The Design of Optimal Collateralized Contracts*

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Abstract

This paper presents a two-period optimal contracting model of collateral. A borrower values a capital good and a composite non-capital good. He privately observes an income shock in the composite good in the second period. Collateralization of both goods occurs in the optimal contract, whereas it does not under full information.

Relative to full information, the capital good in the optimal contract is over-consumed in the initial loan period and under-consumed in the repayment period. The relation between forfeiture of assets and contractual distortion is summarized by a formula showing higher distortions associated with larger increases in forfeited collateral. Forfeiture is decreasing in income at the tails of the income distribution, and low income types forfeit more than high income types.

We obtain a closed form solution in a parameterized model. Forfeited collateral is globally decreasing in income with pooling at the bottom when the borrower’s initial assets are low.

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1 Introduction

Collateralized contracts arise in many types of credit transactions. These include durable goods purchases, consumption loans, and repurchase ("repo") agreements. Typically, such contracts specify a tangible pledge of property to be forfeited in the event the borrower cannot or will not repay the loan.

Collateral plays a central role in the large literature on credit markets. Collateral is a key ingredient, for instance, in explanations of credit rationing (Stiglitz and Weiss (1981), Bester (1985)), debt contracts (Lacker (2001)), bank screening (Besanko and Thakor (1987)), and capital structure (Rampini and Viswanathan (2013)).

Why is collateral used? Rationales vary. Some argue that collateral requirements can prevent borrowers who are otherwise able to repay from strategically defaulting and/or misusing the loaned funds.\footnote{Explanations of this type have a long tradition, dating back at least to Barro (1976).} Others have focused on the role of incomplete information, arguing that a borrower’s assets will be collateralized when they are correlated with his ability to repay. In that case, collateral can be pledged by the borrower as a credible signal of credit-worthiness, or can be used by lenders to screen out high risk borrowers.\footnote{Chan and Kanatas (1985), Bester (1985), and Besanko and Thakor (1987), are early examples of this second rationale.}

In most of the literature, endogenous collateral requires the existence of a critical friction such as limited liability, limited commitment or enforcement, or a restriction on the set of contractual instruments that are available.\footnote{See Section 2 for a more detailed review of this literature.} This paper proposes a mechanism design model of endogenous collateral that drops all such constraints. We separate out capital structure and credit market concerns.

Instead, we focus solely on the role of private information to better understand why collateral is used in some environments, and not in others, and what form a collateralized contract should take. We characterize an \textit{optimal contract} as a solution to a mechanism design problem subject only to information and resource constraints. We then show that under general assumptions such a contract will entail pledges and forfeiture of collateral in certain states of nature.

We posit a familiar contracting scenario with two agents, a lender and a borrower. The lender is risk neutral, the borrower risk averse. There are two periods. In each period, the
borrower values composite consumption and an additional capital good that generates use value for the borrower. A leading example is a household’s consumption of non-housing and housing services.

The borrower is endowed with little or nothing in the first period. In the second period he receives a random income/return in composite consumption units. The capital good is non-stochastic. The realization of composite income is determined from a continuous distribution; low realizations correspond to a negative shocks such as job loss or poor returns on investments. Consequently, the borrower seeks a loan that allows him to smooth across time and across his uncertain income stream.

We assume that the borrower privately observes his realized composite income in the second period. A contract specifies consumption of the two goods in period 1 and contingent consumption of the goods in period 2 given the reported income of the borrower. An optimal contract maximizes the borrower’s expected two-period payoff subject to a standard incentive constraint (IC) and a 0-profit constraint of the lender.

The presence of private information is shown to play a critical role in determining the contractual collateral, i.e., the amount of the borrower’s net worth that can be contractually seized by the lender. Consider first what would happen under full information. Without the IC constraint, an optimal contract entails a transfer of the composite good that offsets the income shock. This results in constant consumption for the borrower in both the composite and the capital good, independently of the state. In other words, the lender fully insures the borrower in the stochastic composite good, and the consumption of the capital good does not vary with the income shock. The capital good is not used in the contract in any meaningful way. It can be shown in this case, that the contracted collateral is zero. In other words, the full information optimal contract is not collateralized.

We characterize the optimal contract under private information. Most fundamentally, we show that unlike in the full information model, all optimal contracts here are collateralized, and the capital good is instrumental in making this so.

A preliminary result shows that all optimal contracts exhibit a distortion in both the intra-temporal marginal rate of substitution between capital and composite consumption and the inter-temporal rate of substitution between first and second period consumption. The intra-temporal distortion, in particular, is a familiar finding in the optimal taxation literature (Mirrlees (1971) and Mirrlees (1976)).
Putting both types of distortions together, we show that when capital and non-capital goods are complements, the borrower will generally over-consume or under-accumulate assets in the initial period anticipating distortions in the repayment period. In the repayment period all income types of the borrower except the highest, and possibly the lowest income type under-consumes the capital good and over-consumes the composite good. In this sense, the composition of seized collateral is excessively tilted toward the capital good.

We also establish some distributional properties of the contract. Optimal contracts in the repayment period are shown to be strictly separating at the top of the income distribution, but may admit pooling and full forfeiture of collateral at the bottom. This implies that, de facto, the optimal contract will not resemble a standard debt contract.

These and other properties are summarized by a simple equation that relates collateralized forfeiture to the distortionary wedge mentioned above. Given an extra dollar of income, higher distortions are shown to be associated with larger increases, or smaller decreases, in forfeiture of collateral. The formula can be used to show that forfeiture is locally regressive — it decreases locally in income at both tails of the income distribution — a form of “anti-insurance.” Furthermore, in any separating contract the collateral forfeited by the poorest agents exceeds that of the richest.

Along these lines, we give a precise characterization of the optimal contract in a parameterized model with log preferences and a uniform distribution. There, the contract can be solved in closed form. When the type distribution is concentrated and/or the borrower’s assets are large enough, then the contract is fully separating. However, when the type distribution is sufficiently diffuse and/or the borrower’s initial assets are low enough, then pooling occurs at the bottom of the income distribution. In either case, forfeiture is shown to be globally regressive — the collateral forfeited is strictly decreasing in income across the entire distribution — and consumption of the capital good is convex in realized income. Combining this with the inter-temporal distortion, this means that, compared with the full information optimum, the optimal contract is especially harsh on low income borrowers as more of their savings ends up being forfeited.

Finally, we describe a method for computing optimal contracts and show that they can be easily displayed in a phase diagram.

The next section, Section 2, gives a brief review of the literature. Section 3 introduces

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4 The no-distortion at the top property is also standard in Mirleesian models.
the baseline model. We introduce the planner’s problem and define what it means for the resulting optimal contract (OC) to be “collateralized.” Section 4 describes the main results. Section 5 presents a closed form solution for the OC in a parametric model. Section 6 describes methods for computing OCs numerically and tackles robustness issues. Section 7 embeds are framework in a GE model. The proofs are contained in an Appendix.

2 The Literature

The present paper builds upon two distinct, though overlapping, literatures. The first is the large literature on Mirleesian contracts with private information. Our model is in many ways a throwback to first-generation models in that literature, dating back to Mirrlees (1971, 1976) himself. Following in the tradition of these models, we specify general preferences for the borrower that are non-separable in the two goods. Our preliminary results characterize intra-temporal distortions similar to those in the Mirleesian contracting literature, and are mostly closely related to Hellwig (2007).

Because our focus is on credit rather than on taxation and labor effort, the random shocks here are additive income shocks rather than common multiplicative productivity shocks in the optimal taxation literature. This seemingly minor restriction leads to a sharp characterization of the optimal contract as it relates to collateral.

A second, related literature concerns the specific role of collateral in credit contracts. These studies characterize contracts under various constraints in order to produce instruments that intuitively resemble collateral. These constraints may be motivated by legal restrictions such as limited liability or limited commitment as with Stiglitz and Weiss (1981), Wette (1983), Lacker (2001), Rampini and Viswanathan (2013). In other cases, the models restrict attention to basic debt instruments in order to focus on particular attributes of the collateralized contract. Examples include Chan and Kanatas (1985), Bester (1985), Besanko and Thakor (1987), and Rampini (2005) who derive discontinuous threshold requirements in debt contracts, Manove et al. (2001) who examine banks’ trade off

\footnote{For tractability, later generations of models often assume separability and even quasi-linearity in preferences. Non-separability is an important generalization for certain types of collateral since durables such as housing and autos are complements to nondurable consumption.}

\footnote{Our assumptions differ somewhat from his and to the best of our knowledge, our results not nested within his or other models.}

\footnote{The literature is vast, though Diamond (1998) gives canonical treatment used in later models.}

\footnote{See references contained therein.}
between screening or collateralizing loans to risky projects, Eisfeldt and Rampini (2009) who evaluate the borrower’s choice of whether to lease or purchase the collateralized asset, and Campbell and Cocco (2015) who study strategic default by borrowers. Similarly, there is a growing literature on the equilibrium implications of collateralized debt contracts including Fostel and Geanakoplos (2008), Geanakoplos (2010), Cao (2011), and Simsek (2013).

Among these, the closest models to our own are Rampini (2005) and Lacker (2001). Both consider a private information contracting model with an additive shock structure. Rampini (2005) incorporates additive shocks to analyze the role of collateral in aggregate default. Incorporating non-monetary default penalties that do not affect lenders’ profits, he derives a cutoff rule in which the default rate can be discontinuous in income. Lacker (2001) examines bilateral credit contracts with private information and a limited commitment assumption. The limited commitment introduces the potential for renegotiation. This bounds both the size of the initial loan and the amount of the capital good that can be credibly collateralized and consumed in his model. As a result, the optimal contract pools types at the upper, rather than lower, end of the distribution.

These constraints no doubt play a significant role in collateralized contracts. Our goal is to see how much mileage can be obtained by private information alone. Thus we allow for full commitment, no constraints on the loan size or collateral, and are still able to derive contracts with collateral. The substantive differences in these contracts highlight the role of information as apart from other constraints in determining the role of collateral.

From this point of view, the collateralized credit contracts in these models would be sub-optimal in the absence of the additional constraints. To the extent that they examine the aggregate and equilibrium implications of the contracts, our results suggest that these implications might change if the optimal contracts are used in equilibrium instead. For example, we show that in general equilibrium the optimal contracts might lead to lower price of collateral relative to the first best level. While the recent papers using ad-hoc collateralized debt contracts such as Geanakoplos (2010), Cao (2011), and Fostel and Geanakoplos (2012) show that the price of collateral is higher than the first-best level.

A purely private information-based notion of liquid collateral appears in an influential paper by Ed Green (1987). Green formulated a dynamic contracting model in which the borrower (agent) possesses private information each period about an income shock

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9This is also true of many successors that his paper brought forth.
to a single composite good. In Section 6 we show that Green’s optimal credit contract has collateral-like features. The consumption trade-offs between periods play a role similar to our trade-offs between composite and capital consumption. In either case, collateralization requires bootstrapping the contract on an “extra” good not subject to shocks in that same period. Our collateralized contract will have different features than Green’s credit contract, owing to the distinct attributes of capital good consumption and to the complementarities between the capital and non-capital good consumption.

3 A Baseline Model

3.1 Overview

We lay out a simple two-period contracting problem with two agents, a lender and a borrower. In the first period, the borrower needs external funding. Depending on interpretation, external funding may be required for a variety of reasons. Businesses have insufficient internal resources to fund an expansion. Entrepreneurs lack funds for investment opportunities. Households lack accumulated wealth to fund durable goods purchases.

The simplest specification of payoff that accommodates the framework is of the form $U_0(c_0, k_0) + \beta U(c, k)$ for the borrower. Composite consumption good $c_0$ is consumed at $t = 0$ and $c$ is consumed $t = 1$. Goods $k_0$ and $k$ are service-generating capital goods that will, eventually, play a critical part in the collateralization of a loan. The capital good is durable. For simplicity we rule out depreciation as it plays no role in the analysis.

The flow payoff functions $U_0, U : \mathbb{R}^2 \to \mathbb{R}$ are assumed strictly concave and strictly increasing in each good. When the borrower is a household, these are natural assumptions. We assume further that (i) $U$ and $U_0$ are twice continuously differentiable, and (ii) $U$ satisfies a weak complementarity condition given by

$$\frac{\partial^2 U}{\partial c \partial k} > \max \left\{ \frac{\partial U}{\partial c} \frac{\partial U}{\partial k}, \frac{\partial U}{\partial k} \frac{\partial U}{\partial c} \right\},$$

for all $c, k$. Obviously, $U$ need not be separable. Because the right-hand side is negative, the inequality in (1) is clearly satisfied when $c$ and $k$ are complements, i.e. $\frac{\partial^2 U}{\partial c \partial k} \geq 0$. This is quite natural when $k$ is housing services and $c$ is non-housing consumption. The inequality, however, also allows for $c$ and $k$ to be substitutes, provided that the interaction
We emphasize that there are no ad hoc bounds on either $c$ or $k$. Lower bounds may be implied for certain payoff functions, for instance when $U$ is the log function.

The borrower starts the initial period with composite wealth $A$. When $A$ is low enough the borrower needs a loan to smooth consumption across the two dates. At the beginning of the second period, the borrower realizes a random income/return $\theta$ of the composite good, distributed according to $F$ on support $[\theta, \bar{\theta}]$. The distribution $F$ admits a continuous density $f$. Though $F$ is common knowledge, the realized value $\theta$ is privately observed only by the borrower. (An alternative interpretation is that a lender in a competitive credit market draws a borrower from population distribution $F$.)

Goods $c_0$, $k_0$, and $k$ are non-stochastic. The spot price of the composite good is normalized to 1 each period, and the prices of the capital goods are $q_0$ and $q$, resp. For its part, the lender is risk neutral and belongs to a perfectly competitive set of intermediaries, all of whom offer loans with a market return of $R$.

### 3.2 A Contracting Problem

A loan contract or simply a contract between the lender and borrower consists of a list $(y_0, k_0, y, k)$ such that $y_0$ and $k_0$ units of composite and capital goods, resp., are offered to the borrower in the initial period, resulting in consumption $c_0 = y_0$ and $k_0$ for the borrower. In the last (repayment) period, consumption is contingent on realized income, and so $y$ and $k$ are functions mapping types $\theta$ to units $y(\theta)$ and $k(\theta)$ of the composite and capital good, resp., transferred from the lender. Here, $y(\theta) < 0$ constitutes a repayment to the lender. The borrower’s second period consumption is $c(\theta) = \theta + y(\theta)$ and $k(\theta)$.

Taking the perspective of the borrower ex ante, we characterize the optimal contract taking perfect competition in the lending sector as given.

**Definition.** An optimal contract (OC) is a list $(y_0, k_0, y, k)$ that solves:

$$V(A, q_0, q, R) = \max_{y_0, k_0, y(\cdot), k(\cdot)} U_0(y_0, k_0) + \beta \int_{\theta} U(\theta + y(\theta), k(\theta)) \, dF(\theta)$$

$^{10}$In Lemma 4 in Appendix A, we show that condition (1) implies that both goods, $c$ and $k$, are strictly normal goods.
subject to
\[ y_0 + q_0 k_0 + \frac{1}{R} \int_{\theta}^{\bar{\theta}} (y(\theta) + qk(\theta)) dF(\theta) \leq \frac{1}{R} qk_0 + A \]  \tag{3}

and
\[ U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\hat{\theta}), k(\hat{\theta})) \quad \forall \theta, \hat{\theta}. \]  \tag{4}

We refer to (2) as the optimal contracting or OC problem. The resource constraint is displayed in (3), the incentive constraint in (4). The problem above resembles a standard Bayesian mechanism design problem in which the resource constraint replaces an interim participation constraint of the borrower. In the resource constraint, \( A \) is the borrower’s initial wealth, \( R > 1 \) is the gross return on capital, and \( q_0 \) and \( q \) are the prices of first and second period capital goods, respectively.\(^{11}\) A no-arbitrage condition requires that \( q_0 > \frac{q}{R} \). The constraint in Equation (3) may be interpreted as a zero profit or participation constraint for a competitive lender.

If \( A \) is large enough, it is possible that the borrower does not borrow at all. The interesting case is when \( A \) is sufficiently small so that in the optimal contract the borrower obtains a loan for first period consumption in the amount \( y_0 + k_0(q_0 - \frac{q}{R}) - A > 0 \). The resource constraint then guarantees that the first period loan is fully repaid in expectation.

The incentive constraint in (4) involves only the continuation contract \((y, k)\) received in period 2. The following characterization of IC is standard.

**Lemma 1.** A continuation contract \((y, k)\) satisfies the incentive constraint if and only if for \( \theta > \theta' \), we have

\[ y(\theta) \leq y(\theta') \quad \text{and} \quad k(\theta) \geq k(\theta'), \]  \tag{5}

\[ \frac{dU^-}{d\theta}(\theta + y(\theta), k(\theta)) \geq \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)), \quad \text{and} \]  \tag{6}

\[ \frac{dU^+}{d\theta}(\theta + y(\theta), k(\theta)) \leq \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) \]  \tag{7}

where \( \frac{dU^-}{d\theta} \) and \( \frac{dU^+}{d\theta} \) are the left and right one-sided derivatives respectively.

**Proof.** Though standard, we provide a proof of Lemma 1 in Appendix B. \( \square \)

The case where the one-sided derivatives in (6) and (7) coincide constitutes the fa-\(^{11}\)Depreciation is implicitly captured in \( q \).
familiar envelope condition. An *incentive compatible* continuation contract is any pair \((y, k)\) satisfying the monotonicity and envelope conditions of Lemma 1. Thus, (5)-(7) can replace the IC constraint in the planner’s problem. Building on a result by Hellwig (2007), one can take this one step further:

**Lemma 2.** Consider any piecewise continuously differentiable contract \((y_0, k_0, y, k)\) that solves an optimal contracting problem in which the IC constraint (4) is replaced only by (5) and (6) of Lemma 1. Then this contract is an optimal contract.

The proof is in Appendix D. We note that, by an application of the Karush-Kuhn-Tucker Theorem, the piecewise continuously differentiable OCs yield higher value than any other incentive compatible contracts, including *discontinuous contracts*. We discuss this point further in Subsection 3.4. Applying the Lemma, the upward incentive constraint given by (7) is dropped; we concern ourselves with only downward incentive constraints that deter any type \(\theta\) from mimicking lower types.

### 3.3 Collateral: A Definition

In order to ascertain when a contract involves “collateral” and how much is required, one can calculate what the borrower will actually forfeit in the contract. To do this, we first identify the borrower’s expected net worth heading into the second period. This is given by \(E[\theta] + qk_0 + B\) where \(qk_0\) is the value of the capital asset and \(B \equiv R(A - q_0k_0 - y_0)\) the value of debt (if \(B < 0\)) or savings (if \(B > 0\)) carried into the second period. The inclusion of expected future income \(E[\theta]\) in his net worth reflects the fact that the borrower can access a functional credit market.\(^{12}\) Next, we identify the type-contingent value of his realized consumption from the contract in that period: \(qk(\theta) + \theta + y(\theta)\). Hence, the value of collateral seized by the lender is

\[
\Gamma(\theta) \equiv \underbrace{E[\theta] + B + qk_0}_{\text{expected net worth}} - \underbrace{(qk(\theta) + \theta + y(\theta))}_{\text{consumption expenditure by type } \theta} \tag{8}
\]

In other words, the seized or forfeited collateral is the difference between the borrower’s expected net worth and the value of his total consumption in the last period. By this definition, forfeiture can be positive or negative.\(^{13}\) When seizure is smaller than the

\(^{12}\)In particular, the credit market must allow the parties to contract on future income.

\(^{13}\)This must be so since \(\int\Gamma(\theta)dF = 0\) when the resource constraint is binding.
pledged amount, the difference is, in effect, returned to the borrower. This is the case in mortgage contracts when, for instance, the value of the house exceeds the outstanding debt on the mortgage, or when one’s realized income is higher than expected. The value of the collateral pledged by the borrower is therefore that maximal amount that the lender can seize in the last period:

\[ \Gamma \equiv \max_{\theta \in [\theta, \bar{\theta}]} \Gamma(\theta) \]  

Definition. A contract is collateralized if \( \Gamma > 0 \).

No other constraints are added to derive a collateralized solution. Both participants can fully commit to the contract ex ante, and there is no renegotiation. There are no liability or enforcement restrictions. There are no borrowing or liquidity constraints, and no moral hazard.\(^{14}\) Asymmetric information alone is the key.

We emphasize that forfeited collateral is not the same as loan repayment. Loan repayment in this model corresponds to \( B \) (debt obligation entering the second period). Indeed, under full information and functional credit markets, both repayment and insurance are present in the model while collateral is not. To see why, consider the following three benchmarks.

1. The full information optimal contract. In the absence of the IC constraint, the solution to (2) corresponds to a full information optimal contract \( (y_{FI}^{F}, k_{FI}^{F}, y_{FI}, k_{FI}) \). One can easily verify that under full information, the contract provides provides full insurance. That is, \( y_{FI}^{F}(\theta) = \bar{c} - \theta \) and \( k_{FI}^{F}(\theta) = \bar{k} \) for some constants \( \bar{c} \) and \( \bar{k} \). Thus the contract fully insures the borrower in the composite good, and the borrower incurs no risk in his consumption of the capital good. The usual optimality conditions equating marginal rates of substitutions to relative prices are satisfied:

\[
\frac{\partial U}{\partial k}\bigg|_{\theta} = q, \quad \frac{\partial U_{0}}{\partial k_{0}} = q_{0} - \frac{q}{R}, \quad \frac{\partial U_{0}}{\partial c_{0}} = \left( \frac{Rq_{0} - q}{q} \right) \frac{\partial U}{\partial k}\bigg|_{\theta}, \quad \text{etc.}
\]

More importantly, the full information contract is not collateralized. To see why, notice that the capital good is not needed to achieve full insurance. The contractual allocation of \( y \) suffices to fully insure the borrower. Since \( k_{FI}^{F}(\theta) = \bar{k} \) is constant and \( y_{FI}^{F}(\theta) = \bar{c} - \theta \), it follows that \( \Gamma_{FI}^{F}(\theta) = \bar{k} \) is constant: \( \Gamma_{FI}^{F}(\theta) = \bar{k} \). Because the resource constraint will be satisfied with equality, we obtain \( \Gamma_{FI}^{F} = 0 \). In other words, value of seized collateral for

\(^{14}\)Without moral hazard or market incompleteness, expected future income can be used as collateral.
every type $\theta$ is zero. No collateral is pledged. None is seized.

2. **The self-funded allocation.** In the absence of any contracting option, the borrower faces autarky. He cannot borrow or save across periods, and his second period consumption satisfies the type-by-type budget constraint $q{k}_{aut}(\theta) + c{aut}(\theta) = q{k}_{0}^{aut} + \theta$. The type-by-type optimality condition also holds under autarky, but without the benefit of complete insurance. While each type’s chosen mix of consumption between the two goods is optimal, consumption differences across types exposes the borrower to ex ante risk.

The self-funded contract is not collateralized by definition. Without access to the credit market, the relevant constraint for the borrower is his realized (as opposed to expected) net worth in the second period. That is, $\theta + q{k}_{0}^{aut}$. Hence, the collateral “forfeited” is $\theta + q{k}_{0}^{aut} - q{k}_{aut}(\theta) - \theta - y{aut}(\theta) = 0$ for all $\theta$.

3. **Non-contractable income.** In this case, the borrower can access the credit market to borrow in the first period but cannot contract on future income. In this case, neither $k$ nor $c$ can condition on realized $\theta$ in which case forfeiture under non-contractable income is $\Gamma^{NC}(\theta) = \tilde{\Gamma}^{NC} = B + q{k}_{0} - q\bar{k} - \bar{y}$ where $\bar{k}$ and $\bar{y}$ are constants. Once again, the binding resource constraint implies $\Gamma^{NC}(\theta) = 0$ for all $\theta$. Again the contract is not collateralized.

The three benchmarks each lay out different scenarios. All have the property that the optimal contract in these cases is not collateralized.

3.4 **The Saddle Point Problem**

To show that under private information, optimal contracts will be collateralized, we require a tractable formulation of the optimal contract (OC) as a solution to a Lagrangian saddle problem.

Let $\lambda \geq 0$ be the multiplier associated with resource constraint (3), and let $\xi(\theta) \in \mathbb{R}$ be the multiplier associated with envelope constraint in (6). As for the monotonicity constraint, one can find a function $j(\theta)$ satisfying

$$k(\theta) = k(\theta) + \int_{\theta}^{\bar{\theta}} j(\theta) d\theta. \quad (10)$$
The monotonicity requirement for \( k(\theta) \) then implies

\[
j(\theta) \geq 0. \tag{11}
\]

Consequently, we let \( \eta(\theta) \) denote the multiplier on the equality constraint in (10), and \( \gamma(\theta) \) the multiplier on the non-negativity constraint in (11).

Because the envelope constraint associated with multiplier \( \eta(\theta) \) is an equality constraint, \( \eta(\theta) \) can be either positive or negative. On the other hand, the constraints associated with \( \lambda, \xi(\theta) \) and \( \gamma(\theta) \) are inequality constraints by Lemma 2, and so \( \lambda \geq 0, \xi(\theta) \geq 0, \) and \( \gamma(\theta) \geq 0 \) are required for all income types \( \theta \). With the multipliers, the Lagrangian function can be stated as,

\[
L(y_0, k_0, y, k, j, k(\theta), \xi, \eta, \gamma, \lambda) \equiv U_0(y_0, k_0) + \\
\beta \left\{ \int_\theta^\theta (U(\theta + y(\theta), k(\theta)) + \lambda (B + qk_0 - y(\theta) - qk(\theta))) f(\theta) \, d\theta \\
+ \int_\theta^\theta \xi(\theta) \left( \frac{\partial U}{\partial \theta}(\theta + y(\theta), k(\theta)) - \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) \right) d\theta \\
+ \int_\theta^\theta \eta(\theta) \left( k(\theta) - k(\theta) - \int_\theta^\theta j(\theta) d\theta \right) d\theta + \int_\theta^\theta \gamma(\theta) j(\theta) d\theta \right\} \tag{12}
\]

where, recall, \( B \equiv R(A - q_0k_0 - y_0) \).

A standard application of the Karush-Kuhn-Tucker Theorem shows that any saddle point of \( L \), i.e. any list \((y_0^*, k_0^*, y^*, k^*, j^*, k^*(\theta); \xi^*, \eta^*, \gamma^*, \lambda^*)\) satisfying

\[
L(y_0, k_0, y, k, j, k(\theta), \xi^*, \eta^*, \gamma^*, \lambda^*) \leq L(y_0^*, k_0^*, y^*, k^*, j^*, k^*(\theta), \xi^*, \eta^*, \gamma^*, \lambda^*) \leq L(y_0^*, k_0^*, y^*, k^*, j^*, k^*(\theta), \xi, \eta, \gamma, \lambda) \tag{13}
\]

corresponds to an OC \((y_0^*, k_0^*, y^*, k^*)\). We emphasize that the OC yields higher value than any other incentive compatible contracts, including discontinuous contracts. Therefore, we restrict our analysis of OCs to the set of continuously differentiable contracts as required in Lemma 2. After characterizing these contracts, we return to the saddle point problem \( L \), and verify the optimality of these contracts.
3.5 The Reformulated Saddle Problem

The saddle problem can be further simplified. As formulated, the controls \( y, k, \) and \( j \) are complicated functionals. The multipliers, however, can be regarded as type-dependent scalars and can thus be optimized point-by-point. Using integration by parts, we arrive at an alternative formulation of the problem that reverses the roles: in this alternative, controls become type-dependent scalars while multipliers are the more complicated functionals. The alternative formulation will prove more useful. Thus, using integration by parts (see Appendix C), the approach yields a reformulated Lagrangian function:

\[
L_R(y_0, k_0, y, k, j, y(\theta), k(\theta), y(\overline{\theta}), k(\overline{\theta}), \xi, \eta, \gamma, \lambda) \equiv U_0(y_0, k_0) + \\
\beta \left\{ \int_{\overline{\theta}}^{\theta} U(\theta + y(\theta), k(\theta)) \left( f(\theta) - \xi'(\theta) \right) d\theta + \lambda \left( B + qk_0 - \int_{\overline{\theta}}^{\theta} (y(\theta) + qk(\theta)) f(\theta) d\theta \right) \\
- \int_{\overline{\theta}}^{\theta} \xi(\theta) \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) d\theta + \xi(\overline{\theta}) U(\overline{\theta} + y(\overline{\theta}), k(\overline{\theta})) - \xi(\theta) U(\theta + y(\theta), k(\theta)) \\
+ \int_{\overline{\theta}}^{\theta} \eta(\theta) \left( k(\theta) - k(\overline{\theta}) \right) d\theta + \int_{\overline{\theta}}^{\theta} j(\theta) \left( \gamma(\theta) - \int_{\overline{\theta}}^{\theta} \eta(\theta) d\theta \right) d\theta \right\}. \\
\]

(14)

The reformulation requires that we separately specify extra controls \( y(\theta), k(\theta), y(\overline{\theta}), k(\overline{\theta}) \) at the boundaries \( \theta \) and \( \overline{\theta} \). The Lagrangian function \( L_R \) admits a saddle point if it is concave in \( (y(\theta), k(\theta), j(\theta), y(\overline{\theta}), k(\overline{\theta}), y(\overline{\theta}), k(\overline{\theta})) \) for all types \( \theta \). In the settings we consider later on, this property is fulfilled. For example, when \( U \) is separable in \( c, k, \) and \( \frac{\partial^2 U}{\partial c^2} > 0 \), then \( \xi(\theta) \geq 0, f(\theta) - \xi'(\theta) > 0, \xi(\theta) = \xi(\overline{\theta}) = 0 \), which imply the concavity of \( L_R \) in the relevant variables.

Any contract that solves \( L_R \) also solves \( L \), and so the solution to the OC problem is unchanged if \( L_R \) replaces \( L \) in the planner’s objective function. In the foregoing analysis, we focus attention on OCs that solve this reformulated problem.

4 Characterizing Optimal Contracts

In this section, we present three main findings. First we establish a preliminary result on the distorting nature of the OC. We show that all optimal contracts will exhibit an intra-
temporal distortion — a wedge between the marginal rate of substitution and the relative prices of the two goods in the second period for almost all types except for the highest type \( \theta \) and possibly for the lowest \( \theta \) as well.

The result is similar to characterizations of distortions found elsewhere in the Mirkeesian contracting literature, although to the best of our knowledge, it isn’t nested within other models. We show the distortion will always be in the direction of under-consumption of capital good \( k \) in the second period, and will be largest for types in the middle of the income distribution. Moreover, the continuation contract will be strictly separating at the top of income distribution. That is, for some open neighborhood of \( \tilde{\theta} \), \( k \) will be strictly increasing, and \( y \) strictly decreasing in \( \theta \). If pooling occurs, it is bounded away from the top of the distribution. Thus, the OC will not function like a standard debt contract.

Second and more fundamentally, we show that, unlike the full information model, all optimal contracts under incomplete information are collateralized. We show specifically how the value of forfeited collateral, \( \Gamma(\theta) \), relates to the distortionary wedge mentioned above. On the margin, higher distortions are shown to be associated with larger increases, or smaller decreases, in forfeiture of collateral ("anti-insurance"). The formula can be used to show that forfeiture is regressive at the tails — it decreases locally in income in intervals of both the bottom and top of the income distribution. When the OC is fully separating, forfeiture at the bottom is in fact higher than forfeiture at the top.

Third, we analyze the inter-temporal distortion in the OC between first and second periods. When capital and non-capital goods are complements, the borrower will generally over-consume or under-accumulate assets in the first period due the anticipated distortion in the repayment period.

### 4.1 Collateralization and Intra-temporal Distortion

The first order conditions for the saddle point problem \( \mathcal{L}_R \) are as follows. The first order conditions for initial period consumptions \( y_0 \equiv c_0 \) and \( k_0 \) are

\[
\frac{\partial U_0}{\partial c_0} = \beta \lambda R \quad \text{and} \quad \frac{\partial U_0}{\partial k_0} = \beta \lambda (q_0 - \frac{q}{R})
\]  

(15)
The first order condition in $y(\theta)$ is
\[\frac{\partial U}{\partial c}(\theta) (f(\theta) - \zeta'(\theta)) - \lambda f(\theta) = \frac{\partial^2 U}{\partial c^2}(\theta) \zeta(\theta)\] (16)

The first order condition in $k(\theta)$ is
\[\frac{\partial U}{\partial k}(\theta) (f(\theta) - \zeta'(\theta)) + \eta(\theta) - \lambda q f(\theta) = \frac{\partial^2 U}{\partial c \partial k}(\theta) \zeta(\theta)\] (17)

The first order conditions in controls $y(\bar{\theta}), y(\tilde{\theta}),$ and $k(\tilde{\theta})$ are
\[\frac{\partial U}{\partial c}(\theta) \zeta(\theta) = \frac{\partial U}{\partial c}(\bar{\theta}) \zeta(\bar{\theta}) = \frac{\partial U}{\partial k}(\tilde{\theta}) \zeta(\tilde{\theta}) = 0.\] (18)

The FOC in $k(\theta)$ is
\[\int_{\theta}^{\tilde{\theta}} \eta(\tilde{\theta}) d\tilde{\theta} - \zeta(\tilde{\theta}) = 0.\] (19)

Finally, the first order condition in $j(\theta)$ is
\[-\int_{\theta}^{\tilde{\theta}} \eta(\tilde{\theta}) d\tilde{\theta} + \gamma(\theta) = 0.\] (20)

All told, the Karush-Kuhn Tucker equations in the saddle problem yield a system of thirteen equations and thirteen unknowns. The first order conditions (15)-(20) on the controls yield nine equations (there are two equations in (15) and three in (18)). The constraints (3), (6), (10), and (11) yield the other four.

Combining first order conditions (16) and (17), one arrives at
\[\frac{\partial U}{\partial k} / \frac{\partial U}{\partial c}(\theta) = q \left[ \frac{\partial^2 U}{\partial c \partial k}(\theta) \zeta(\theta) / q - \eta(\theta) / q + \lambda f(\theta) \right] \equiv q \Delta(\theta)\] (21)

Equation (21) equates the marginal rate of substitution for each type to a distorted relative price of capital in the second period. The bracketed term “[.]” is the distortion applied to the price of capital. A distortion exists whenever $\Delta(\theta)$ differs from one. When $\Delta(\theta) > (\leq) 1$, there is relative under-consumption (over-consumption) of the capital good $k$, and over-consumption (under-consumption) of the composite good $c$ in the second (re-)payment period.
A preliminary result below mirrors some standard results in Mirleesian models of optimal taxation.

**Proposition 1.** Let \((y^*_0, k^*_0, y^*, k^*)\) be an optimal contract (OC). Then:

(i) A distortion exists at each income type \(\theta \in (\bar{\theta}, \theta_0)\), and in all such distortions the borrower consumes too little of the capital good relative to that of the composite good in the second period, i.e.,
\[
\frac{\partial U}{\partial k}(\theta) > q, \quad \text{and so} \quad \Delta^*(\theta) > 1.
\]

(ii) No distortion exists for the highest income type \(\bar{\theta}\), i.e., \(\frac{\partial U}{\partial k}(\bar{\theta}) = q\), and if the OC is strictly separating in a neighborhood of the lowest income type \(\theta_0\), then no distortion exists for that type as well.

(iii) There exists \(\epsilon > 0\) such that both \(y^*\) and \(k^*\) are strictly monotone on the interval \((\bar{\theta} - \epsilon, \bar{\theta})\).

(iv) The value of the transfer \(y^*(\theta) + qk^*(\theta)\) given to type \(\theta\) is weakly decreasing in \(\theta\), and strictly so on an interval of types.

**Proof.** Appendix E.

According to Part (i), distortions exist in \((\bar{\theta}, \theta_0)\), and will always be in the direction of relative under-consumption of \(k\), and over-consumption of \(c\). Part (ii) and (ii) show that no distortion exists at the top and, for strictly separating contracts, at the bottom as well. Part (iii) asserts that the OC must be separating at the upper end of the realized income distribution. This means that the OC is not pooling at the top and is therefore not a standard debt contract. Parts (iii) and Part (iv) imply that partial insurance is provided in every OC.

Most of Proposition 1 is familiar in the optimal tax literature. Various forms of (i)-(iii) appear elsewhere, though typically under more stringent assumptions. For example, most of the papers in the literature, such as Mirrlees (1971) and Seade (1982), assume that the monotonicity constraint never binds.\(^{15}\) Difficulties arise when the monotonicity constraint can potentially bind, i.e. \(\gamma(\theta) > 0\) and \(\eta(\theta) \neq 0\) for some \(\theta\), as shown in Proposition 4 below. Hence, we have to have to deal with this issue explicitly. The most closely related result in the prior literature (that we are aware of) appears in Hellwig (2007) who

\(^{15}\)Under this assumption, Werning (2000) provides a simple and elegant argument for distortion.
works with general, non-separable preferences of the form $u(\theta, y, k)$ and allow for potentially binding monotonicity constraint. Our proof strategy is similar to Hellwig’s, though the two models are not nested.\footnote{Mapped into our model, Hellwig’s Assumption 3.7 requires that $k$ and $\theta$ are weak substitutes at the lower bound of consumption. In our case, there need not be a lower consumption bound, and in any event $k$ and $\theta$ may be strict complements in our model.}

Figure 1 illustrates the, by now familiar, intuition. Ideally, the OC would provide at least partial insurance in the form of a larger transfer value $T(\theta) \equiv y^*(\theta) + qk^*(\theta)$ for lower types $\theta$. In order to do this while deterring mimicry by higher types, the OC must introduce an intra-temporal distortion in the allocation assigned to $\theta$ (point $B$ in the Figure). Since type $\bar{\theta}$ will not be mimicked, no such distortion for $\bar{\theta}$ is required.

The novel bit here is the precise characterization of the distortion function $\Delta(\theta)$ in Equation (21). The function $\Delta(\theta)$ is difficult to characterize partly because payoffs are not quasi-linear or even separable across $c$ and $k$ (although later, we do examine the special case of separable payoffs). In all the examples we compute later, the optimal $\Delta^*(\theta)$ is shown to be single peaked with an interior maximum.

The intuition for this is roughly the following. Under strict single crossing, distortions with under-consumption of $k$ are very costly to high types. For this reason, the cost of deterring mimicry by these types is low — small distortions provide sufficient deterrent. As for low types, they are harder to punish but at the same time the gains from mimicry - superior indemnification - are low. Consequently, low types do not require a large distortion either. Hence, it is intermediate types who incur the largest distortion. The gains from indemnification remain relatively large, while the cost of deterrence is also (relatively) large.

Using the distortion result in Proposition 1, we establish the main result of this section: all optimal contracts are collateralized. Furthermore, a simple and intuitive formula relates forfeited collateral to the distorted cost of capital.

**Proposition 2.** Let $(y^*_0, k^*_0, y^*, k^*)$ be an optimal contract. Then:

(a) the optimal contract is collateralized: $\Gamma^* > 0$,

(b) there exists $\epsilon > 0$ such that the $\Gamma^*$ is decreasing in the intervals $[\theta, \theta + \epsilon)$ and $(\bar{\theta} - \epsilon, \bar{\theta}]$, and if the OC is fully separating then the value of the collateral seized from the lowest income type exceeds that of the highest income type: $\Gamma^*(\theta) > \Gamma^*(\bar{\theta})$. 

\[ A = (k^*(\bar{\theta}), y^*(\bar{\theta})) \]

\[ B = (k^*(\theta), y^*(\theta)) \]

**Figure 1: Insurance-distortion tradeoff**
(c) for each type $\theta$,

$$
\Gamma^*(\theta) \ = \ \Gamma^*(\bar{\theta}) \ + \ \int_{\theta}^{\bar{\theta}} q \frac{dk^*}{d\theta} (\Delta^*(\bar{\theta}) - 1) d\bar{\theta} \ - \ (\theta - \bar{\theta}). \quad (22)
$$

**Proof. Appendix F.**

In the parameterized model laid out in the next Section, Part (b) can be strengthened to show that the forfeiture function $\Gamma^*(\theta)$ is globally regressive (strictly decreasing everywhere).\textsuperscript{17} To prove Part (a), it suffices to find at least two types $\theta'$ and $\theta''$ for whom the values of seized collateral (as defined in (8)) are different. This is easy if the contract contains a pooling region since $\Gamma^*(\theta)$ is strictly decreasing everywhere in this region. If the contract is strictly separating everywhere, then we utilize the fact that there is no distortion at the bottom (Part (ii) in Prop. 1). In this case one can show $\Gamma^*(\theta) > \Gamma^*(\bar{\theta})$ as asserted in the second statement of Part (b). Indeed, if $\Gamma^*(\theta) \leq \Gamma^*(\bar{\theta})$, i.e. the value of forfeiture of low income types was smaller than for high income types, then the no-distortion result would imply $U(c^*(\theta), k^*(\theta)) \geq U(c^*(\bar{\theta}), k^*(\bar{\theta}))$ contradicting the strictly increasing payoffs in $\theta$ implied by the envelope constraint.

The first statement in Part (b) asserts that forfeiture is decreasing locally around $\theta$ and $\bar{\theta}$. In these portions of the income distribution, seized collateral is locally regressive in income. The argument for this is shown below using the result in Part (c).

For Part (c), Equation (22) is derived in Appendix F. It describes the explicit relationship between forfeiture and the distortion. To our knowledge this linkage between collateral and the distortion is novel. The derivation is straightforward, combining the local IC constraint with the distortion equation in (21). To gain a better understanding of what it means, it helps to evaluate $\Gamma^*$ at the margin. By totally differentiating Equation (22) and re-arranging terms, we obtain

$$
q \frac{dk^*}{d\theta} (\Delta^*(\theta) - 1) \ = \ 1 + \frac{d\Gamma^*(\theta)}{d\theta} \quad (23)
$$

Marginal forfeiture can now be interpreted as a “tax wedge” as follows. Consider an extra “dollar” $d\theta$ of realized income. What does the borrower gain if he truthfully reveals

\textsuperscript{17}We have been unable to prove global monotonicity of $\Gamma^*$, nor can we rule it out. We suspect that it is not generally true: it may be that seized collateral sometimes increases in realized income. We do prove global regressivity of $\Gamma^*$ in the parametric model in Section 5, and in the result below, we show that it’s also true in local neighborhoods of the highest and lowest type.
the extra dollar of income? He receives an incremental consumption of capital \( \frac{dk^*}{d\theta} \), (the marginal propensity to consume (MPC) capital ). With asymmetric information, however, the value of this consumption in the optimal contract is distorted. The net value of this distortion is \( q(\Delta^*(\theta) - 1) \) which by Proposition 1 is positive. Hence, the left hand side of (23) is

\[
q \frac{dk^*}{d\theta}(\Delta^*(\theta) - 1) = \text{Net distortion in value of MPC}.
\]

This is the value to the borrower when he receives and truthfully reveals an “extra dollar” of income.\(^{18}\)

Now consider what would happen if instead the borrower hid the dollar. The right-hand side of (23) represents the value gained by hiding the extra dollar. Typically (though possibly not always), this value is reduced by forfeiture since \( \frac{d\Gamma^*(\theta)}{d\theta} < 0 \). This means that hiding an extra dollar of income yields the borrower a gain of less than a dollar due to higher forfeiture required of lower types. The right-hand side represents that portion of a hidden dollar left to the borrower after accounting for seized assets by the lender.

Putting everything together, the marginal value of hiding income must be offset by the marginal value of revealing it. Collateralized forfeiture is required in this offset. By integrating (23) up to \( \theta \), we obtain (22) which posits that the forfeited collateral for type \( \theta \) is the difference between pledged collateral \( \Gamma^*(\theta) \) and the average loss from the net distortion in MPC among types that \( \theta \) might be inclined to mimic.

It is easy to verify from (23) that \( \frac{d\Gamma^*(\theta)}{d\theta} \geq -1 \) for all \( \theta \in [\bar{\theta}, \bar{\theta}] \). Combining this result with Proposition 1, it follows that

\[
\frac{d\Gamma^*(\bar{\theta})}{d\theta} = -1 \quad \text{and} \quad \frac{d^2\Gamma^*(\bar{\theta})}{d\theta^2} < 0,
\]

and

\[
\frac{d\Gamma^*(\bar{\theta})}{d\theta} = -1 \quad \text{and} \quad \frac{d^2\Gamma^*(\bar{\theta})}{d\theta^2} \geq 0
\]

with strict inequality if the contract is separating in a neighborhood of \( \bar{\theta} \). This establishes the first part of (b). Because forfeiture must average out to zero over the entire distribution, it must be negative somewhere yielding a windfall to certain income types. The implicit subsidy to these types is required in order to satisfy IC constraints.

The second derivatives address what happens locally to the marginal forfeiture of an

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\(^{18}\)Clearly the borrower receives an incremental consumption of the composite good as well, but the IC constraint may be used to express everything in terms of incremental capital consumption.
extra dollar of income. For low income levels very close to \( \theta \), the marginal forfeiture of an extra dollar of income is weakly increasing (strictly so if there is no pooling at the bottom). Incrementally larger portions of extra income go to the lender. For high income levels close to \( \bar{\theta} \), the marginal forfeiture of an extra dollar of income is strictly decreasing. Incrementally smaller portions of extra income go to the lender.

### 4.2 Collateralization and Inter-temporal Distortion

From the first order equations (15), one can verify that intra-temporal consumption in the initial period \( t = 0 \) is undistorted: \( \frac{\partial U_0}{\partial k_0} = q_0 - \frac{q}{R} \). The marginal rate of substitution in date 0 equals the user cost of capital. However, an inter-temporal distortion comes from the informational distortion (21) in the second period. Combining first order conditions in \( c_0 \) and \( k_0 \) with those in the continuation problem we obtain,

\[
\frac{1}{R\beta} \int \frac{1}{\partial U/\partial c} (\theta) dF(\theta) = \frac{1}{\partial U_0/\partial c_0} \left( 1 - \int \frac{\partial^2 U/\partial c^2}{\partial U/\partial c} (\theta) \xi(\theta) d\theta \right) \tag{24}
\]

\[
\frac{1}{\beta(q_0 - \frac{q}{R})} \int \frac{1}{\partial U/\partial c} (\theta) dF(\theta) = \frac{1}{\partial U_0/\partial k_0} \left( 1 - \int \frac{\partial^2 U/\partial c^2}{\partial U/\partial c} (\theta) \xi(\theta) d\theta \right) \tag{25}
\]

\[
\frac{1}{R\beta} \int \frac{1}{\partial U/\partial k} (\theta) dF(\theta) = \frac{1}{\partial U_0/\partial c_0} \left( 1 - \int \frac{\partial^2 U/\partial c \partial k}{\partial U/\partial k} (\theta) \xi(\theta) d\theta \right) \tag{26}
\]

\[
\frac{q}{\beta(q_0 - \frac{q}{R})} \int \frac{1}{\partial U/\partial k} (\theta) dF(\theta) = \frac{1}{\partial U_0/\partial k_0} \left( 1 - \int \frac{\partial^2 U/\partial c \partial k}{\partial U/\partial k} (\theta) \xi(\theta) d\theta \right) \tag{27}
\]

When \( U \) is additively separable between \( c \) and \( k \), one obtains \( \frac{\partial^2 U}{\partial c \partial k} = 0 \) in which case (26) and (27) reduce to

\[
\frac{1}{R\beta} \int \frac{1}{\partial U/\partial k} (\theta) dF(\theta) = \frac{1}{\partial U_0/\partial c_0} \tag{28}
\]

and

\[
\frac{1}{\beta(q_0 - \frac{q}{R})} \int \frac{1}{\partial U/\partial k} (\theta) dF(\theta) = \frac{1}{\partial U_0/\partial k_0} \tag{29}
\]

which are recognizable as inverse Euler equations, common in dynamic mechanism de-
The application of Jensen’s Inequality to (28) and (29) implies \( \frac{\partial U_0}{\partial c_0} < R\beta \int \frac{\partial U}{\partial k}(\theta) dF(\theta) \) and \( \frac{\partial U_0}{\partial k_0} < \beta \frac{q_0 - q}{q} \int \frac{\partial U}{\partial k}(\theta) dF(\theta) \). In other words, compared to the full information optimum, the planner puts too much weight on current consumption \( k_0 \) and \( y_0 \) relative to future consumption of capital. This is primarily due to the imperfect insurance from second period incentive constraints.

Here, because \( U \) may not be separable in our model, an additional distortion arises due to income effects of a change in the distorted relative price \( q\Delta(\theta) \) for type \( \theta \) when initial consumption is varied. Taking, as an example, the inter-temporal trade off between \( k_0 \) and \( k \), we rewrite (27) to obtain

\[
\frac{\partial U_0}{\partial k_0} = \beta \left( \frac{q_0 - q}{R} \right) \int \frac{\partial U}{\partial k}(\theta) dF(\theta) - d_{1,k} - d_{2,k} \tag{30}
\]

where

\[
d_{1,k} = \beta \left( \frac{q_0 - q}{q} \right) \left[ \int \frac{\partial U}{\partial k}(\theta) dF(\theta) - \left( \int \frac{1}{\partial U/\partial k}(\theta) dF(\theta) \right)^{-1} \right] > 0
\]

and

\[
d_{2,k} = \beta \left( \frac{q_0 - q}{q} \right) \left( \int \frac{1}{\partial U/\partial k}(\theta) dF(\theta) \right)^{-1} \left( \int \frac{\partial^2 U/\partial c \partial k}{\partial U/\partial k}(\theta) \xi(\theta) d\theta \right) \geq 0
\]

Equation (30) displays the double distortion explicitly. Here, \( d_{1,k} \) represents the distortion mentioned above. The planner commits less savings to a period in which consumption is volatile. The second distortion \( d_{2,k} \) comes from the income effect off the relative price distortion \( q\Delta(\theta) \). Here, \( d_{2,k} > 0 \) if \( c \) and \( k \) are complements and \( d_{2,k} < 0 \) if \( c \) and \( k \) are substitutes as Assumption 1 allows. The distortion \( d_{2,k} \) is zero only if there are no income effects (i.e., \( U \) is separable in \( c \) and \( k \)). A similar expression appears in the trade off between \( c_0 \) and \( k \). Since \( d_{2,k} > 0 \) would be expected when, say, \( k \) is housing, the “double distortion” leads to a more severe under-consumption of the second period’s capital good than would be the case in the standard incentive contract with separable utility.

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\[\text{sign.}^{19}\]

See, for instance, Rogerson (1985), Golosov et al. (2003), and Farhi and Werning (2012) and references therein. Note that the capital good in our model plays a role similar to the consumption good in the Mirlees model. In either case, the unobserved shock does not directly apply to it. This is the reason that inverse Euler equations hold in both models when \( U \) is separable. By contrast, the non-capital consumption in our model is substantively closer to the labor/leisure choice in the Mirlees models since the shock enters consumption of those goods directly.
The trade-offs represented by (24) and (25) also exhibit a double distortion, even with separable $U$. Unlike the future capital good, the two distortions may have different signs. Rewriting (25) for instance, one obtains

$$\frac{\partial U_0}{\partial k_0} = \beta (q_0 - \frac{q}{R}) \left( \int \frac{\partial U}{\partial c}(\theta) dF(\theta) - d_{1,c} - d_{2,c} \right)$$

where

$$d_{1,c} = \beta (q_0 - q/R) \left[ \int \frac{\partial U}{\partial c}(\theta) dF(\theta) - \left( \int \frac{1}{\partial U/\partial c}(\theta) dF(\theta) \right)^{-1} \right] > 0$$

and

$$d_{2,c} = \beta (q_0 - q/R) \left( \int \frac{1}{\partial U/\partial c}(\theta) dF(\theta) \right)^{-1} \left( \int \frac{\partial^2 U/\partial c^2}{\partial U/\partial c}(\theta) \xi(\theta) d\theta \right) < 0.$$

The distortion $d_{1,c}$ pushes the borrower in the direction of over-consumption in the first period just as before. However, a second distortion $d_{2,c}$ pushes toward under-consumption in the first period. The intuition is that the planner wish to increase savings in order to improve insurance options. This is done indirectly since increased savings relaxes incentive constraints. The degree to which this can work is determined by weighted average value of risk aversion (see last term). The larger the average risk aversion, the greater is the effect of relaxing IC on insurance. Which distortion $d_{1,c}$ or $d_{2,c}$ dominates is unknown. The net effect is therefore ambiguous. To summarize,

**Proposition 3.** Any optimal contract $(y^*_0, k^*_0, y^*, k^*)$ exhibits inter-temporal distortions $d_{1,k}, d_{2,k}, d_{1,c}$ and $d_{2,c}$. In the Euler equation (30), the distortions $d_{1,k}$ and $d_{2,k}$ are positive and nonnegative, respectively, when $c$ and $k$ are weak complements. In that case both distortions lead to over-consumption of initial capital $k_0$ relative to future capital consumption $k$. In (31), the distortions have opposite signs: $d_{1,c} > 0$ and $d_{2,c} < 0$. The net effect of these distortions lead to over (under)-consumption of initial $k_0$ relative to future non-capital consumption $c$ if $|d_{1,c}| > (\leq) |d_{2,c}|$.

5 Closed Form Solutions in a Log-Uniform Model

In this Section we present a parametric “log-uniform” model in which the contract can be solved in closed form, and capital good consumption in the repayment period is convex in income. We show that if the distribution is sufficiently concentrated around the mean
and/or the borrower holds sufficiently large assets initially, the OC is strictly separating. In these cases the cost of satisfying the incentive constraint is low. Partial insurance is then provided for all types, and full default does not occur except for the lowest type. By contrast, when the distribution is sufficiently diffuse and/or the borrower’s initial assets are low enough, then pooling occurs at the bottom of the income distribution. Full default therefore occurs in some interval of income types below a threshold. In both cases, forfeiture is globally regressive — it is strictly decreasing in income over the entire distribution.

Let $f(\theta) = \bar{f} = \frac{1}{\theta - \bar{\theta}}$ and $U = \frac{1}{2} \log c + \frac{1}{2} \log k$. The log-uniform case simplifies the problem by assuming no complementarities between the two goods. By allowing for curvature of payoffs in the two goods, however, it provides a more nuanced contract than under the standard quasi-linear setup. We abstract from the first period and focus attention on the continuation OC as a function of the multiplier $\lambda$, itself determined by first period consumption.

The result below characterizes a closed form solution to the OC that, depending on parameter values, is either strictly separating, or pools types only at the lower end of the distribution. In either case, we show that that consumption of the capital good is convex in the income type, so that the individual’s consumption of capital good $k$ relative to his realized income $\theta$ is increasing.

Given the log utility function, the first order conditions, (16) and (17) become

$$
\frac{1}{2c(\theta)} \left( \bar{f} - \xi'(\theta) \right) - \lambda \bar{f} = -\frac{1}{c(\theta)^2} \xi(\theta)
$$

and

$$
\frac{1}{2k(\theta)} \left( \bar{f} - \xi'(\theta) \right) + \eta(\theta) - \lambda q \bar{f} = 0.
$$

We first conjecture and verify existence of a separating equilibrium for some set of parameter values. Under full separation, the monotonicity constraint does not bind. The multipliers associated with the monotonicity constant therefore vanish: $\gamma(\theta) \equiv 0$ and $\eta(\theta) \equiv 0$. Using this restriction, we combined the two first order conditions, and then differentiate with respect to $\theta$ to obtain,

$$
2qk \left( k'(\theta)c(\theta) + k(\theta)c'(\theta) \right) - 2q \frac{d}{d\theta} \left\{ (c(\theta))^2 \right\} = 2qk(\theta) - 1. \quad (32)
$$
Now, the incentive constraint implies

\[ \frac{1}{2c(\theta)} (c'(\theta) - 1) + \frac{1}{2k(\theta)} k'(\theta) = 0. \] (33)

Combining this restriction with (32) and (33), we obtain

\[ 2\lambda \frac{d}{d\theta} \left\{ (c(\theta))^2 \right\} = 1, \] implying

\[ c(\theta) = \sqrt{(c(\theta))^2 + \frac{\theta - \theta^2}{2\lambda}}. \] (34)

Consequently,

\[ y(\theta) = \sqrt{(c(\theta))^2 + \frac{\theta - \theta^2}{2\lambda}} - \theta \] (35)

Notice that \( y \) is weakly decreasing, as required, if and only if \( c(\theta) \geq \frac{1}{4\lambda} \).

Integrating (33), using (34), we obtain

\[ \log(k(\theta)) = \log(k(\theta)) + 4\lambda c(\theta) - 4\lambda c(\theta) - \log(c(\theta)) + \log(c(\theta)) \] (36)

We now assume that \((y, k)\) is a separating OC, i.e., \( y \) and \( k \) are strictly monotone and we will verify this condition later. By Proposition 1 (Part ii), since \( k \) is strictly increasing, there is no distortion at either boundary \( \underline{\theta} \) or \( \bar{\theta} \). We therefore have

\[ c(\theta) = qk(\theta) \quad \text{and} \quad c(\bar{\theta}) = qk(\bar{\theta}) \] (37)

The no-distortion-at-the-bottom condition is combined with (36) to produce

\[ k(\theta) = D \frac{\exp(4\lambda c(\theta))}{c(\theta)} \] (38)

where \( D \), a constant, is equal to \( q(c(\theta))^2 \exp(-4\lambda c(\theta)) \).

It is not difficult to show that \( k(\theta) \) globally is strictly convex in \( \theta \). In other words, the ratio of capital good consumption to realized income is increasing.

Notice that \( k \) and \( y \) are ultimately functions of \( c(\theta) \). One can solve for \( c(\theta) \) by evaluat-
ing (36) at $\theta = \bar{\theta}$ and using the no-distortion-at-the-top condition in (37) to obtain

$$\log \left( c(\theta)^2 + \frac{\bar{\theta} - \theta}{2\lambda} \right) = \log(c(\bar{\theta})^2) + 4\lambda \left( (c(\bar{\theta}))^2 + \frac{\bar{\theta} - \theta}{2\lambda} \right)^{1/2} - 4\lambda c(\bar{\theta}).$$ (39)

The result below establishes parametric restrictions under which this construction is valid; $k$ and $y$ are, in fact, strictly separating, i.e. $c(\theta) \geq \frac{1}{4\lambda}$. It also asserts that when these parametric restrictions are not satisfied, the optimal collateralized contract has a pooling region at the bottom of the income distribution, and is strictly separating everywhere else.

**Proposition 4.** There exists $\delta > 0$ such that

1. If $\lambda(\bar{\theta} - \theta) \leq \delta$, then the optimal continuation contract $(y^*, k^*)$ is strictly separating and satisfies (34), (35), (38), and (39). Notably,
   
   (a) $k^*$ is increasing and strictly convex in $\theta$, $y^*$ is decreasing and strictly concave, and
   
   (b) $\Gamma^*(\theta)$, the collateral seized from income type $\theta$, is decreasing in $\theta$.

2. If $\lambda(\bar{\theta} - \theta) > \delta$, then there exists a cutoff type $\theta^* \in (\bar{\theta}, \bar{\theta})$ such that the optimal continuation contract $(y^*, k^*)$ is separating in the interval $[\theta^*, \bar{\theta}]$ and is pooling in the interval $[\bar{\theta}, \theta^*)$. Notably,
   
   (a) above the cutoff $\theta^*$, $k^*$ is strictly convex in $\theta$, $y^*$ is strictly concave,
   
   (b) Collateral seized is decreasing in $\theta$.

**Proof.** Appendix G.

We refer to OCs that satisfy Part (2) of the Proposition as *semi-pooling*. The Proposition identifies a partition of the parameter set into those that generate separating OCs, and those that generate semi-pooling ones. Both the support and the multiplier $\lambda$ play a critical role. A larger support increases the incentive to misreport in a separating equilibrium since the consumption of the high income types cannot be increased too much without violating the resource constraint. As a result, incentives can be only be brought into line by a pooling contract at the lower end. In that case there is no further gain from mimicry below the threshold $\theta^*$.

---

20 In fact, the parameter $\delta$ can be pinned down precisely: $\delta = (e - 1)/8$ where $e$ is the unique scalar that satisfying $\log(e) = e^{1/2} - 1$. ($e \approx 12.34$).
An increase in \( \lambda \) also diminishes the range in which the separating contract exists. In the Proposition, the multiplier \( \lambda \) the resource constraint is treated as a parameter. However, \( \lambda \) is an implicit solution to the resource constraint evaluated at \( y \) and \( k \), given budget \( B \). As a shadow price of the constraint, \( \lambda \) is decreasing in \( B \).

In Appendix G we extend the analysis above to show that the separating part of the semi-pooling contract in Part 2 satisfies

\[
y(\theta) = \sqrt{(c(\theta^*))^2 + \frac{\theta - \bar{\theta}}{2\lambda} - \theta} \tag{40}
\]

and

\[
k(\theta) = D^* \exp\left(\frac{4\lambda c(\theta)}{c(\theta)^{1/2}}\right) \tag{41}
\]

where \( D^* \), in this case, is equal to \( k(\theta^*) c(\theta^*) \exp(-4\lambda c(\theta^*)) \).

Notice that (40) and (41) are analogues of (35) and (38). They have the same functional form with the pooling threshold \( \theta^* \) replacing the lower bound \( \underline{\theta} \) of the support in the functions. The pooling consumptions \( k(\theta^*) \) and \( y(\theta^*) \) can be found by evaluating (40) and (41) at \( \theta = \bar{\theta} \) and using the no-distortion condition \( q c(\bar{\theta}) = k(\bar{\theta}) \) to define three equations with three unknowns \( c(\bar{\theta}), k(\bar{\theta}), \) and \( \theta^* \) (see Appendix G for details).

In the semi-pooling contract, \( k \) consumption is globally convex in \( \theta \); it is constant below the threshold \( \theta^* \) and strictly convex above the \( \theta^* \). In this sense, the semi-pooling contract more closely resembles the familiar debt contract in home mortgages.

We note that in the existing literature that uses quasi-linear utility functions, pooling (equivalently “bunching”) is verified using ironing argument when hazard rates are not monotonic and/or when participation constraints exist. Examples include Myerson (1981), Lollivier and Rochet (1983), Jullien (2000), Noldeke and Samuelson (2007), and many others. In Proposition 4, under general utility functions, pooling might happen even with monotone hazard rates and without a participation constraint. The result suggests that the rationale for pooling in an optimal contract may be very different when one departs from the quasi-linear paradigm.

The continuation optimal collateralized contract is illustrated in two parametric special cases. We set parameter values

\[
q = 1 \quad \lambda = 1 \quad f'/f = 1
\]
The allocations, distortions, and forfeitures are then displayed in Figure 2 for two different supports. Each column displays a continuation OC, a distribution of the distortionary wedge, and the distribution of the value of forfeiture/seizure of the asset across the realized incomes.

In the first column of Figure 2 the support is \([\theta, \bar{\theta}] = [-0.5, 0.5]\). The OC in this case is strictly separating, as per Part 1 of Proposition 4.

In the second column \([\theta, \bar{\theta}] = [-1.5, 1.5]\) so that the second support is twice as broad as the first. In this case, the OC is semi-pooling, as per Part 2 of the Proposition. Poorer income below a cutoff of around \(-1.3\) are pooled, while types above that threshold strictly
In both cases, the distortion is a non-monotonic function of income. The wedge is largest for middle income types, and smallest for the very poor and very rich. This is largely due to the differing information incentives and the cost mitigating bad incentives across types.

From the perspective of the full information optimum, the very poor have no incentive to lie. Hence, there is no reason to distort their consumption. The very rich, by contrast have the greatest incentive to lie. To align incentives of this group, one might suppose that a large distortion is required. However, it is more costly to distort the consumption of the rich than the poor, since they take up a larger share of the resource constraint. Hence, the largest distortion is assigned to the middle income types — the types who are most likely to mimicked by high income types, a priori. Nevertheless, the distribution of forfeited collateral indicates that the brunt of incentive costs is borne by low income types.

6 Computing Optimal Contracts

The previous section saw that an explicit closed form solution for the optimal contract exists in the log uniform case. Here we give an algorithm for computing one in the general case following the same guidelines.

6.1 An Algorithm

Given strict concavity of $U$, Lemma 3 in the Appendix A is used to prove there exists a one-to-one function $H : \mathbb{R}^2_+ \to \mathbb{R}^2$ satisfying

$$
\begin{bmatrix}
    c \\
    k
\end{bmatrix}
= H \left( \frac{\partial U(c,k)}{\partial c}, \frac{\partial U(c,k)}{\partial k} \right)
= \begin{bmatrix}
    H^c \left( \frac{\partial U(c,k)}{\partial c}, \frac{\partial U(c,k)}{\partial k} \right) \\
    H^k \left( \frac{\partial U(c,k)}{\partial c}, \frac{\partial U(c,k)}{\partial k} \right)
\end{bmatrix}
$$

The function $H$ associates marginal utilities of each of the two goods to those values of the consumption that generated them. Using the change of variables: $x = \frac{\partial U(c,k)}{\partial c}$ and
\[
\begin{align*}
    z = \frac{\partial U(c,k)}{\partial k}, \text{ we obtain} \\
    \begin{bmatrix}
    \frac{\partial^2 U(c,k)}{\partial c^2} & \frac{\partial^2 U(c,k)}{\partial c \partial k} \\
    \frac{\partial^2 U(c,k)}{\partial c \partial k} & \frac{\partial^2 U(c,k)}{\partial k^2}
    \end{bmatrix} = \\
    \begin{bmatrix}
    \frac{\partial H^c(x,z)}{\partial x} & \frac{\partial H^c(x,z)}{\partial z} \\
    \frac{\partial H^k(x,z)}{\partial x} & \frac{\partial H^k(x,z)}{\partial z}
    \end{bmatrix}^{-1} \\
    = \\
    \begin{bmatrix}
    \frac{\partial H^c(x,z)}{\partial x} & \frac{\partial H^c(x,z)}{\partial z} \\
    \frac{\partial H^k(x,z)}{\partial x} & \frac{\partial H^k(x,z)}{\partial z}
    \end{bmatrix} \\
    \cdot \\
    \begin{bmatrix}
    \frac{\partial H^c(x,z)}{\partial x} & -\frac{\partial H^c(x,z)}{\partial z} \\
    -\frac{\partial H^k(x,z)}{\partial x} & \frac{\partial H^k(x,z)}{\partial z}
    \end{bmatrix} \\
    = \\
    \begin{bmatrix}
    J^k_z(x,z) & -J^c_z(x,z) \\
    -J^k_z(x,z) & J^c_z(x,z)
    \end{bmatrix}.
\end{align*}
\]

Assuming that the monotonicity constraint is not binding locally, the first order conditions in \( y \) and \( k \), (16) and (17), can now be expressed as

\[
    x(\theta) (f(\theta) - \zeta'(\theta)) - \lambda f(\theta) = J^k_z(x,z) \zeta(\theta),
\]

and

\[
    z(\theta) (f(\theta) - \zeta'(\theta)) - \lambda q f(\theta) = -J^c_z(x,z) \zeta(\theta).
\]

From these two equations, we can solve for \( \zeta(\theta) \) as

\[
\begin{align*}
    \zeta(\theta) &= \lambda f(\theta) K(x(\theta), z(\theta)), \\
    \text{where} \\
    K(x, y) &= \frac{q x - z}{z J^k_z(x,z) + x J^c_z(x,z)}.
\end{align*}
\]
Plugging in this solution of $\xi(\theta)$ back to (42), and divide both side by $f(\theta)$, we obtain

$$x(\theta) \left( 1 - \lambda \frac{f'(\theta)}{f(\theta)} K(x(\theta), z(\theta)) - \lambda \left( \frac{\partial K}{\partial x} x'(\theta) + \frac{\partial K}{\partial z} z'(\theta) \right) \right) - \lambda$$

$$= \lambda K(x(\theta), z(\theta)) K(x(\theta), z(\theta)) .$$

(45)

From the incentive constraint, we obtain another equation

$$\frac{\partial U}{\partial c} \left( \frac{dc}{d\theta} - 1 \right) + \frac{\partial U}{\partial k} \frac{dk}{d\theta} = 0,$$

or

$$x(\theta) \left( \frac{\partial H^c}{\partial x} x'(\theta) + \frac{\partial H^c}{\partial z} z'(\theta) - 1 \right) + z(\theta) \left( \frac{\partial H^k}{\partial x} x'(\theta) + \frac{\partial H^k}{\partial z} z'(\theta) \right) = 0. \quad (46)$$

(45) and (46) form a system of differential equations in $x(\theta)$ and $z(\theta)$. Under the full separating assumption, Proposition 1 shows that there is no distortion at the top and the bottom. Therefore,

$$\frac{\partial U}{\partial k} \left( \frac{dk}{d\theta} \right) = q = \frac{\partial U}{\partial c} \left( \frac{dc}{d\theta} \right).$$

Equivalently,

$$\frac{z(\theta)}{x(\theta)} = q = \frac{z(\bar{\theta})}{x(\bar{\theta})}. \quad (47)$$

We can use standard boundary value problem, e.g. MATLAB’s BVP functions, to solve numerically for $x(\theta), z(\theta)$ as a solution to the ODE system (45)-(46) and the boundary condition (47).

6.2 The Phase Diagram

When the density $f$ belong to the exponential family, i.e. satisfies $\frac{f'}{f} = \text{const}$ and equivalently,

$$f(\theta) = \phi \exp(-\psi\theta), \quad (48)$$

---

21 An alternative algorithm similar to the one used in the Mirrlees’ optimal nonlinear income tax literature solves for $x(\theta), y(\theta)$ as functions of $\xi(\theta), \xi'(\theta)$ from equations (42) and (43). The local incentive constraint, (46), then yields a second order differential equations in $\xi(\theta)$ with the boundary conditions $\xi(\theta) = \xi'(\theta) = 0$. Another algorithm often used in that literature, as described in Scheuer (2013, Appendix B), iterates on the optimal marginal tax rate formulae. These formulae are not available in our setting.
for some constants $\phi$ and $\psi$, the ODE system (45)-(46) becomes autonomous in $x$ and $z$, i.e independent of $\theta$. We can then use the phase diagram to analyze this system for characterizing optimal contracts.

The log-uniform model fits in this case. Recall that Figure 2 displayed solutions for two different supports, $\bar{\theta} - \underline{\theta} = 1$ and $\bar{\theta} - \underline{\theta} = 3$. The phase diagram for these two supports is displayed in Figure 3. Figure 3 displays the strictly separating “solutions” for both $\bar{\theta} - \underline{\theta} = 1$ and $\bar{\theta} - \underline{\theta} = 3$. Both start at higher marginal utilities corresponding to the contract continuation $(y^*(\underline{\theta}), k^*(\underline{\theta}))$ at the bottom. Because the phase diagram is constructed from a BVP assuming away the monotonicity constraint, only $\bar{\theta} - \underline{\theta} = 1$ corresponds to a true solution. When $\bar{\theta} - \underline{\theta} = 3$, the phase solution generates a non-monotonicity in $k^*$, and hence is not a real solution to the OC.

The actual solution when $\bar{\theta} - \underline{\theta} = 3$ is displayed in Figure 4. There, bottom end pooling requires an ironing argument (see Proposition 4 proof for details).

The phase diagram gives complementary insights into the OCs. For example, Figure 3 shows that $\dot{x} < 0$ when $x, z > 0$. Therefore $c^*(\theta)$ is strictly increasing in $\theta$. Further analysis of the phase diagrams reveals that this is the case for the log-log utility function with weakly higher weight on the composite good. However, this is not the case when lower weight is put on the composite good as in Example 2 below.

For more general utility functions, we use numerical analysis to display phase diagrams corresponding to OCs below.

### 6.3 Numerical Results

In this subsection, we use the algorithm described above to study the optimal contracts in two different settings. In the first one, we study the optimal contract under standard utility functions over housing and non-housing consumption. In the second one, we apply our solution method to the multi-period model.

#### 6.3.1 Housing and Non-Housing Consumption

We make use of the algorithm developed in the previous section to examine the properties of optimal contracts under several specifications of the utility function and type
Figure 3: Phase Diagram and Two Solutions
$\bar{\theta} - \theta = 3$

Figure 4: Phase Diagram for Semi-Pooling Contract
distribution.

We focus on the case of exponential type distribution, (48), to take advantage of the phase diagram in analyzing the ODE system (45)-(46). We consider the general specification of the utility function with constant elasticity of substitution (CES) between non-housing and housing consumption:

$$U(c, k) = \left( \alpha c^{\frac{\epsilon - 1}{\epsilon}} + (1 - \alpha) k^{\frac{\epsilon - 1}{\epsilon}} \right) \frac{e^{(1-\epsilon)\sigma}}{1 - \sigma} - 1,$$

(49)

where $\epsilon$ is the CES coefficient and $\sigma$ is the constant relative risk-aversion coefficient. When $\epsilon \to 1$ and $\sigma \to 1$, 

$$U(c, k) = \alpha \log(c) + (1 - \alpha) \log(k).$$

(50)

Examples are as follows.

**Example 1.** In this example, we consider the case with utility function (50) with $\alpha = \frac{1}{2}$ as in the log-uniform case considered in Proposition 4. However, $\phi = 1$ instead of $\phi = 0$ as in the proposition. Figures 5 and 6 in Appendix I depict the phase diagram, allocations, distortion, and forfeiture for this case. Despite the difference in the type distribution, the OC in this case qualitatively resembles the one with uniform distribution in Proposition 4.

**Example 2.** In this example, we consider the case with utility function (50) with less weight on non-housing consumption, $\alpha = 0.1$. We keep the uniform distribution of types. Figures 7 and 8 in Appendix I depict the phase diagram, allocations, distortion, and forfeiture for this case. One property of the optimal contract that stands out in this case is that $c(\theta)$ is non-monotone in $\theta$ near $\bar{\theta}$. This is surprising since $c$ is a normal good. This can be explained by the decreasing distortion near $\bar{\theta}$. While higher types $\theta$ receive more total budget $c(\theta) + qk(\theta)$ (which corresponds to decreasing forfeiture), they also face less distortion, or effectively higher relative price of non-housing consumption. Therefore, these types can potentially decrease non-housing consumption as we increase $\theta$.

**Example 3.** In this example, we consider the case with CES function (49) with $\alpha = 0.5$, and $\epsilon = 0.5$, $\sigma = 5$ as used in Piazzesi et al. (2007). We keep the uniform distribution of types. Figures 9 and 10 in Appendix I depict the phase diagram, allocations, distortion, and forfeiture for this case. Due to the low elasticity of substitution, the distortion in the OC is much smaller than in the previous examples. Forfeiture is also smaller but to a lesser extent. However, the shape of the allocation and forfeiture remain similar. In particular, forfeiture is strictly decreasing in $\theta$. 

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One property stands out from Examples 1-3 is that housing consumption $k(\theta)$ is convex in $\theta$. This property is shown analytically in Proposition 4. When utility function is separable and has CRRA in $c,k$, i.e. when $\epsilon = \frac{1}{\sigma}$ in (49), we show a similar result. We call a contract exhibiting end-point convexity if $\frac{\partial k}{\partial \theta}(\bar{\theta}) < \frac{\partial k}{\partial \theta}(\underline{\theta})$.

**Proposition 5.** In the class of collateralized contracting problems with CRRA payoffs $U = \alpha c^{1-\sigma} + (1-\alpha) k^{1-\sigma}$, any continuation OC exhibits end-point convexity.

**Proof.** Appendix H. □

The result can be extended without much trouble to CRRA payoffs with different exponential weights $\sigma_1$ and $\sigma_2$. Without further restriction on payoffs and distributions, little can be said about the behavior of $k^*$ in the middle of the distribution.

The convexity of housing consumption is, moreover, consistent with mortgage contracts in housing. Our preliminary empirical analysis using data from the Consumer Expenditure Surveys shows that among households with mortgages, housing consumption is roughly convex in income, and this is not generally true of households without mortgages.

### 6.3.2 Collateral in Multiple Period Contracts

This subsection shows that a notion of collateral arises naturally in multi-period optimal contracts with only a single consumption good each period. Examples include Townsend (1982) and Green (1987). We argue that collateral is present in these models, taking an intangible form in the guise of future credit.

Assume that a consumer receives I.I.D. income shocks $\theta_i$ in two periods 1, 2. The distribution of $\theta$ has finite support $[\underline{\theta}, \bar{\theta}]$ and continuous density. At time 0, the consumer signs a three period optimal contract with a competitive, risk-neutral bank. The optimal contract can be written as

$$\max_{y_0, y_1(\cdot), y_2(\cdot, \cdot)} \mathbb{E} \left[ u(y_0) + \beta u(\theta_1 + y_1(\theta_1)) + \beta^2 u \left( \theta_2 + y_2(\theta_1, \theta_2) \right) \right]$$

subject to

$$\mathbb{E} \left[ y_0 + \frac{1}{R} y_1(\theta_1) + \frac{1}{R^2} y_2(\theta_1, \theta_2) \right] \leq A.$$
and incentive constraints

\[ \mathbb{E} \left[ u(y_0) + \beta u(\theta_1 + y_1(\theta_1)) + \beta^2 u(\theta_2 + y_2(\theta_1, \theta_2)) \right] \]

\[ \geq \mathbb{E} \left[ u(y_0) + \beta u(\theta_1 + y_1(\theta_1')) + \beta^2 u(\theta_2 + y_2(\theta_1', \theta_2(\theta_1', \theta_1, \theta_2))) \right] \]

for all realizations \((\theta_1, \theta_2)\) and reporting strategies \((\theta_1', \theta_2', \ldots)\).

Let

\[ U(c_1, y_2) \equiv u(c_1) + \beta \mathbb{E}_{\theta_2} u(\theta_2 + y_2) \]  \hspace{1cm} (51)

Since \(\theta_2\) is realized in the last period, incentive compatibility implies that \(y_2\) depend only on \(\theta_1\). Therefore the optimal contracting problem can be re-written as:

\[
\max_{y_0, y_1(\cdot), y_2(\cdot)} \mathbb{E} \left[ u(y_0) + \beta U(\theta_1 + y_1(\theta_1), y_2(\theta_1)) \right]
\]

subject to

\[ \mathbb{E} \left[ y_0 + \frac{1}{R} y_1(\theta_1) + \frac{1}{R^2} y_2(\theta_1) \right] \leq A \]

and

\[ U(\theta_1 + y_1(\theta_1), y_2(\theta_1)) \geq U(\theta_1 + y_1(\theta'_1), y_2(\theta'_1)), \]

for all \(\theta_1, \theta'_1 \in [\underline{\theta}, \overline{\theta}]\).

Green (1987) calls the variable \(y_2\) “future credit.” In the context of the present model, \(y_2\) corresponds to the capital good \(k\), a collateralizable asset. Since \(\frac{\partial^2 u}{\partial c \partial y_2} \equiv 0\), we can apply our results to characterize the behavior of credit and distortions in the optimal contract. In particular, Lemma 1, Proposition 1, and Proposition 3 show that \(y_2\) is increasing in \(\theta_1\) and for all \(\theta_1 \in (\underline{\theta}, \overline{\theta})\):

\[ u'(\theta_1 + y_1(\theta_1)) < \beta R \mathbb{E}_1 u'(\theta_2 + y_2(\theta_1)), \]

and from the Inverse Euler equations in Proposition 3,

\[ u'(y_0) < (\beta R)^2 \mathbb{E}_0 u'(\theta_2 + y_2(\theta_1)) \]

and

\[ u'(y_0) \geq \beta R \mathbb{E}_0 u'(\theta_1 + y_1(\theta_1)). \]
The following example provides a numerical example of the optimal contract.

**Example 4.** In this example, we consider the CARA utility function used in Green (1987)

\[ u(c) = - \frac{1}{\gamma} \exp(-\gamma c) \]

From the definition of \( U \) in equation (51), we have

\[ U(c, y_2) = u(c) + \tilde{\beta} u(y_2) \]

where \( \tilde{\beta} = \beta \mathbb{E}_1(\theta_2) \)

Figures 11 and 12 in Appendix I depict the phase diagram, allocations, distortion, and forfeiture for the case with \( R = 1, \tilde{\beta} = 1 \) and the distribution of income is uniform between 0 and 3.

In contrast to the numerical examples in the previous subsection, with the CARA utility function, there is no natural lower bound on consumption. However, as shown in Figure 12, the optimal consumption (credit) \( y_2 \) still appears to be convex in \( \theta_1 \).

The phase diagram in Figure 11 suggests that, when \( \bar{\theta} - \theta \) is sufficiently large, the optimal contract cannot be fully separating. The optimal contract is state-invariant over any pooling interval. In the log uniform model, we know that such an interval exists near \( \theta \) for certain parameter values. In this case uninsurable income a la Huggett (1993) and Aiyagari (1994) is derived endogenously for low income types.

### 7 General Equilibrium

So far, we have shown that the capital good in the OC is relatively under-consumed in the second period. This holds for a fixed exogenous price \( q \). In a general equilibrium model with large numbers of lenders and borrowers, \( q \) is endogenously determined. We show in a weighted log model that \( q \) may be lower than the full information price. Markets, in essence, partially compensate for the distortion \( \Delta(\theta) \). Numerical simulations for the log-log uniform model in Subsection 5 compares the variation of the two prices across income.

To simplify the analysis we focus on the second period. Consumers are endowed with one unit of capital which, for concreteness, we refer to as housing. The banking sector is perfectly competitive. Banks offer contracts to the consumers in exchange for
consumers’ deposits before income shocks are realized. Banks maximize profit subject to an endogenous outside options of the consumers. Both consumers and banks have access to a market for housing at unit price $q$. Each bank solves

$$\Pi(q, U) = \max_{\{y, k\}} \left\{ q \cdot 1 - \int_{\theta} (y(\theta) + qk(\theta)) dF(\theta) \right\}$$

subject to

$$\int_{\theta} U(\theta + y(\theta), k(\theta)) dF(\theta) \geq U,$$

and

$$U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\theta'), k(\theta')),$$

for all $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$.

We define a competitive equilibrium as following.\footnote{The competitive equilibrium concept here is similar to the “self-confirming policy equilibrium” concept developed in Rothschild and Scheuer (2013, 2016) in which the optimal incentive compatible Mirleesian tax policies are designed taking wages as given, and in equilibrium wages are determined to clear the labor markets}

**Definition.** A competitive equilibrium is a price $q$, an endogenous outside option $U$ and an optimal contract $\{y^*(\theta), k^*(\theta)\}_{\theta \in [\underline{\theta}, \overline{\theta}]}$ such that at this optimal contract, banks make zero profit

$$\Pi(q, U) = 0$$

and the housing and composite good markets clear: $\int_{\theta}^\overline{\theta} k^*(\theta)dF(\theta) = 1$, and $\int_{\theta}^\overline{\theta} y^*(\theta)dF(\theta) = 0$.

It is easy to show that the outside option and the equilibrium optimal contracts solves

$$U = \max_{\{y, k\}} \int_{\theta}^\overline{\theta} U(\theta + y(\theta), k(\theta)) dF(\theta)$$

subject to

$$q = \int_{\theta}^\overline{\theta} (y(\theta) + qk(\theta)) dF(\theta)$$

and

$$U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\theta'), k(\theta')).$$
The following proposition shows that in general equilibrium the distortion towards under consumption of housing shown in Proposition 1 leads to lower house price.

**Proposition 6.** Assume that \( U(c, k) = \alpha \log(c) + (1 - \alpha) \log(k) \). In general equilibrium under asymmetric information the house price is strictly less than the equilibrium house price under full information.

In the competitive equilibrium under full information

\[
\frac{1 - \alpha}{\alpha} (\theta + y^{FI}(\theta)) = q^{FI} k^{FI}(\theta)
\]

for all \( \theta \). Integrating this equality from \( \underline{\theta} \) to \( \overline{\theta} \) and using the market clearing conditions, we arrive at

\[
q^{FI} = \frac{1 - \alpha}{\alpha} \mathbb{E}[\theta].
\]

However, under asymmetric information, by the distortion result in Proposition 1

\[
\frac{1 - \alpha}{\alpha} (\theta + y^{*}(\theta)) > q^{*} k^{*}(\theta)
\]

for all \( \theta \in (\underline{\theta}, \overline{\theta}) \). After integrating this inequality from \( \underline{\theta} \) to \( \overline{\theta} \) and using the market clearing conditions, we obtain

\[
q^{*} < \frac{1 - \alpha}{\alpha} \mathbb{E}[\theta] = q^{FI}.
\]

When \( \alpha = 0.5 \) and the distribution of types is uniform, Proposition 4 provides a closed form solution to the optimal contract given any house price. The following example uses this closed form solution to calculate numerically the house price as a function of the mean income.

**Example 5.** Consider the case with log-log utility function with equal weight on composite goods and housing and uniform distribution as in Subsection 5. Figure 13 shows the competitive equilibrium house price, compared to the full information competitive equilibrium price, when we vary \( \mathbb{E}[\theta] \) while keeping \( \overline{\theta} - \underline{\theta} = 2 \). We observe that the competitive equilibrium price always lie below the full information price and the difference between the two prices is larger at lower levels of \( \theta \). This result might speak to the low house price during the last financial crisis 2007-2009 and the subsequent recession.
8 Summary

This paper proposes a theory of endogenous collateral based on optimal contracting with private information. The optimal contract requires the use of asset forfeiture (collateral) as a means to implement the “second best” distortion between durable capital and non-capital composite consumption. The distortion entails under-consumption of capital and overconsumption of non-capital goods in the period in which the income shock is realized. This, in turn, leads to too little savings and accumulation of collateralizable capital.

We show that all optimal contracts under asymmetric information are collateralized, while optimal contracts with full information are not. We show that forfeiture is strictly decreasing in income at the tails of the distribution, and in the log-uniform model, is strictly decreasing everywhere. This regressive feature of collateral limits the amount of insurance the borrower can receive through the financial system.

A few important issues remain unresolved. First, our understanding of the distributional attributes of collateralized contracts is incomplete. Beyond the log uniform model, we cannot say if forfeiture is always decreasing in income or how the composition of collateralized assets changes in $\theta$. One would like to know more about how the contract differs across ex ante heterogeneous agents.

Second, in the two period model here, the dynamics are highly circumscribed. We show that the three period version of the Green (1987) model maps to ours when payoffs are additively separable. This leads us to think that in an infinite horizon extension of our model with income shocks each period, Green’s use of an intangible statistic — accumulated credit/debt — to allocate consumption in the OC might still be relevant even when tangible capital is available.

Finally, the baseline theory takes prices as given, and so our baseline describes a locally optimal contract between the two contracting parties. The extension to a GE model suggests that perfect competition can “partially mitigate” the distortion and thus reduce forfeiture. It remains unclear to what extent this is so.

These questions indicate that the theory is far from complete, but also suggest a number of viable paths forward.
References


Appendix

A  Preliminaries: Implications of Assumption 1

For the proofs in this section, we re-write condition (1) as

\[ \frac{\partial U^2}{\partial c^2} \frac{\partial U}{\partial k} - \frac{\partial^2 U}{\partial c \partial k} \frac{\partial U}{\partial c} < 0 \]  

(52)

and

\[ \frac{\partial U^2}{\partial k^2} \frac{\partial U}{\partial c} - \frac{\partial^2 U}{\partial c \partial k} \frac{\partial U}{\partial k} < 0 \]  

(53)

for all \( c, k \).

The first condition (52) is equivalent to the Strict Single Crossing Condition (SSCC) for \( \hat{U}(y, k, \theta) \equiv U(\theta + y, k) \).

Indeed,

\[ \frac{\partial}{\partial \theta} \left( \frac{\partial \hat{U}}{\partial k} \right) = \frac{\partial^2 U}{\partial k \partial c} \frac{\partial U}{\partial c} - \frac{\partial^2 U}{\partial c^2} \frac{\partial U}{\partial k} \right) > 0. \]  

(54)

Lemma 4 also shows that this condition is equivalent to the normality of capital good, \( k \). Similarly, the second condition (53) is equivalent to the normality of the composite good, \( c \).

The following lemma is a generalization of Dixit and Seade (1979) and is important for the change of variables used in Section 6.

Lemma 3. If \((c_0, k_0), (c_1, k_1)\) satisfy

\[ \frac{\partial U(c_0, k_0)}{\partial c} = \frac{\partial U(c_1, k_1)}{\partial c}, \]

\[ \frac{\partial U(c_0, k_0)}{\partial k} = \frac{\partial U(c_1, k_1)}{\partial k}, \]

then

\[ (c_0, k_0) = (c_1, k_1). \]
Proof. If \((c_0, k_0) \neq (c_1, k_1)\), by strict concavity of \(U\):

\[
U(c_0, k_0) - U(c_1, k_1) < \frac{\partial U(c_1, k_1)}{\partial c}(c_0 - c_1) + \frac{\partial U(c_1, k_1)}{\partial k}(k_0 - k_1).
\]

Similarly

\[
U(c_1, k_1) - U(c_0, k_0) < \frac{\partial U(c_0, k_0)}{\partial c}(c_1 - c_0) + \frac{\partial U(c_0, k_0)}{\partial k}(k_1 - k_0)
= \frac{\partial U(c_1, k_1)}{\partial c}(c_1 - c_0) + \frac{\partial U(c_1, k_1)}{\partial k}(k_1 - k_0).
\]

Therefore

\[
U(c_0, k_0) - U(c_1, k_1) > \frac{\partial U(c_1, k_1)}{\partial c}(c_0 - c_1) + \frac{\partial U(c_1, k_1)}{\partial k}(k_0 - k_1),
\]

which contradicts the earlier inequality. Thus \((c_0, k_0) = (c_1, k_1)\). \(\square\)

**Lemma 4.** Conditions (52) and (53) are satisfied for all \(c, k\) if and only if given any vector of prices \(p > 0\) an increase in income increases both \(c\) and \(k\) consumption, i.e. \(c\) and \(k\) are normal goods.

Proof. First we show the sufficient condition, i.e. if \(c, k\) are normal goods then (52) and (53) are satisfied for all \(c, k\). Indeed, given \(c, k\), consider the following income and prices:

\[
m = \frac{\partial U(c, k)}{\partial c}c + \frac{\partial U(c, k)}{\partial k}k \quad \text{and} \quad p = \left[\begin{array}{c}
\frac{\partial U(c, k)}{\partial c} \\
\frac{\partial U(c, k)}{\partial k}
\end{array}\right] > 0.
\]

It is easy to verify that \((c, k)\) is the optimal consumption at \(m\) and \(p\).

Let \(D(m, p)\) denote the demand function associated with the utility function \(U\). The standard algebra derivations for \(D\) implies

\[
\frac{\partial D}{\partial m} = \left[\begin{array}{cc}
\frac{\partial^2 U(c, k)}{\partial c^2} & \frac{\partial^2 U(c, k)}{\partial c \partial k} \\
\frac{\partial^2 U(c, k)}{\partial c \partial k} & \frac{\partial^2 U(c, k)}{\partial k^2}
\end{array}\right]^{-1} \left[\begin{array}{c}
p' \\
p'
\end{array}\right] > 0,
\]

where

\[
p' = \left[\begin{array}{c}
\frac{\partial^2 U(c, k)}{\partial c^2} & \frac{\partial^2 U(c, k)}{\partial c \partial k} \\
\frac{\partial^2 U(c, k)}{\partial c \partial k} & \frac{\partial^2 U(c, k)}{\partial k^2}
\end{array}\right]^{-1} \left[\begin{array}{c}
p \\
p
\end{array}\right].
\]
because both housing and consumption are normal goods. Since \( p = \begin{bmatrix} \frac{\partial U(c,k)}{\partial c} \\ \frac{\partial U(c,k)}{\partial k} \end{bmatrix} \) and

\[
\begin{bmatrix}
\frac{\partial^2 U(c,k)}{\partial c^2} & \frac{\partial^2 U(c,k)}{\partial c \partial k} \\
\frac{\partial^2 U(c,k)}{\partial c \partial k} & \frac{\partial^2 U(c,k)}{\partial k^2}
\end{bmatrix}^{-1}
\]

\( p < 0 \), (from the concavity of \( U \)), the last inequality yields

\[
\begin{bmatrix}
\frac{\partial^2 U(c,k)}{\partial c^2} & \frac{\partial^2 U(c,k)}{\partial c \partial k} \\
\frac{\partial^2 U(c,k)}{\partial c \partial k} & \frac{\partial^2 U(c,k)}{\partial k^2}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\partial U(c,k)}{\partial c} \\
\frac{\partial U(c,k)}{\partial k}
\end{bmatrix}
< 0.
\]

By direct algebra, the last inequality is equivalent to (52) and (53), given that \( \left( \frac{\partial^2 U(c,k)}{\partial c \partial k} \right)^2 > 0 \) due to the concavity of \( U \).

The necessary condition is proved in exactly the same way. \( \square \)

## B Proof of Lemma 1

We divide the proof of Lemma 1 in two parts. In the first part, Lemma 5, we show the necessary condition, i.e. any incentive compatible contract must satisfy the monotonicity condition (5) and the envelope conditions (6) and (7). In the second part, Lemma 6, we show that the two properties guarantee incentive compatibility.

**Lemma 5.** Given any incentive compatible contract, for \( \theta > \theta' \), we have \( y(\theta) \leq y(\theta') \) and \( k(\theta) \geq k(\theta') \). In addition, the envelope conditions (6) and (7) are satisfied for all \( \theta \).

**Proof.** From the IC constraint for type \( \theta \):

\[
U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\theta'), k(\theta')) ,
\]

we rule out the possibility that \( y(\theta') \geq y(\theta) \) and \( k(\theta') \geq k(\theta) \), with at least on strictly inequality. Similarly from the IC constraint for type \( \theta' \), we rule out the possibility that
\( y(\theta') \leq y(\theta) \) and \( k(\theta') \leq k(\theta) \), with at least one strict inequality. Therefore to obtain this lemma, we just need to eliminate the possibility that \( y(\theta) \geq y(\theta') \) and \( k(\theta) \leq k(\theta') \) with at least one strict inequality.

We show this by contradiction. Suppose that it is true. Let

\[ \check{U}(\check{y}, k, \theta) \equiv U(\theta - \check{y}, k). \]

Then \( \check{U} \) satisfies the Strict Single Crossing Condition (SSCC) because it satisfies the Spence-Mirlees condition (see Milgrom and Shannon (1994) for the exact definition of these conditions):

\[
\frac{\partial}{\partial \theta} \left( \frac{\partial \check{U}/\partial \check{y}}{\partial \check{U}/\partial k} \right) = \frac{\partial}{\partial \theta} \left( -\frac{\partial U(\theta - \check{y}, k)}{\partial c} \right) > 0,
\]

where the last inequality is equivalent to (52). By Milgrom and Shannon (1994, Theorem 3), since

\[ (-y(\theta), k(\theta)) < (-y(\theta'), k(\theta')) , \]

and

\[ \check{U} (-y(\theta'), k(\theta'), \theta') \geq \check{U} (-y(\theta), k(\theta), \theta') , \]

we have

\[ \check{U} (-y(\theta'), k(\theta'), \theta) > \check{U} (-y(\theta), k(\theta), \theta) , \]

or equivalently

\[ U (\theta + y(\theta'), k(\theta')) > U (\theta + y(\theta), k(\theta)) , \]

which contradicts the IC constraint for \( \theta \). Therefore by contradiction, we obtain the monotonicity property.

Now we turn to the envelope conditions. For \( \theta' < \theta \), we write the IC constraint for type \( \theta \) as

\[
U (\theta + y(\theta), k(\theta)) - U (\theta' + y(\theta'), k(\theta')) \geq U (\theta + y(\theta'), k(\theta')) - U (\theta' + y(\theta'), k(\theta')).
\]

Dividing both-side by \( \theta - \theta' \) and take the limit \( \theta' \to \theta \), we obtain

\[
\frac{dU}{d\theta}(\theta + y(\theta), k(\theta)) \geq \frac{\partial U}{\partial c} (\theta + y(\theta), k(\theta)).
\]

The second half of the envelope condition is obtained similarly by considering the IC
constraint for type $\theta'$.  

\[ \text{Lemma 6. Any piecewise differentiable allocation that satisfies the monotonicity condition, (5) and envelope conditions (6) and (7) satisfies the incentive constraint.} \]

\[ \text{Proof. We show that if the first part of (6) holds, then the global downward IC property holds, i.e. for } \theta' < \theta \]

\[ U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\theta'), k(\theta')). \]

Indeed, the inequality is equivalent to

\[ \int_{\theta'}^{\theta} \frac{d}{d\theta} (U(\theta + y(\theta'), k(\theta))) \ d\theta \geq 0. \quad (55) \]

We use the differential form of the (left) Envelope Condition:

\[ \frac{\partial U(\tilde{\theta} + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial c} \frac{dy^-}{d\tilde{\theta}} + \frac{\partial U(\tilde{\theta} + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial k} \frac{dk^-}{d\tilde{\theta}} \geq 0 \]

to estimate the intergrand above as:

\[ \frac{d^-}{d\tilde{\theta}} (U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))) = \left( \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial c} \frac{dy^-}{d\tilde{\theta}} + \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial k} \frac{dk^-}{d\tilde{\theta}} \right) \]

\[ \geq \left( \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial c} \left( - \frac{\partial U(\tilde{\theta} + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial c} \frac{\partial U(\tilde{\theta} + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial k} \right) + \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial k} \right) \frac{dk^-}{d\tilde{\theta}}. \]

Because of the SSCC, (54),

\[ \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial k} \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial c} > \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial k} \frac{\partial U(\theta + y(\tilde{\theta}), k(\tilde{\theta}))}{\partial c}', \]

and because of the monotonicity property $\frac{dk^-}{d\tilde{\theta}} \geq 0$. Therefore $\frac{d^-}{d\tilde{\theta}} (U(\theta + y(\tilde{\theta}), k(\tilde{\theta})))$ is positive, thus the integral (55) is positive. So the global downward IC property holds.

Similarly, we can show that if the second part of (7) holds then the global upward IC property holds.

\[\square\]
C Integration-by-Parts for the Reformulated Saddle Problem

The integration-by-parts arguments needed to derive the reformulated saddle problem are given by

\[
\int_{\theta}^{\bar{\theta}} \zeta(\theta) \frac{dU}{d\theta}(\theta + y(\theta), k(\theta)) d\theta = \\
\zeta(\bar{\theta}) U(\bar{\theta} + y(\bar{\theta}), k(\bar{\theta})) - \zeta(\theta) U(\theta + y(\theta), k(\theta)) - \int_{\theta}^{\bar{\theta}} U(\theta + y(\theta), k(\theta)) \zeta'(\theta) d\theta
\]

and

\[
\int_{\theta}^{\bar{\theta}} \eta(\theta) \left( k(\theta) - k(\bar{\theta}) - \int_{\theta}^{\bar{\theta}} j(\bar{\theta}) d\bar{\theta} \right) d\theta = \\
\int_{\theta}^{\bar{\theta}} \eta(\theta) \left( k(\theta) - k(\bar{\theta}) \right) d\theta - \int_{\theta}^{\bar{\theta}} j(\theta) \left( \int_{\theta}^{\bar{\theta}} \eta(\bar{\theta}) d\bar{\theta} \right) d\theta.
\]

D Proof of Lemma 2

Since we consider a "weakly relaxed problem" in which only downward incentive compatibility and monotonicity are imposed, the multiplier \(\zeta(\theta)\) on the local downward IC constraint (6) is positive by assumption. We then show that the optimal solution to this "weakly relaxed problem" also satisfies the global incentive constraint, therefore it is also the optimal solution to the original problem.

Let

\[
v(\theta) = U(\theta + y(\theta), k(\theta)).
\]

The local downward IC constraint (6) can be written as (for simplicity we assume differentiability)

\[
v'(\theta) - \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) \geq 0.
\]  
(56)
We also require that \( k(\cdot) \) is non-decreasing in \( \theta \), i.e. (10) and (11) hold. As shown in the proof of Lemma 6, the local downward IC constraint (56) and monotonicity constraint imply the global downward IC constraint.

We write the Lagrangian of this “weakly relaxed problem” in the second period as

\[
\mathcal{L}_C = \int_{\bar{\theta}}^{\tilde{\theta}} U (\theta + y(\theta), k(\theta)) \, dF(\theta) \\
+ \lambda \left( B - \int_{\bar{\theta}}^{\tilde{\theta}} (y(\theta) + qk(\theta)) \, dF(\theta) \right) \\
+ \int_{\bar{\theta}}^{\tilde{\theta}} \xi(\theta) \left( v'(\theta) - \frac{\partial U}{\partial c} (\theta + y(\theta), k(\theta)) \right) \, d\theta \\
+ \int_{\bar{\theta}}^{\tilde{\theta}} \eta(\theta) \left( k(\theta) - k(\bar{\theta}) - \int_{\bar{\theta}}^{\theta} j(\tilde{\theta}) \, d\tilde{\theta} \right) \, d\theta + \int_{\bar{\theta}}^{\tilde{\theta}} \gamma(\theta) j(\theta) \, d\theta.
\]

Because of the constraints (56) and (11), \( \xi \geq 0 \) and \( \gamma \geq 0 \).

We use the integral by parts in Appendix C and let \( \mu(\theta) = -\xi'(\theta) \) to write \( \mathcal{L}_C \) as

\[
\mathcal{L}_C = \int_{\bar{\theta}}^{\tilde{\theta}} U (\theta + y(\theta), k(\theta)) \, (f(\theta) + \mu(\theta)) \, d\theta \\
+ \lambda \left( B - \int_{\bar{\theta}}^{\tilde{\theta}} (y(\theta) + qk(\theta)) \, f(\theta) \, d\theta \right) \\
- \int_{\bar{\theta}}^{\tilde{\theta}} \xi(\theta) \frac{\partial U}{\partial c} (\theta + y(\theta), k(\theta)) \, d\theta + \xi(\tilde{\theta}) v(\tilde{\theta}) - \xi(\bar{\theta}) v(\bar{\theta}) \\
+ \int_{\bar{\theta}}^{\tilde{\theta}} \eta(\theta) \left( k(\theta) - k(\bar{\theta}) - \int_{\bar{\theta}}^{\theta} j(\tilde{\theta}) \, d\tilde{\theta} \right) \, d\theta - \int_{\bar{\theta}}^{\tilde{\theta}} j(\theta) \left( \int_{\bar{\theta}}^{\tilde{\theta}} \eta(\tilde{\theta}) \, d\tilde{\theta} + \gamma(\theta) \right) \, d\theta.
\]

We use shortcuts \( \frac{\partial U}{\partial c}(\theta), \frac{\partial^2 U}{\partial c^2}(\theta) \) to write the F.O.Cs as follow.

F.O.C. in \( y(\theta) \)

\[
\frac{\partial U}{\partial c}(\theta) (\mu(\theta) + f(\theta)) - \lambda f(\theta) = \frac{\partial^2 U}{\partial c^2}(\theta) \xi(\theta). \tag{57}
\]

F.O.C. in \( k(\theta) \)

\[
\frac{\partial U}{\partial k}(\theta) (\mu(\theta) + f(\theta)) + \eta(\theta) - \lambda q f(\theta) = \frac{\partial^2 U}{\partial c \partial k}(\theta) \xi(\theta). \tag{58}
\]
F.O.C. in $j(\theta)$

\[ \gamma(\theta) - \int_{\theta}^{\bar{\theta}} \eta(\bar{\theta}) d\bar{\theta} = 0. \quad (59) \]

F.O.C. in $k(\bar{\theta})$

\[ U_k(\bar{\theta}) \xi(\bar{\theta}) = 0. \quad (60) \]

F.O.C. in $y(\bar{\theta})$:

\[ U_c(\bar{\theta}) \xi(\bar{\theta}) = 0. \quad (61) \]

F.O.C. in $k(\bar{\theta})$:

\[ -U_k(\bar{\theta}) \xi(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} \eta(\theta) d\theta = 0. \quad (62) \]

Because $U_k(\bar{\theta}) > 0$, (60) implies that $\xi(\bar{\theta}) = 0$. Similarly, because $U_c(\bar{\theta}) > 0$, (61) implies $\xi(\bar{\theta}) = 0$.

Combining this result with (62) yields

\[ \int_{\theta}^{\bar{\theta}} \eta(\theta) d\theta = 0. \quad (63) \]

Therefore, from (59),

\[ \gamma(\bar{\theta}) = 0. \quad (64) \]

Also from (59),

\[ \gamma(\theta) = 0. \quad (65) \]

Armed with these properties, Lemma 9 below show that in the optimal solution of the "weakly relaxed problem," the local incentive constraint is satisfied. Lemma 6 then shows that the global incentive constraint is satisfied. Lemma 9 uses the following two Lemmas.

**Lemma 7.** In an optimal solution to the "weakly relaxed problem", if $k(\theta)$ is constant over some interval $[\theta_1, \theta_2] \in [\theta, \bar{\theta}]$ then $y(\theta)$ is constant over the same interval.

**Proof.** Assume that $k(\theta) = k^*$ over $[\theta_1, \theta_2] \in [\theta, \bar{\theta}]$. By downward incentive compatibility, $y(\theta)$ is non-decreasing over $[\theta_1, \theta_2]$.
We show the result in this lemma by contradiction. Assume that \( y(\theta) \) is not constant over the same interval because \( y \) is continuous, there exists a non-degenerate subinterval \([\theta', \theta''] \in [\theta_1, \theta_2]\) such that \( y(.) \) is strictly increasing over this interval.

In this interval
\[
v'(\theta) = \frac{d}{d\theta} (U(\theta + y(\theta), k^*)) = \frac{\partial U}{\partial c} (\theta + y(\theta), k^*) + \frac{\partial U}{\partial c} (\theta + y(\theta), k^*) \frac{dy}{d\theta} > \frac{\partial U}{\partial c} (\theta + y(\theta), k^*).\]

Therefore (56) does not bind, i.e. \( \zeta(\theta) = 0 \) for \( \theta \in [\theta', \theta''] \). Since \( \mu = -\zeta', \mu(\theta) = 0 \) for \( \theta \in [\theta', \theta''] \). (57) then implies
\[
\frac{\partial U}{\partial c} (\theta + y(\theta), k^*) = \lambda
\]
for \( \theta \in [\theta', \theta''] \). Differentiate both sides with respect to \( \theta \), we have
\[
\frac{\partial^2 U}{\partial c^2} (\theta + y(\theta), k^*) \left( 1 + \frac{dy}{d\theta} \right) = 0.
\]

This is a contradiction since \( \frac{\partial^2 U}{\partial c^2} < 0 \) and \( \frac{dy}{d\theta} > 0 \). \( \square \)

**Lemma 8.** In an optimal solution to the "weakly relaxed problem," for each \( \theta^* \in (\theta, \bar{\theta}) \), if \( \zeta(\theta^*) = 0 \) then \( \gamma(\theta^*) > 0 \).

**Proof.** We show this result by contradiction. Assume that \( \gamma(\theta^*) = 0 \).

We rewrite (57) as
\[
\zeta''(\theta) + \frac{\partial^2 U}{\partial c^2} (\theta) \zeta(\theta) = f(\theta) \left( 1 - \frac{\lambda}{\frac{\partial U}{\partial c} (\theta)} \right)
\]
where \( \frac{\partial U}{\partial c} (\theta), \frac{\partial^2 U}{\partial c^2} (\theta), \frac{\partial^2 U}{\partial c d k} (\theta) \) are shortcuts. Using the fact that \( \zeta(\theta) = 0 \), we obtain
\[
\zeta(\theta) = \exp \left( - \int_\theta^0 \frac{\partial^2 U}{\partial c^2} (\bar{\theta}) d\bar{\theta} \right) g(\theta),
\]
where
\[ g_1(\theta) = \int_\theta^0 \exp \left( \int_\theta^{\theta_1} \frac{\partial^2 U}{\partial c^2}(\hat{\theta}) \ d\hat{\theta} \right) \left( 1 - \frac{\lambda q}{\partial U/\partial c(\theta)} \right) dF(\theta_1). \]

Because \( g \geq 0 \), \( g_1(\theta) \geq 0 \) for all \( \theta \in (\theta, \theta_\bar{\theta}) \). In addition, \( g_1(\theta^*) = 0 \). Therefore \( \theta^* \) is a local minimum of \( g_1 \). Therefore, \( g'_1(\theta^*) = 0 \) and \( g''_1(\theta^*) \geq 0 \). By the definition of \( g_1 \), this is equivalent to,

\[ 1 - \frac{\lambda q}{\partial U/\partial c(\theta^*)} = 0 \]

and

\[ \frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\partial U/\partial c(\theta)} \right) \geq 0 \]

and \( \theta = \theta^* \). Equivalently, at \( \theta = \theta^* \)

\[ x = \frac{d}{d\theta} \left\{ \frac{\partial U}{\partial c}(\theta) \right\} \geq 0. \quad (66) \]

Similarly, we rewrite (58) as

\[ \xi'(\theta) + \frac{\partial^2 U}{\partial c \partial k}(\theta)\xi(\theta) = f(\theta) \left( 1 - \frac{\lambda q}{\partial U/\partial k(\theta)} + \frac{\eta(\theta)}{\partial U/\partial k(\theta) f(\theta)} \right). \]

Again, since \( \xi(\theta) = 0 \), we have

\[ \xi(\theta) = \exp \left( - \int_\theta^0 \frac{\partial^2 U}{\partial c \partial k}(\hat{\theta}) \ d\hat{\theta} \right) g_2(\theta), \]

where

\[ g_2(\theta) = \int_\theta^0 \exp \left( \int_\theta^{\theta_1} \frac{\partial^2 U}{\partial c \partial k}(\hat{\theta}) \ d\hat{\theta} \right) \left( 1 - \frac{\lambda q}{\partial U/\partial k(\theta_1)} + \frac{\eta(\theta_1)}{\partial U/\partial k(\theta_1) f(\theta_1)} \right) dF(\theta_1). \]

Because \( g \geq 0 \), \( g_2 \geq 0 \). In addition, \( g_2(\theta^*) = 0 \). Therefore \( \theta^* \) is a local minimum of \( g_2 \). Thus, \( g'_2(\theta^*) = 0 \) and \( g''_2(\theta^*) \geq 0 \). By the definition of \( g_2 \), this is equivalent to

\[ 1 - \frac{\lambda q}{\partial U/\partial k(\theta^*)} + \frac{\eta(\theta^*)}{\partial U/\partial k(\theta^*) f(\theta^*)} = 0 \]

(67)
and at $\theta = \theta^*$:

$$
\frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\partial U / \partial k (\theta)} + \frac{\eta (\theta)}{\partial U / \partial k (\theta)} f(\theta) \right) \geq 0. \quad (68)
$$

Notice also that $\gamma \geq 0$ and $\gamma' (\theta) = -\eta (\theta)$ and $\gamma'' (\theta) = -\eta' (\theta)$. In addition $\gamma (\theta^*) = 0$. Therefore $\theta^*$ is a local minimum of $\gamma$. Thus $\gamma' (\theta^*) = -\eta (\theta^*) = 0$ and $\gamma'' (\theta^*) = -\eta' (\theta^*) \geq 0$. Plugging the first equality into (67) implies

$$
1 - \frac{\lambda q}{\partial U / \partial k (\theta^*)} = 0.
$$

Plugging the second inequality into (68) implies

$$
\frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\partial U / \partial k (\theta^*)} \right) + \frac{1}{\partial U / \partial k (\theta^*)} \frac{d}{d\theta} (\eta (\theta)) \geq 0. \quad (69)
$$

Since $\frac{d}{d\theta} (\eta (\theta)) \leq 0$, the inequality above implies that $\frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\partial U / \partial k (\theta^*)} \right) \geq 0$ at $\theta = \theta^*$. Therefore,

$$
z = \frac{d}{d\theta} \left\{ \frac{\partial U}{\partial k} (\theta^*) \right\} \geq 0. \quad (70)
$$

However, (66) and (70) contradict the normality of $c$ and $k$.

Indeed by total differentiation

$$
z = \frac{\partial^2 U}{\partial k \partial c} \frac{dc}{d\theta} + \frac{\partial^2 U}{\partial k^2} \frac{dk}{d\theta}
$$

and

$$
x = \frac{\partial^2 U}{\partial c^2} \frac{dc}{d\theta} + \frac{\partial^2 U}{\partial c \partial k} \frac{dk}{d\theta}.
$$

So

$$
\begin{bmatrix}
\frac{dc}{d\theta} \\
\frac{dk}{d\theta}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 U(c,k)}{\partial c^2} & \frac{\partial^2 U(c,k)}{\partial c \partial k} \\
\frac{\partial^2 U(c,k)}{\partial c \partial k} & \frac{\partial^2 U(c,k)}{\partial k^2}
\end{bmatrix}^{-1} \begin{bmatrix}
x \\
z
\end{bmatrix}.
$$
Besides,

\[
\frac{d}{d\theta} \{ U(c(\theta), k(\theta)) \} = \left[ \begin{array}{cc} \frac{\partial U}{\partial c} & \frac{\partial U}{\partial k} \\ \frac{dc}{d\theta} & \frac{dk}{d\theta} \end{array} \right] = \frac{\partial U(c(\theta), k(\theta))}{\partial c} > 0.
\]

On the other hand

\[
\left[ \begin{array}{cc} \frac{\partial U}{\partial c} & \frac{\partial U}{\partial k} \\ \frac{dc}{d\theta} & \frac{dk}{d\theta} \end{array} \right] = \left[ \begin{array}{cc} \frac{\partial^2 U(c,k)}{\partial c^2} & \frac{\partial^2 U(c,k)}{\partial c \partial k} \\ \frac{\partial^2 U(c,k)}{\partial c \partial k} & \frac{\partial^2 U(c,k)}{\partial k^2} \end{array} \right]^{-1} \left[ \begin{array}{c} x \\ z \end{array} \right] \leq 0,
\]

since, by (1)

\[
\left[ \begin{array}{cc} \frac{\partial U}{\partial c} & \frac{\partial U}{\partial k} \\ \frac{dc}{d\theta} & \frac{dk}{d\theta} \end{array} \right] \left[ \begin{array}{cc} \frac{\partial^2 U(c,k)}{\partial c^2} & \frac{\partial^2 U(c,k)}{\partial c \partial k} \\ \frac{\partial^2 U(c,k)}{\partial c \partial k} & \frac{\partial^2 U(c,k)}{\partial k^2} \end{array} \right]^{-1} < 0
\]

and \(x, z \geq 0\). This is the desired contradiction. \(\square\)

Given Lemma 7 and Lemma 8, it is relatively straightforward to show the main result.

**Lemma 9.** In the optimal solution to the "weakly relaxed problem," for all \(\theta \in (\theta, \bar{\theta})\),

\[ v'(\theta) - \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) = 0. \]

**Proof.** We show this result by contradiction. If there exists \(\theta^*\) such that this is not true:

\[ v'(\theta^*) > \frac{\partial U}{\partial c}(\theta^* + y(\theta^*), k(\theta^*)). \tag{71} \]

Then \(\zeta(\theta^*) = 0\). By Lemma 8, \(\gamma(\theta^*) > 0\), therefore by continuity \(\gamma(\theta) > 0\) in some neighborhood of \(\theta^*\). So \(k(\theta) = k^*\) in this neighborhood. By Lemma 7, \(y(\theta) = y^*\) in this neighborhood. This however contradicts (71). \(\square\)

We have established that the optimal solution to the "weakly relaxed problem" satisfies the incentive compatibility constraint, therefore it is also an optimal solution to the original problem.


E Proof of Proposition 1

Lemma 2 implies that the multiplier $\xi(\theta)$ in the Lagrangians (12) and (14) is positive for all $\theta$. With this property, we are now ready to prove the properties in Proposition 1.

Proof of Proposition 1, Parts (i) and (ii). Consider $\theta^* \in (\bar{\theta}, \tilde{\theta})$. There are two cases:

Case 1: $\gamma'(\theta^*) > 0$ then there is pooling at $\theta^*$. Let $\theta^{**}$ denote the left most point, such that there is pooling from $\theta^{**}$ to $\theta^*$. Formally

$$\theta^{**} = \inf \{ \theta \in [\bar{\theta}, \theta^*] : \gamma(\theta_1) > 0 \text{ for all } \theta_1 \in (\theta, \theta^*) \}.$$  

Then $\gamma(\theta^{**}) = 0$ and $\gamma(\theta) > 0$ for all $\theta \in (\theta^{**}, \theta^*)$ (this comes from definition of $\theta^{**}$ if $\theta^{**} > \bar{\theta}$. If $\theta^{**} = \bar{\theta}$, then this is also true since $\gamma(\bar{\theta}) = 0$). Consequently,

$$\gamma'(\theta^{**}) = -\eta(\theta^{**}) \geq 0.$$  

We show that

$$\frac{\partial U}{\partial k}(\theta^{**} + y(\theta^{**}), k(\theta^{**})) \geq q. \quad (72)$$

From (57) and (58) at $\theta^{**}$

$$\xi(\theta^{**}) \left( \frac{\partial U}{\partial c}(\theta^{**}) \frac{\partial^2 U}{\partial c \partial k}(\theta^{**}) - \frac{\partial U}{\partial k} (\theta^{**}) \frac{\partial^2 U}{\partial c^2} (\theta^{**}) \right)$$

$$= \lambda f(\theta^{**}) \left( \frac{\partial U}{\partial k}(\theta^{**}) - q \frac{\partial U}{\partial c}(\theta^{**}) \right) + \eta(\theta^{**}) \frac{\partial U}{\partial c}(\theta^{**}).$$

Together with $\xi(\theta^{**}) \geq 0$, $\eta(\theta^{**}) \leq 0$ and (52) at $\theta^{**}$, we obtain (72).

Now, since there is pooling over $(\theta^{**}, \theta^*)$, and because of (54) we have

$$\frac{\partial U}{\partial k}(\theta^* + y(\theta^*), k(\theta^*)) > \frac{\partial U}{\partial c}(\theta^* + y(\theta^*), k(\theta^*)) \geq q.$$

Case 2: $\gamma(\theta^*) = 0$. By Lemma 8 $\xi(\theta^*) > 0$. 

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Since $\gamma(\theta^*) = 0$ and $\gamma(\theta) \geq 0$ for all $\theta$,

$$\gamma'(\theta^*) = -\eta(\theta^*) \geq 0.$$  

From (57) and (58) at $\theta^*$

$$\zeta(\theta^*) \left( \frac{\partial U}{\partial c} (\theta^*) \frac{\partial^2 U}{\partial c \partial k} (\theta^*) - \frac{\partial U}{\partial k} (\theta^*) \frac{\partial^2 U}{\partial c^2} (\theta^*) \right)$$

$$= \lambda f(\theta^*) \left( \frac{\partial U}{\partial k} (\theta^*) - q \frac{\partial U}{\partial c} (\theta^*) \right) + \eta(\theta^*) \frac{\partial U}{\partial c} (\theta^*).$$

Together with $\zeta(\theta^*) > 0$, $\eta(\theta^*) \leq 0$ and (52) at $\theta^*$, we obtain

$$\frac{\partial U}{\partial k} (\theta^* + y(\theta^*), k(\theta^*)) > q.$$  

Similarly, if there is no pooling at the bottom then there is no distortion at the bottom. By the continuity of $c$ and $k$, there is also (weakly) positive distortion at the bottom.

**Proof of Proposition 1, Part (iii).** Assume by contradiction that this property does not hold, i.e. there is pooling at the top. Let $[\theta^*, \bar{\theta}]$ denote the maximum pooling interval. If $\theta^* > \bar{\theta}$, the continuity of $\theta^*$, $\gamma(\theta^*) = 0$ and (58) at $\bar{\theta}$ then implies that

$$\frac{\partial U}{\partial c} (\bar{\theta} + y(\bar{\theta}), k(\bar{\theta})) = q.$$  

Evaluating (57) and (58) at $\bar{\theta}$, we have

$$\frac{\partial U}{\partial c} (\bar{\theta}) (\mu(\bar{\theta}) + f(\bar{\theta})) - \lambda f(\bar{\theta}) = 0.$$  

This implies $(\mu(\bar{\theta}) + f(\bar{\theta})) > 0$. And

$$\frac{\partial U}{\partial k} (\theta) (\mu(\bar{\theta}) + f(\bar{\theta})) + \eta(\bar{\theta}) - \lambda q f(\theta) = 0.$$  

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Since $\gamma(\bar{\theta}) = 0$ and $\gamma(\theta) \geq 0$ for all $\theta$, $\eta(\bar{\theta}) = -\gamma'(\bar{\theta}) \geq 0$. Therefore
\[
\frac{\partial U}{\partial k} (\bar{\theta}) \leq q. \tag{73}
\]
From (57) and (58) at $\theta^*$
\[
\zeta(\theta^*) \left( \frac{\partial U}{\partial c} (\theta^*) \frac{\partial^2 U}{\partial c^2} (\theta^*) - \frac{\partial U}{\partial k} (\theta^*) \frac{\partial^2 U}{\partial c^2} (\theta^*) \right) = \lambda f(\theta^*) \left( \frac{\partial U}{\partial k} (\theta^*) - q \frac{\partial U}{\partial c} (\theta^*) \right) + \eta(\theta^*) \frac{\partial U}{\partial c} (\theta^*). 
\]
Because $\gamma(\theta^*) = 0$ and $\gamma(\theta) \geq 0$ for all $\theta \geq \theta^*$, $\eta(\theta^*) = -\gamma'(\theta^*) \leq 0$. In addition we have $\zeta(\theta^*) \geq 0$ and (52) at $\theta^*$, so
\[
\frac{\partial U}{\partial k} (\theta^*) \geq q. \tag{74}
\]
Combining (73) and (74), we obtain:
\[
\frac{\partial U}{\partial k} (\theta^*) \geq \frac{\partial U}{\partial k} (\bar{\theta}). 
\tag{75}
\]
This is a contradiction since there is pooling over $[\theta^*, \bar{\theta}]$ and therefore
\[
\frac{d}{d\theta} \left( \frac{\partial U}{\partial c} (\theta) \right) = \frac{\partial}{\partial \theta} \left( \frac{\partial U}{\partial k} (\theta) \right) > 0, 
\]
for $\theta \in [\theta^*, \bar{\theta}]$, which implies
\[
\frac{\partial U}{\partial k} (\theta^*) < \frac{\partial U}{\partial k} (\bar{\theta}), 
\]
contradicting (75). 

\hspace{1cm} \Box

\textbf{Proof of Proposition 1, Part (iv).} From the local IC condition, we have
\[
\frac{dy}{d\theta} \frac{\partial U}{\partial c} + \frac{dk}{d\theta} \frac{\partial U}{\partial k} = 0.
\]
Since $\frac{dk}{d\theta} \geq 0$ and $\frac{\partial U}{\partial k} \geq q \frac{\partial U}{\partial c}$, by Part (i), the last inequality implies $\frac{d(y+qk)}{d\theta} \leq 0$. Therefore, the transfer is weakly decreasing in $\theta$. 

\hspace{1cm} \Box


\section*{F Proof of Proposition 2}

The proofs of Parts (a) and (b) are given in the paper.

For Part (c), observe that

\[
\frac{d\Gamma^*(\theta)}{d\theta} = -\frac{d}{d\theta}(c^*(\theta) + qk^*(\theta)) \iff \frac{dy^*}{d\theta} + q\frac{dk^*}{d\theta} = -1 - \frac{d\Gamma^*(\theta)}{d\theta} \iff -\frac{dy^*}{d\theta} = q\frac{dk^*}{d\theta} + 1 + \frac{d\Gamma^*(\theta)}{d\theta}
\]

By IC and the distortion result of Proposition 1,

\[
-\frac{dy^*}{d\theta} / \frac{dk^*}{d\theta} = q\Delta(\theta).
\]

Combining these two gives (23). Integrating this equation from $\theta$ to $\bar{\theta}$, we obtain (22).

\section*{G Proof of Proposition 4}

\textit{Proof of Proposition 4 Part 1.} Let $s = \frac{\bar{\theta} - \theta}{2\lambda(c(\bar{\theta}))}$, we rewrite (39) as:

\[
\log \left( \frac{1 + s}{1 + \sqrt{1 + s}} \right) \frac{\sqrt{s}}{s} = 2\sqrt{2\lambda (\bar{\theta} - \theta)}.
\]

After lengthy algebras, we show that the left hand side is strictly increasing in $s$ and is equal to 0 at $s = 0$ and to $\infty$ at $s = \infty$. So there exists a unique solution $s^*$ to this equation.

Now, from the IC constraint, we have

\[
\frac{1}{k(\theta)}k'(\theta) = \frac{1}{c(\theta)} \left(1 - \frac{1}{4\lambda c(\theta)}\right).
\]

Therefore, in order for $k$ to be increasing in $\theta$, we require $4\lambda c(\theta) \geq 1$. Because $c$ is increasing in $\theta$, this is equivalent to $4\lambda c(\bar{\theta}) = 4\lambda qk(\bar{\theta}) \geq 1$. From the definition of $s^*$,
\[ s^* \leq 8\lambda (\overline{\theta} - \theta). \] Equivalently
\[ \log \left( 1 + 8\lambda (\overline{\theta} - \theta) \right) \left( 1 + \sqrt{1 + 8\lambda (\overline{\theta} - \theta)} \right) \sqrt{8\lambda (\overline{\theta} - \theta)} > 2\sqrt{2\lambda (\overline{\theta} - \theta)}. \]

After lengthy algebras, we show that
\[ \log \left( 1 + t \right) \left( 1 + \sqrt{1 + t} \right) \]

is strictly decreasing in \( t \), and it is equal to 2 at \( t = 0 \) and to 0 at \( t = \infty \). Therefore, the above inequality is equivalent to

\[ 8\lambda (\overline{\theta} - \theta) < t^*, \]

where \( t^* \) is uniquely determined by

\[ \frac{(1 + \sqrt{1 + t}) \log (1 + t)}{t} = 1. \] (76)

After some algebra manipulation, it is easy to see that \( t^* = \delta - 1 \), where \( \delta \) is defined in the Proposition. The numerical value of \( t^* \) is 11.3402.

For Part (1a), the convexity of \( k^* \) and the concavity of \( y^* \) can be obtained easily by twice differentiating the closed form expressions for (38) and (35). Part (1b) is immediate since \( k^*, c^* \) are both strictly increasing in \( \theta \).

Proof of Proposition 4 Part 2. As stated in the Proposition, we look for continuous allocations \( k(.), y(.) \) pooling over \([\theta, \theta^*]\) and separating over \([\theta^*, \overline{\theta}]\), together with the multipliers \( \mu(.), \gamma(.), \eta(.) \) that satisfy the F.O.Cs given in Subsection 4.1, and reformulated in Appendix D.

From (57), because \( f(.), y(.), k(.), \xi(.) \) are all continuous in \( \theta \), \( \mu(\theta) \) is continuous in \( \theta \).

From (58), because \( f(.), y(.), k(.), \xi(.) \), \( \mu(.) \) are all continuous in \( \theta \), \( \eta(\theta) \) is continuous in \( \theta \).
F.O.C in $y(\theta)$, for $\theta \in [\theta, \theta^*]$:

$$\frac{1}{2(\theta + y)}(\mu(\theta) + \bar{f}) - \lambda \bar{f} = -\frac{1}{2(\theta + y)}\zeta(\theta)$$

(77)

Therefore

$$\frac{1}{2(\theta + y)}\bar{f} - \lambda \bar{f} = \frac{d}{d\theta} \left(\frac{1}{2(\theta + y)}\zeta(\theta)\right)$$

So

$$\frac{1}{2(\theta + y)}\zeta(\theta) = \frac{\bar{f}}{2} \left(\log(\theta + y) - \log(\bar{y})\right) - \lambda \bar{f}(\theta - \bar{y})$$

or

$$\zeta(\theta) = \frac{\bar{f}}{2} \left(\log(\theta + y) - \log(\bar{y})\right) - 2\lambda \bar{f}(\theta - \bar{y})(\theta + y)$$

(78)

and

$$\mu(\theta) = -\zeta'(\theta) = -\bar{f} \left(\log(\theta + y) - \log(\bar{y})\right) - \bar{f} + 2\lambda \bar{f}(\theta + y) + 2\lambda \bar{f}(\theta - \bar{y})$$

F.O.C in $k$:

$$\frac{1}{2k}(\mu(\theta) + \bar{f}) + \eta(\theta) - \bar{f} \lambda q = 0.$$  

(79)

At $\theta = \theta^*$, $\eta(\theta^*) = 0$, so

$$\mu(\theta^*) + \bar{f} - 2\bar{f} \lambda q = 0.$$

Therefore, from the earlier expression for $\mu(\theta)$:

$$-\log(c^*) + \log(c^* - (\theta^* - \bar{y})) + 2\lambda c^* + 2\lambda (\theta^* - \bar{y}) - 2\lambda \bar{y} = 0,$$

(80)

where $c^* = \bar{y} + \theta^*$.

Integrating (79) from $\theta$ to $\theta^*$, we obtain

$$\frac{1}{2k}(\zeta(\theta) - \zeta(\theta^*) + \bar{f}(\theta^* - \bar{y})) + \gamma(\theta) - \gamma(\theta^*) - \bar{f} \lambda q (\theta^* - \bar{y}) = 0.$$

Given that $\gamma(\theta^*) = \gamma(\bar{y}) = 0$ and $\zeta(\bar{y}) = 0$ and from the earlier expression for $\zeta$, this is equivalent to:

$$-c^* (\log c^* - \log(c^* - (\theta^* - \bar{y}))) + 2\lambda (\theta^* - \bar{y}) c^* + (\theta^* - \bar{y}) - 2\lambda \bar{y} \lambda k (\theta^* - \bar{y}) = 0$$

(81)
Lastly, using the assumption that the contract is fully separating over \([\theta^*, \bar{\theta}]\) and, thus there is no distortion at the top, we obtain:

\[
\log \left( \left( c^* \right)^2 + \frac{\bar{\theta} - \theta^*}{2\lambda} \right) = \log(qk) + 4\lambda \sqrt{\left( c^* \right)^2 + \frac{\bar{\theta} - \theta^*}{2\lambda} - 4\lambda c^* + \log c^*}. \tag{82}
\]

We show that there is a solution \((y, \bar{k}, \theta^*)\) to (80)-(82).

From the first two equations, (80) and (81), we have

\[
c^* - \log c^* + \log \left( c^* - \frac{(\theta^* - \bar{\theta})}{\theta^* - \bar{\theta}} \right) + 2\lambda c^* + 1 = -\log c^* + 2\log c^* - (\theta^* - \bar{\theta}) + 2\lambda c^* + 2\lambda (\theta^* - \bar{\theta}) .
\]

Let \(\zeta = \frac{\theta^* - \bar{\theta}}{c^*}\), this expression simplifies to:

\[
\frac{\log(1 - \zeta)}{\zeta} + 1 = \log(1 - \zeta) + 2\lambda c^* \zeta
\]

So \(c^*\) is a function of \(\zeta\)

\[
c^* = \hat{c}(\zeta) \equiv \frac{\log(1 - \zeta)}{\zeta} - \log(1 - \zeta) + 1 \over 2\lambda \zeta
\]

Using Taylor expansion

\[
\hat{c}(\zeta) = \frac{1}{2\lambda} \sum_{n=0}^{\infty} \frac{\zeta^n}{(n+1)(n+2)}, \tag{83}
\]

which is strictly increasing in \(\zeta\).

Notice that \(\lim_{\zeta \to 0} \hat{c}(\zeta) = \frac{1}{4\lambda}\) and \(\lim_{\zeta \to 1} \hat{c}(\zeta) = \frac{1}{2\lambda}\). Because \(8\lambda(\bar{\theta} - \theta) > t^* > 4\),

\[
1 < 2\lambda(\bar{\theta} - \theta).
\]

So

\[
\lim_{\zeta \to 1} \zeta \hat{c}(\zeta) = \frac{1}{2\lambda} < (\bar{\theta} - \theta),
\]

which implies at \(\zeta = 1\), \(\theta^* = \bar{\theta} + \lim_{\zeta \to 1} \zeta \hat{c}(\zeta) < \bar{\theta}\).
Solving for $q_k$ from (80) and (82), we obtain another equation:

$$\left(c^* + \frac{\overline{\theta} - \theta}{2\lambda} - \frac{c^* \zeta}{2\lambda}\right) \exp\left(-4\lambda \sqrt{(c^*)^2 + \frac{\overline{\theta} - \theta}{2\lambda} - \frac{c^* \zeta}{2\lambda}}\right) \frac{\exp(4\lambda c^*)}{c^*} = \log\left(\frac{1}{2\lambda}\right) + c^* + c^* \zeta. \tag{84}$$

From the earlier expression for $c^*$, (83), this corresponds to one equation, and one unknown in $\zeta$. Let $\Delta(\zeta)$ denote the difference between the RHS and LHS of this equation.

We will show that $\Delta(0) < 0$ and $\Delta(1) > 0$.

It is easy to show that $\lim_{\zeta \uparrow 1} \Delta(\zeta) = +\infty > 0$, since $\lim_{\zeta \uparrow 1} \hat{c}(\zeta) = \frac{1}{2\lambda} < \infty$ and $\lim_{\zeta \uparrow 1} \log(1 - \zeta) = -\infty$.

Now at $\zeta = 0$, $\hat{c}(0) = \frac{1}{4\lambda}$, so

$$\Delta(0) = \left((\hat{c}(0))^2 + \frac{\overline{\theta} - \theta}{2\lambda}\right) \exp\left(-4\lambda \sqrt{(\hat{c}(0))^2 + \frac{\overline{\theta} - \theta}{2\lambda}}\right) \frac{\exp(4\lambda \hat{c}(0))}{\hat{c}(0)} - \hat{c}(0).$$

Let $\tilde{s} = \frac{\overline{\theta} - \theta}{2\lambda (\hat{c}(0))^2}$, after lengthy algebra, we desired inequality $\Delta(0) < 0$ as:

$$\frac{\log(1 + \tilde{s})}{\sqrt{\tilde{s}}} \frac{(1 + \sqrt{1 + \tilde{s}})}{\sqrt{\tilde{s}}} < 2\sqrt{2\lambda (\overline{\theta} - \theta)}. \tag{85}$$

Consider the solution to the relaxed problem in Part 1 of this Proposition, $c_R(\theta)$ and $k_R(\theta)$ ($k_R$ might not be increasing). At $\theta$,

$$s^* = \frac{\overline{\theta} - \theta}{2\lambda (c_R(\theta))^2}$$

and

$$\frac{\log(1 + s^*)}{\sqrt{s^*}} \frac{(1 + \sqrt{1 + s^*})}{\sqrt{s^*}} = 2\sqrt{2\lambda (\overline{\theta} - \theta)}.$$ 

As shown in the first part of this Proposition, the RHS of (85) is strictly increasing in $\tilde{s}$. Therefore (85) is equivalent to

$$\tilde{s} < s^*$$

or, by definition of $\tilde{s}$ and $s^*$,

$$\hat{c}(0) = \frac{1}{4\lambda} > c_R(\theta).$$

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Indeed this is the case since $8\lambda(\bar{\theta} - \bar{\theta}) > t^*$.

So there exists $\zeta^* \in (0,1)$ such that $\Delta(\zeta^*) = 0$. We determine $\theta^*, y$ and $k$ as $\theta^* = \hat{\theta}(\zeta^*)z^* + \hat{\theta}$, $y = \hat{\theta}(\zeta^*) - \theta^*$ and $k$ is a function of $c^*$ and $\zeta^*$ as in either (80) or (82). It is easy to verify that $\left(\theta^*, y, k\right)$ solves (80)-(82).

We can verify that for all $\theta \in (\theta, \theta^*)$:

$$
\gamma(\theta) = \frac{1}{2k} \left(-\tilde{z}(\theta) + \bar{f}(\theta - \bar{\theta})\right) - \bar{f}\lambda q(\theta - \bar{\theta}) > 0.
$$

We first show by contradiction that there is no local minimum of $\gamma$ in $(\theta, \theta^*)$. Assume by contradiction that there exists a local minimum $\tilde{\theta} \in (\theta, \theta^*)$. Then $\gamma'(\tilde{\theta}) = 0$ and $\gamma''(\tilde{\theta}) \geq 0$. From the expression for $\gamma$:

$$
\gamma''(\theta) = -\frac{1}{2k} \tilde{g}''(\theta) = -\frac{1}{2k} \tilde{f} \left(\frac{1}{\theta + y} - 4\lambda\right)
$$

is increasing in $\theta$. Therefore $\gamma''(\theta) > \gamma''(\tilde{\theta}) \geq 0$ for $\theta \in (\tilde{\theta}, \theta^*)$. From the definition of $\theta^*$, $\gamma'(\theta^*) = -\eta(\theta^*) = 0$. So $\gamma'(\tilde{\theta}) < 0$ which contradicts the property that $\gamma'(\tilde{\theta}) = 0$. So $\gamma$ does not have a local minimum in $(\theta, \theta^*)$.

At $\tilde{\theta}$, $\gamma(\tilde{\theta}) = 0$ and by construction, at $\theta^*$, $\gamma(\theta^*) = 0$. So $\gamma \geq 0$ for all $\theta \in [\theta, \theta^*]$. In addition, $\gamma(\theta) > 0$ for all $\theta \in (\theta, \theta^*)$ (otherwise, $\gamma$ would have a local minimum in $(\tilde{\theta}, \theta^*)$.

Given $\theta^*, y, k$, we determine the allocation over $[\theta^*, \bar{\theta}]$ using the derivation in Subsection 5

$$
c(\theta) = \sqrt{(c^*)^2 + \frac{\theta - \theta^*}{2\lambda}}
$$

and

$$
\log(k(\theta)) = \log(k) + 4\lambda \sqrt{(c^*)^2 + \frac{\theta - \theta^*}{2\lambda}} - 4\lambda c^* - \frac{1}{2} \log \left((c^*)^2 + \frac{\theta - \theta^*}{2\lambda}\right) + \log (c^*). \quad (86)
$$

Since $c^* = \hat{\theta}(\zeta^*) > \hat{\theta}(0) = \frac{1}{4\pi}, k(.)$ is strictly increasing over $(\theta^*, \bar{\theta})$.

All the first order conditions are satisfied, therefore $\{y^*, k^*\}$ as constructed above is an
OC. Parts (2a) and (2b) regarding the shape of the allocation and distortion are derived similarly to the fully separating case in Part 1.

\[\square\]

**H Proof of Proposition 5**

To prove Proposition 5, we apply the general analysis in Subsection 6.1 to this special case with separable utility function:

\[U(c, k) = u_1(c) + u_2(k)\]

Then

\[x = u_1'(c)\]
\[z = u_2'(k),\]

and

\[c = H^c(x) = (u_1')^{-1}(x)\]
\[k = H^k(z) = (u_2')^{-1}(z).\]

Furthermore,

\[\frac{\partial^2 U(c, k)}{\partial c^2} = \frac{1}{(H^c)'(x)}\]

and

\[\frac{\partial^2 U(c, k)}{\partial c \partial k} = 0.\]

From the expression for \(\xi\), (44),

\[\xi(\theta) = \Lambda f(\theta) K(x(\theta), z(\theta)),\]
where, in this case,

\[ K(x, z) = -\frac{z - qx}{z} (H^c)'(x) \]

\[ = - (H^c)'(x) + \frac{qx}{z} (H^c)'(x). \]

Therefore

\[ \frac{\partial K}{\partial x} = -(H^c)''(x) + \frac{q}{z} (H^c)'(x) + \frac{qx}{z} (H^c)''(x), \]

and

\[ \frac{\partial K}{\partial z} = -\frac{q}{z^2} (H^c)'(x). \]

The differential equations (45) and (46), for \(x(\theta)\) and \(z(\theta)\) can be re-written as:

\[ x \left( 1 - \lambda \frac{f'}{f} K(x, z) - \lambda \left( \frac{\partial K}{\partial x} x' + \frac{\partial K}{\partial z} z' \right) \right) - \lambda \]

\[ = \frac{1}{(H^c)'(x)} \lambda K(x, z). \tag{87} \]

and

\[ x \left( (H^c)'(x)x' - 1 \right) + z \left( (H^c)'(z)z' \right) = 0. \tag{88} \]

Armed with these results, we can show the following lemma.

**Lemma 10.** Assuming that \( \frac{(H^c)'(x)}{(H^k)'(qx)} \) is weakly increasing in \( x \). We have \( \frac{dk}{d\theta}(\tilde{\theta}) > \frac{dk}{d\theta}(\theta) \).

**Proof.** As shown in Proposition 1, there is no pooling at the top, i.e. \( \frac{dk(\theta)}{d\theta} > 0 \). Therefore, if the desired inequality is immediately satisfied if there is pooling at the bottom, i.e. \( \frac{dk(\theta)}{d\theta} = 0 \).

We just have to show the inequality if there is no pooling at the bottom. In this case, at \( \theta = \theta \) or \( \tilde{\theta} \), \( z = qx \). Therefore, at \( \theta = \theta \) or \( \tilde{\theta} \), \( K(x, z) = 0 \) and

\[ \frac{\partial K}{\partial x} = \frac{1}{x} (f^e)'(x), \]

and

\[ \frac{\partial K}{\partial z} = -\frac{1}{x} (f^e)'(x). \]
Conditions (87) and (88) become
\[ x - \lambda (H^c)'(x)x' \left( 1 + \frac{(H^c)'(x)}{(H^k)'(z)} \right) + \frac{\lambda (H^c)'(x)}{q(H^k)'(z)} - \lambda = 0, \]
and
\[ (H^c)'(x)x' + q(H^k)'(z)z' = 1. \]
After algebra manipulations:
\[ (H^c)'(x)x' = \frac{x - 2\lambda}{\lambda \left( q + \frac{(H^c)'(x)}{q(H^k)'(qx)} \right)} + 1. \]
Therefore
\[ \frac{dk}{d\theta} = (H^k)'(z)z' \]
\[ = \frac{1}{q} \left( 1 - (H^c)'(x)x' \right) \]
\[ = \frac{1}{q} \frac{2\lambda - x}{\lambda \left( q + \frac{(H^c)'(x)}{q(H^k)'(qx)} \right)}. \]

Since \( \frac{(H^c)'(x)}{q(H^k)'(qx)} \) is weakly increasing in \( x \), \( \frac{dk}{d\theta} \) is strictly decreasing in \( x \). From the assumption that the optimal contract is separating at the top and at the bottom, \( k(\bar{\theta}) > k(\theta) \), so \( x(\bar{\theta}) = \frac{1}{q} z(\bar{\theta}) < x(\theta) = \frac{1}{q} z(\theta) \). Therefore, \( \frac{dk}{d\theta}(\bar{\theta}) > \frac{dk}{d\theta}(\theta) \). 

\textbf{Proof of Proposition 5.} This proposition is a direct application of Lemma 10. Indeed, given the functional forms,
\[ c = H^c(x) = \frac{(x/\alpha)^{-1/\sigma}}{\alpha}, \]
\[ k = H^k(z) = \frac{(z/(1-\alpha))^{-1/\sigma}}{\alpha}. \]

Therefore, \( \frac{(H^c)'(x)}{q(H^k)'(qx)} \) is constant in \( x \). When the CRRA coefficient in the composite
good is strictly greater than the the CRRA coefficient in capital good then \( \frac{(H^c)'(x)}{q(H^k)'(qx)} \) is strictly increasing in \( x \). In both cases, we can apply Lemma 10 to obtain the desired result.

\[ \square \]

I  Figures for Numerical Examples

\[ 70 \]
Figure 5: Phase Diagram for Example 1
Figure 6: Allocations, Distortion, and Forfeiture for Example 1
Figure 7: Phase Diagram for Example 2
Figure 8: Allocations, Distortion, and Forfeiture for Example 2
Figure 9: Phase Diagram for Example 3
Figure 10: Allocations, Distortion, and Forfeiture for Example 3
Figure 11: Phase Diagram for Example 4
Figure 12: Allocations, Distortion, and Forfeiture for Example 4
Figure 13: General Equilibrium