefs are homogeneous. The reason is again that the collateral constraints are often not binding due to risk-aversion. Figures 4 and 3 plot the equity premium and portfolio choice of agent 1 as functions of agent 1’s financial wealth share. Equity premium is very high when the collateral constraint is binding. The last two rows of Table 3 show the small probabilities of binding collateral constraints and the small collateral value as a fraction of asset price under either agent 1’s belief or agent 2’s belief.

2 Online Appendix F: Dynamic Leverage Cycles

This appendix uses this definition of leverage in Remark 3 to investigate how leverage varies over the business cycles. Subsections 4.2 through 4.4 capture some realistic behavior of asset price including the debt deflation channel, leverage defined in Appendix C is not consistent with what we observe in financial markets: high leverage in good times and low leverage in bad times, as documented in Geanakoplos (2010). In order to gen-
Figure 3: Portfolio Choice under Belief Heterogeneity
Figure 4: Equity Premium under Belief Heterogeneity
Moments & Data | CM | HL | $m = 0$ | $m = 0, \pi^1$ | $m = 0, \pi^2$
--- | --- | --- | --- | --- | ---
Consumption Growth & Average | 0.020 | 0.018 | 0.018 | 0.019 | 0.027 | 0.020
& Standard deviation | 0.030 | 0.028 | 0.044 | 0.041 | 0.048 | 0.057
Bond return & Average | 0.008 | 0.080 | 0.077 | 0.078 | 0.084 | 0.080
& Standard deviation | 0.026 | 0.009 | 0.012 | 0.012 | 0.011 | 0.018
Stock return & Average | 0.089 | 0.082 | 0.079 | 0.079 | 0.086 | 0.076
& Standard deviation | 0.173 | 0.029 | 0.032 | 0.030 | 0.029 | 0.033
Collateral constraint & Pr(binding constraint) | | | | 0.042 | 0.085 | 0.001
& Collateral value | | | | 0.002 | 0.006 | 0.005

Table 3: Summary Statistics with Belief Heterogeneity

To formalize this idea, I assume that in the good state, $s = G$, next period’s dividend has low variance. However, when a bad shock hits the economy - $s = GB$ or $BB$ - the variance of next period dividend increases. In this dynamic setting, the formulation translates to a dividend process that depends not only on the current exogenous shock but also on the last period exogenous shock: if the last period shock is good, dividend is 1 for current good shock and 0.8 for current bad shock; if the last period shock is bad, dividend is 1 for current good shock and 0.2 for current bad shock. Therefore, in the Markov chain, we need to use three exogenous states instead of the two exogenous states in the Section 4:

$$s \in \{G, GB, BB\}.$$  

Figure 5, left panel, shows that in the good state (G), the variance of next period dividend is low: $d = 1$ or 0.8. However in the bad states (GB or BB), the variance of the next period dividend is higher: $d = 1$ or 0.2. The right panel of the figure shows the evolution through
time of the exogenous states using Markov chain representation. Even though we have three exogenous states in this set-up, each state has only two immediate successors. So we can still use Proposition 6 to show that, in any history, there is only one leverage level in the economy. Moreover, in order to generate realistic levels of leverage observed in financial markets, i.e., around 20, we need to set \( m = 0.05 \) in the collateral constraint (4):

\[
\phi_{t+1}^h + 0.95\theta_{t+1}^h \min_{s^{t+1} | s^t} \left( q \left( s^{t+1} \right) + d \left( s_{t+1} \right) \right) \geq 0.
\]

The uncertainty structure generates high leverage in the good state, \( s_t = G \) and low leverage in the bad states \( s_t = GB \) or \( BB \). Figure 6 shows this pattern of leverage. The bold green line represents the leverage levels in the good state \( s = G \) as a function of the financial wealth share of the optimists. The two lines, blue and dashed-dotted red, represent the leverage levels in bad states \( s = GB \) or \( BB \) respectively.

In addition to the fact that increased uncertainty significantly decreases leverage emphasized in Geanakoplos (2010), we also learn from Figure 6 that financial wealth distribution is another determinant of leverage. Figure 6 shows that leverage decreases dramatically from good states to bad states. However, in contrast to the static version in Geanakoplos (2010), we can quantify the relative contributions of the changes in wealth distribution and the changes in uncertainty to the changes in leverage over the business
cycles. Figure 6 shows that changes in the wealth distribution contribute relatively little to the changes in leverage at higher levels of $\omega_t^O$. At lower levels of $\omega_t^O$, this version of dynamic leverage cycles generates a pattern of leverage build-up in good times. As shown in Figure 7, good shocks increase leverage as they increase the wealth of the optimists relative to the wealth of the pessimists and leverage is increasing in the wealth of the optimists when $\omega_t^O > 0.1$.

Notice, however, that even though leverage decreases significantly from 17 to 12 when a bad shock hits the economy, the leverage level is still too high compared to what was
Figure 7: Wealth Dynamics when $s_t = G$
observed during the last financial crisis. Gorton and Metrick (2012) show that leverage in some classes of assets actively declined to almost 1. Admittedly, there are some other channels, such as counter-party risk, absent from this paper that might have caused the rapid decline in leverage. This rapid change in leverage is another important question for future research.
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