Amplification and Asymmetric Effects without Collateral Constraints

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Abstract

The seminal contribution by Kiyotaki and Moore (1997) has spurred a vast literature on the importance of collateral constraints in propagating and amplifying shocks to the economy. However, the use of collateral constraints implicitly assumes state non-contingent debt, i.e., markets are incomplete. It is possible that a large part of the amplification effect of collateral constraints is actually due to market incompleteness. We study a simple model with plausible parameter values and solve it with and without collateral constraints and find that indeed market incompleteness by itself plays a quantitatively significant role in the amplified and asymmetric responses of the economy, including land price and output, to exogenous shocks.

Keywords: Incomplete Markets; Collateral Constraint; Amplification; Asymmetric Effects; Dynamic General Equilibrium; Global Nonlinear Methods

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1 Introduction

The mechanisms by which exogenous shocks to the economy propagate and amplify over time are of utmost importance in understanding the positive and normative effects of business cycles. The seminal contribution by Kiyotaki and Moore (1997) identifies a prominent mechanism - the collateral constraint channel - and has spurred a vast literature on the importance of collateral constraints including Iacoviello (2005) and Mendoza (2010). However, more recent important papers on the same topic, such as Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013), generate propagation and amplification without imposing any collateral constraint. The common feature of these papers is that agents borrow through state non-contingent debt, i.e., markets are incomplete. It is possible that a large part of the amplification effect of collateral constraints is actually due to market incompleteness. In this paper, we build a simple, calibrated model with heterogeneous agents and aggregate shocks and solve the model with and without collateral constraints in order to understand the qualitative and quantitative importance of market incompleteness relative to collateral constraints in propagating and amplifying shocks to the economy.

The model is a simplified version of Iacoviello (2005). There are two types of risk-averse agents, entrepreneurs and households, in a one-consumption good economy with a fixed supply of an asset: land. There is a land market in which all agents participate. The entrepreneurs can combine land with labor supplied by the households to produce consumption good. This production process is subject to aggregate stochastic productivity shocks. The households earn wages from working for the entrepreneurs and decide how much to consume in consumption good and in land (housing). We assume that the entrepreneurs are less patient than the households, and thus they tend to borrow from the households. In this economy, we consider three alternative financial market structures. In the first one, the benchmark model - the model with incomplete markets, Model 1 - the entrepreneurs can borrow from the households using state non-contingent debt only subject to the no-Ponzi condition. In the second model - the model with a collateral constraint and incomplete markets, Model 2 - debt is still state non-contingent but the entrepreneurs are subject to the collateral constraint that constrains borrowing to be less than a fraction of the expected value of the entrepreneurs' land holding. In the last model - the model with a collateral constraint and complete markets, Model 3 - the entrepreneurs can sell

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1We also consider a model with a slightly different collateral constraint - Model 2b - that constrains borrowing to be less than a fraction of the value of the entrepreneurs’ land holding evaluated at the current land price, instead of the expected future land price. This constraint using current asset price is often used in the collateral constraint literature, such as Mendoza (2010) and Bianchi (2011).
a complete set of state-contingent securities to the households subject to a collateral con-
straint on each state-contingent security. We use the Markov equilibrium definition and
global nonlinear method developed by Kubler and Schmedders (2003) and Cao (2010) to
solve for the equilibrium in each model.

We find that, in the benchmark incomplete markets model, Model 1, the equilibrium
dynamics exhibit amplification and asymmetric responses to symmetric exogenous pro-
ductivity shocks. For example, land price and output increase after an increase in pro-
ductivity by less than they decrease after a decrease in productivity of the same size.
The amplification and asymmetric effects in this model are due to the net worth effect as
follows. An initial negative shock decreases land productivity and consequently land
price. Because of market incompleteness, the value of debt holding of the entrepreneurs,
which is determined from the last period, remains unchanged. Therefore the decrease in
land price leads to a decrease in the net worth of the entrepreneurs. This decrease in the
net worth of the entrepreneurs reduces their demand for land for production. Only the
entrepreneurs can use land to produce, therefore the reduction in their land demand fur-
ther depresses the price of land, and further lowers their net worth, setting off a vicious
cycle of falling land price and falling net worth. Thus the effects of the negative produc-
tivity shock on land price and production are amplified. Moreover, because land price is
a concave function of the net worth of the entrepreneurs, a negative productivity shock
has larger effects than a positive productivity shock through the net worth channel.

In Model 2 and Model 2b with collateral constraints and incomplete markets, similar
to the findings in the collateral constraint literature, the dynamics of the economy can be
divided into two regions. In the first region, the collateral constraint is not binding and
in the second region, the collateral constraint is binding or close to binding. In the second
region, land price and output of the economy are much more sensitive to changes in the
net worth of the entrepreneurs. In the second region, we find that the collateral constraint
strengthens the asymmetric and amplified responses to exogenous productivity shocks
(compared to incomplete markets alone). However, we also observe that the probability
that we are in the second region decreases rapidly in the size of the shocks due to the

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2More precisely it is the relative net worth effect, in which relative net worth corresponds to net worth
divided by land price.
3Brunnermeier and Sannikov (2014) call this channel amplification through prices.
4In Appendix C, we study a two-period version of our model with log utilities to derive analytically
the net worth effect. We also show that, under complete markets, this effect disappears because the en-
trepreneurs can perfectly insure their (relative) net worth against productivity shocks.
5The amplification through relative net worth effect is similar to the fire-sale phenomenon described in
Shleifer and Vishny (1997) because the entrepreneurs (the specialists) are the only agents in the economy
who can use land to produce output. As a result, they are the only natural buyers. So when all entrepreneurs
reduce their land demand, both land price and output fall significantly.
precautionary saving motive of the entrepreneurs.

In both incomplete markets and collateral constraint models, the economy exhibits amplified and asymmetric responses to symmetric exogenous shocks. Nevertheless, quantitatively, the responses are only slightly smaller in the benchmark incomplete markets model compared to the collateral constraint model. These results suggest that market incompleteness alone accounts for a significant part of the asymmetric and amplified responses to shocks in the collateral constraint model.

To further understand this point, we consider the equilibrium in the last model with collateral constraints and complete markets, in which the entrepreneurs have access to a complete set of state-contingent securities but the sale of these securities has to be collateralized by land. In the long run the economy converges to a single level of wealth distribution, i.e., we have some sort of dynamically complete insurance. At this level of wealth distribution, there is no amplification nor asymmetry effects of exogenous shocks. These results demonstrate that collateral constraints have to be coupled with markets incompleteness in order to generate significant amplification and asymmetric responses to shocks.

The main theoretical results of the paper can be summarized by Table 1. As we will present in details in Section 2, the productivity shock can take on three possible values denoted as expansion (high), normal, and recession (low), where high and low shocks are each 3% away from normal. Starting from the normal state, we compute the percent changes in land price and output when the aggregate shock in the next period switches to expansion or recession respectively. These changes depend on the wealth distribution between entrepreneurs and households, thus we report the average changes over the stationary distribution of the entrepreneurs’ share of wealth.

We report our results under different models - Model 1 (row 1), Model 2 and Model 2b (rows 2 and 3), Model 3 (row 4), and the model with complete markets, Model 4 presented in Appendix A, (row 5) - with the main parameters calibrated to the U.S. economy from Iacoviello (2005). We compare the responses of land price and output to shocks in different models to the responses in the model with complete markets, Model 4.

First, Table 1 shows the asymmetric effect in models with market incompleteness, Models 1,2 and 2b, i.e. positive shocks increase land price and output by less than neg-

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6Furthermore, the gaps between the responses in the benchmark model and the collateral constraint model decrease as we increase the size of the shocks. Because the larger the size of the shocks is, the stronger the precautionary motive of the entrepreneurs becomes, which leads to less borrowing and less often binding collateral constraint. See Tables 3 and 4 in Section 3.

7Land price and output remain almost unchanged if the productivity shock stays in the normal state.

8If we condition on lower values of the entrepreneurs’ wealth share, the responses are much larger.
Table 1: Average land price and output changes after a 3% increase or 3% decrease in aggregate productivity

<table>
<thead>
<tr>
<th>Type of Friction</th>
<th>Land price</th>
<th></th>
<th>Output</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expansion</td>
<td>Recession</td>
<td>Expansion</td>
<td>Recession</td>
</tr>
<tr>
<td>incomplete markets (Model 1)</td>
<td>3.21%</td>
<td>-3.45%</td>
<td>3.25%</td>
<td>-3.44%</td>
</tr>
<tr>
<td>collateral constraint (IM - Model 2)</td>
<td>3.30%</td>
<td>-3.76%</td>
<td>3.15%</td>
<td>-3.50%</td>
</tr>
<tr>
<td>collateral constraint (alt, IM - Model 2b)</td>
<td>3.32%</td>
<td>-3.83%</td>
<td>3.17%</td>
<td>-3.58%</td>
</tr>
<tr>
<td>collateral constraint (CM - Model 3)</td>
<td>3.01%</td>
<td>-2.99%</td>
<td>2.97%</td>
<td>-2.97%</td>
</tr>
<tr>
<td>complete markets (Model 4)</td>
<td>3.00%</td>
<td>-3.00%</td>
<td>2.97%</td>
<td>-2.97%</td>
</tr>
</tbody>
</table>

Note: This table shows the long run average response of land price and output after productivity changes from normal to either high (Expansion) or low (Recession). Productivity can take on one of three realizations, (normal, high, and low), where low and high realizations are 3% away from normal.

Negative shocks decrease land price and output. For example, in the benchmark model, Model 1, a 3% fall in productivity generates a larger response in output (−3.44%) than the response to an increase of the same size in productivity (+3.25%). Second, an amplification effect is also present in all models with market incompleteness. For example, in Model 1, the responses in output (3.25% to a positive shock and −3.44% to a negative shock) are larger than the size of the shocks (3%). The asymmetric and amplification effects are slightly stronger in Models 2 and 2b with both incomplete markets and collateral constraints than in the benchmark model, Model 1. Lastly, the asymmetric and amplification effects are absent with complete markets, with (Model 3) or without collateral constraints (Model 4).

Lastly, the benchmark model with incomplete markets and its variations with collateral constraints imply that the responses of land price and output to productivity shocks are not only asymmetric and amplified but also depend on the wealth share of the entrepreneurs. We find that this implication is broadly consistent with the data in the U.S. with the entrepreneurs correspond to the non-financial business sector. In particular, from the observation that in the model, the real estate leverage of ratio, i.e. total debt (net of credit) divided by total value of the land holding of the entrepreneurs, is strictly decreasing in the entrepreneurs’ wealth share, we use the real estate leverage ratio of the non-financial business (corporate and non-corporate) sector as a proxy for the entrepreneurs’ wealth share in the model. We construct the time series for the real estate leverage of the non-financial business sector in the U.S. from the flow of fund data. One of our empirical

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9 Output here is defined as total amount of consumption good produced by the entrepreneurs using land and labor. The definition omits the imputed rental value of land from the housing consumption of the households. The same results hold, however, when we add this imputed rental value of land to the current definition of output (see footnote 26 for further details).

10 The magnitude of amplification is similar to the one in Iacoviello (2005).

11 This connection from model to data is used in a recent paper, Liu, Wang, and Zha (2013).
findings is that changes in TFP have the largest effects on land price and output when
the changes in TFP are low and when the real estate leverage ratio of the non-financial
business sector is high, which is exactly what our benchmark model predicts. Our em-
pirical findings complement the recent results in Guerrieri and Iacoviello (2015a), which
emphasizes the asymmetric responses of economic activities, including consumption and
employment, to changes in house prices.\textsuperscript{12}

The paper is related to the vast literature on the effects of collateral constraints in addi-
tion to the papers cited above. In particular, the benchmark model is a simplified version
of Iacoviello (2005). But instead of relying on log-linearization as in Iacoviello (2005), we
solve for the global nonlinear dynamics of the equilibrium.\textsuperscript{13} To do so, we use the concept
of Markov equilibrium and the numerical method developed by Kubler and Schmedders
(2003) and Cao (2010) but extend it to a production economy with elastic labor supply,
housing consumption, and the natural borrowing limit.\textsuperscript{14} This global nonlinear solution
also allows us to quantitatively assess the accuracy of the log-linearization solution. An-
other methodological contribution of this paper is that we extend the numerical method
in Kubler and Schmedders (2003) and Cao (2010) to allow for a wide range of financial
markets structure including incomplete markets with exogenous borrowing constraints,
and collateral constraints with complete markets.

In a small open economy framework, Mendoza (2010) also compares the equilibrium
under collateral constraint versus the equilibrium under an exogenous borrowing con-
straint limit and finds that exogenous borrowing limit weakens the amplification effect
on Tobin’s Q by a factor of 5.75. The difference between our results and his comes from
the fact that the supply of the collateral asset (capital) is elastic in Mendoza (2010), while
it is completely inelastic in our model. Inelastic supply of the asset implies more volatility
in the price of the asset and gives more room for negative shocks to be amplified by the
reduction in the demand of the natural holders of the asset induced the reduction in their
wealth, even in the absence of a collateral constraint.\textsuperscript{15} Moreover, Appendix B shows
that imposing an exogenous borrowing constraint, as done in Mendoza (2010), reduces
significantly the amplification and asymmetric effects of incomplete markets.

\textsuperscript{12}Different from their analysis, in our models and empirical exercise, we assume that both land price
and output (and consumption) are endogenously determined by exogenous TFP shocks.

\textsuperscript{13}Guerrieri and Iacoviello (2015b) suggests a way to adapt the log-linearization method in a piecewise
fashion to handle occasionally binding constraints.

\textsuperscript{14}Indeed, because housing is a durable good, we need to make a change to the timing of production
compared to the timing in Iacoviello (2005), without changing key economic forces, in order to apply the

\textsuperscript{15}Boldrin, Christiano, and Fisher (2001) show that when capital supply is flexible, it is impossible to
match the observed volatility and the equity premium in equity prices in the U.S.
The paper is organized as follows. Section 2 presents the benchmark incomplete markets model and the solution method, as well as reasonable parameters to analyze the solution of this benchmark model. Section 3 studies the collateral constraint model and compares it to the benchmark model. Section 4 studies the complete markets model with a collateral constraints and compares it to the collateral constraint model. Section 5 presents some empirical evidence. Section 6 concludes. Additional proofs and constructions are presented in the appendix.

2 Benchmark Model

The model is based on Iacoviello (2005). It is one of simplest models of a production economy in which a durable asset (land) is used as collateral to borrow and as an input in production. Moreover the model is calibrated to the U.S. economy.\(^\text{16}\) The solution method, intuition, and results based on this model should carry over to similar models.

2.1 Economic Environment

Consider an economy inhabited by two types of agents: entrepreneurs and households who are both infinitely lived and of measure one. There is one consumption good. Entrepreneurs produce the consumption good by hiring household labor and combining it with land. Households consume the consumption good and land (housing), and supply labor to the entrepreneurs.

We adopt the standard notation of uncertainty. Time is discrete and runs from 0 to infinity. In each period, an aggregate shock \(s_t\) is realized. We assume that \(s_t\) follows a finite-state Markov chain. Let \(s^t = (s_0, s_1, \ldots, s_t)\) denote the history of realizations of shocks until date \(t\). To simplify the notation, for each variable \(x\), we use \(x_t\) as a shortcut for \(x_t(s^t)\).

Households maximize a lifetime utility function given by

\[
\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \frac{(c'_t)^{1-\sigma_2} - 1}{1 - \sigma_2} \right) + \frac{(h'_t)^{1-\sigma_h} - 1}{1 - \sigma_h} - \frac{1 - \eta}{\eta} (L'_t)^{\eta} \right\},
\]

where \(\mathbb{E}_0 \left[ . \right] \) is the expectation operator, \(\beta \in (0, 1)\) is the discount factor, \(c'_t\) is consumption

\(^{16}\)Cordoba and Ripoll (2004) is another extension of Kiyotaki and Moore (1997) to nonlinear production function and concave utility functions. However, the paper only analyzes the case without aggregate shocks. In the Online Appendix, we show that our global nonlinear method applies to their model with aggregate shocks as well. However we choose Iacoviello (2005) to illustrate the importance of land in the economy.
at time $t$, $h'_t$ is the holding of land. $L'_t$ denotes the hours of work. Households can trade in the market for land as well as a state non-contingent bond market. The budget constraint of the households is

$$c'_t + q_t(h'_t - h'_{t-1}) + p_t b'_t \leq b'_{t-1} + w_t L'_t.$$  

(2)

Given land in the utility function of households, implicitly $h'_t \geq 0$.

Entrepreneurs use a Cobb-Douglas constant-returns-to-scale technology that uses land and labor as inputs. They produce consumption good $Y_t$ according to

$$Y_t = A_t h^\nu_t L^{1-\nu}_t,$$  

(3)

where $A_t$ is the aggregate productivity which depends on the aggregate state $s_t$, $h_t$ is real estate input, and $L_t$ is labor input.

In contrast to Iacoviello (2005), the production function uses the contemporaneous land holding of the entrepreneurs instead of the land holding from the previous period. This minor modification turns out to be crucial to apply the concept of Markov equilibrium in Subsection 2.2 and the solution method in Subsection 2.3.2. However this modification does not affect key economic forces.\footnote{In a continuous time version of this model presented in the Online Appendix, similar to the models in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013), the timing assumption becomes immaterial.}

We want the entrepreneurs to borrow from the households so we assume that the entrepreneurs discount the future at the rate $\gamma < \beta$. The entrepreneurs maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \frac{(c_t)^{1-\sigma_1} - 1}{1 - \sigma_1}$$  

(4)

subject to the budget constraint

$$c_t + q_t(h_t - h_{t-1}) + p_t b_t \leq b_{t-1} + Y_t - w_t L_t.$$  

(5)

Output $Y_t$ is produced by combining land and labor using the production function (3). Given the production function of the entrepreneurs, implicitly $h_t \geq 0$.

In this benchmark model, we do not impose a collateral constraint as in Iacoviello (2005), so we need to impose no-Ponzi scheme conditions on the entrepreneurs and households, i.e., for all $t$

$$\lim_{T \to \infty} \mathbb{E}_t \left[ \prod_{t'=t}^{T-1} p_{t'} \right] b_T \geq 0$$

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and
\[ \lim_{T \to \infty} \mathbb{E}_t \left[ \left( \prod_{t'=t}^{T-1} p_{t'} \right) b_T' \right] \geq 0. \]

To finish the description of the model, here we assume that the only source of uncertainty is the aggregate productivity \( A_t \). It is straightforward to extend the model to incorporate other sources of uncertainty such as uncertainty in the housing preference parameter \( j \).

### 2.2 Equilibrium

The definition of a sequential competitive equilibrium for this economy is standard.

**Definition 1.** A competitive equilibrium is sequences of prices \( \{p_t, q_t, w_t\}_{t=0}^{\infty} \) and allocations \( \{c_t, h_t, b_t, L_t, c_t', h_t', b_t', L_t'\} \) such that (i) the \( \{c_t', h_t', b_t', L_t'\} \) maximize (1) subject to budget constraint (2) and the no-Ponzi condition and \( \{c_t, h_t, b_t, L_t\} \) maximize (4) subject to budget constraint (5) and the no-Ponzi condition, and the production technology (3), given \( \{p_t, q_t, w_t\} \) and initial asset holdings \( \{h_{-1}, b_{-1}, h_{-1}', b_{-1}'\} \); (ii) land, bond, labor, and good markets clear: \( h_t + h_t' = H, b_t + b_t' = 0, L_t = L_t', c_t + c_t' = Y_t \).

Let \( \omega_t \) denote the normalized financial wealth of the entrepreneurs:
\[ \omega_t = \frac{q_t h_{t-1} + b_{t-1}}{q_t H}, \quad (6) \]

and \( \omega_t' \) denote the normalized financial wealth of the households:
\[ \omega_t' = \frac{q_t h_{t-1}' + b_{t-1}'}{q_t H}. \]

By the land and bond market clearing conditions, we have \( \omega_t' = 1 - \omega_t \) in any competitive equilibrium. Therefore in order to keep track of the normalized financial wealth distribution between the entrepreneurs and the households, \( (\omega_t, \omega_t') \), in equilibrium, we only need to keep track of \( \omega_t \). To simplify the language, we use the term *wealth distribution* for normalized financial wealth distribution.

As in Cao (2010), we define Markov equilibrium as follows.

**Definition 2.** A Markov equilibrium is a competitive equilibrium in which prices and allocations at time \( t \), as well as the wealth distribution at time \( t + 1 \) under different realizations of the exogenous shocks \( s_{t+1} \) depend only on the wealth distribution at time \( t \), \( \omega_t \) as well as the exogenous state \( s_t \).
This Markov equilibrium definition features the endogenous state variable $\omega_t$ that depends on land price $q_t$ (which by itself depends on the state variable). This equilibrium was first studied in Duffie et al. (1994). Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013) use the same type of equilibrium definition in their continuous time models.\(^{18}\) We are going to use the algorithm developed in Cao (2010) to compute this Markov equilibrium.\(^{19}\)

### 2.3 Solution

In this Subsection, we first show the equations that characterize a competitive equilibrium. Under natural borrowing limits, the model does not have a steady-state in the absence of uncertainty because of the difference in the discount factors of the households and the entrepreneurs. However, out of steady state, uncertainty prevents the entrepreneurs from borrowing too much because of the precautionary saving motive. Thus, a Markov equilibrium exists with globally bounded amount debt held by the entrepreneurs.

In later sections, when we introduce tighter borrowing constraints, either endogenous collateral constraint or exogenous borrowing limit, a steady-state exists and the Markov equilibrium converges to the steady state when uncertainty vanishes.

#### 2.3.1 Equilibrium Conditions

Given their housing holding at time $t$, $h_t$, the entrepreneurs choose labor demand $L_t$ to maximize profit

$$\max_{L_t} \left\{ Y_t - w_t L_t \right\}$$

\(^{18}\)In the Online Appendix, we present a continuous time version of our paper and make this connection explicit. However, there are two important differences. First, because of persistent TFP shocks (instead of I.I.D. depreciation shocks to capital stock as in Brunnermeier and Sannikov (2014) or I.I.D. return shocks on dividend as in He and Krishnamurthy (2013)), in our Markov equilibrium, we need to keep track of both wealth distribution and the current exogenous shock. Second, we allow for general utility functions, instead of linear or log utility functions as in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013).

\(^{19}\)We can also define Markov equilibrium using more natural state variables such as $h_{t-1}, b_{t-1}$, as in most of the New Keynesian literature using linearization methods. Our solution method applies for such concepts of equilibrium as well. However, our current Markov equilibrium concept allows us to summarize the dynamic equilibrium in one endogenous state variable, $\omega_t$, which greatly facilitates numerical computations and offers intuitive representations of how wealth distribution affects land price and output in simple graphs.
subject to the production technology given in (3). The first order condition (F.O.C.) in \( L_t \) implies
\[
 w_t = (1 - v) A_t h_t^v L_t^{-v}, \tag{7}
\]
i.e. \( L_t = \left( \frac{(1 - v) A_t}{w_t} \right) \frac{1}{v} h_t \) and
\[
 Y_t - w_t L_t = \pi_t h_t
\]
where \( \pi_t = v A_t \left( \frac{(1 - v) A_t}{w_t} \right)^{\frac{1 - v}{v}} \) is profit per unit of land.

The first-order conditions in to \( h_t \) and \( b_t \) in the maximization problem of the entrepreneurs imply
\[
 (\pi_t - q_t) c_t^{-\sigma_1} + \gamma \mathbb{E}_t [q_{t+1} c_{t+1}^{-\sigma_1}] = 0 \tag{8}
\]
and
\[
 - p_t c_t^{-\sigma_1} + \gamma \mathbb{E}_t \left[ c_{t+1}^{-\sigma_1} \right] = 0. \tag{9}
\]

Similarly, the F.O.C.s for the households are
\[
 h'_t : - q_t c_t'^{-\sigma_2} + jh_t'^{-\sigma_1} + \beta \mathbb{E}_t \left[ q_{t+1} c_{t+1}'^{-\sigma_2} \right] = 0 \tag{10}
\]
\[
 b'_t : - p_t c_t'^{-\sigma_2} + \beta \mathbb{E}_t \left[ c_{t+1}'^{-\sigma_2} \right] = 0 \tag{11}
\]
\[
 L'_t : w_t c_t'^{-\sigma_2} = \eta L_t^{\eta - 1} \tag{12}
\]

The first order conditions in \( h_t \) and \( h'_t \) shed light on the determinants of land price. We rewrite (8) as
\[
 q_t = \pi_t + \gamma \mathbb{E}_t \left[ q_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma_1} \right]
 = \mathbb{E}_t \left[ \sum_{s=0}^{\infty} \gamma^s \left( \frac{c_{t+s}}{c_t} \right)^{-\sigma_1} \pi_{t+s} \right].
\]
The right hand side of this equation shows that, from the entrepreneurs’ point of view, land price is the net present discounted value of present and future profit from production using land and the discount factor depends on the marginal utility of the entrepreneurs.
Similarly, we re-write (10) as
\[
q_t = \frac{j (h'_t)^{-\sigma_h}}{c_t^{-\sigma_2}} + \beta \mathbb{E}_t \left[ q_{t+1} \left( \frac{c'_{t+1}}{c'_t} \right)^{-\sigma_2} \right] \\
= \mathbb{E}_t \left[ \sum_{s=0}^{\infty} \beta^s \left( \frac{c'_{t+s}}{c'_t} \right)^{-\sigma_2} j (h'_{t+s})^{-\sigma_h} \right].
\]

From the point of view of the households, land price is the present discounted value of current and future marginal utility from housing.

Despite the fact that we do not impose any constraint on the entrepreneurs’ borrowing except for the no-Ponzi condition, the following lemma shows that, in equilibrium, the financial wealth of the entrepreneurs is endogenously bounded from below.

**Lemma 1.** In any competitive equilibrium, thus any Markov equilibrium, we must have \( \omega_t \geq 0 \) for all \( t \) and \( s^t \).

**Proof.** We prove this result by contradiction. Suppose that in a competitive equilibrium, there is \( t \) and \( s^t \) such that \( \omega_t(s^t) < 0 \). Given the formula for the profit maximization of the entrepreneurs above and the definition of financial wealth \( \omega_t \), the budget constraint (5) can be re-written as
\[
c_t + (q_t - \pi_t) h_t + p_t b_t \leq q_t \omega_t H.
\]

Pick a \( \lambda > 1 \), and consider an alternative trading and consumption plan \( \{ \tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'} \}_{t'=0}^{\infty} \) for the entrepreneurs which is the same as the initial plan for \( t' < t \) but for \( t' \geq t \):
\[
\{ \tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'} \}_{t'=t+1}^{\infty} = \{ \lambda c_{t'}, \lambda h_{t'}, \lambda b_{t'} \}_{t'=t+1}^{\infty}
\]
and
\[
\tilde{c}_t = \lambda c_t - (\lambda - 1) q_t \omega_t > \lambda c_t, \\
\tilde{h}_t = \lambda h_t, \\
\tilde{b}_t = \lambda b_t.
\]

This alternative plan \( \{ \tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'} \}_{t'=t+1}^{\infty} \) clearly delivers strictly higher utility to the entrepreneurs while satisfying all the constraints, including the no-Ponzi condition. This contradicts the fact that the initial plan is optimal. Therefore \( \omega_t \geq 0 \) for all \( t \) and \( s^t \). \( \square \)

We interpret this lower bound of the entrepreneurs’ wealth as their natural borrowing limit.
2.3.2 Global Nonlinear Method

We also solve the exact nonlinear equilibrium of the model using the algorithm in Kubler and Schmedders (2003) and Cao (2010). In particular, we solve for the Markov equilibrium in this economy. The original algorithm in Kubler and Schmedders (2003) is for endowment economies. Cao (2010) extends this algorithm to a production economy with capital accumulation. In the current paper, we show that the original algorithm works similarly when we add labor choice as well as housing consumption decision of the households.

Our algorithm looks for a Markov equilibrium mapping from the financial wealth distribution, \( \omega_t \) - defined in (6), and aggregate shock, \( s_t \), to land price, \( q_t \) and bond price, \( p_t \), the allocation \( \{c_t, h_t, b_t, c_t', h_t', b_t', L_t'\} \) and wage \( w_t \), as well as future financial wealth distributions, \( \omega_{t+1}(s_{t+1}) \), depending on the realization of future aggregate shocks, \( s_{t+1} \). Indeed given the mapping from \( \omega_{t+1} \) to \( \{q_{t+1}, c_{t+1}, c_{t+1}'\} \), for each \( \omega_t \) and \( s_t \), we can solve for \( \{p_t, q_t, w_t\} \) and \( \{c_t, h_t, b_t, c_t', h_t', b_t', L_t'\} \) and \( \omega_{t+1} \) using the equations (8), (9), (10), (11), (12), the land and bond market clearing conditions, as well as the future financial wealth distribution for each future state. Here, we follow the procedure in Cao (2010), instead of the one in Kubler and Schmedders (2003), in solving for \( \omega_{t+1} \) simultaneously with other unknowns. The additional equations needed to solve for \( \omega_{t+1} \) are equation (6) applied to each of the future state \( s_{t+1} \): \( \omega_{t+1} = \frac{q_{t+1}(\omega_{t+1}, s_{t+1}) h_t + b_t}{q_{t+1}(\omega_{t+1}, s_{t+1}) L_t} \), in which the mapping from future wealth distribution and exogenous state to land price, \( q_{t+1}(\omega_{t+1}, s_{t+1}) \), is determined from the previous iteration of the algorithm. It is easy to verify that the number of unknowns are exactly the same as the number of equations.

The algorithm starts by solving for the equilibrium mapping for 1-period economy. Then given the initial mapping for \( T \)-period economy (from period 0 to 1), we can solve for the initial mapping for \( (T + 1) \)-period economy following the procedure described above. The algorithm converges when the initial mappings for \( T \)-period economy and \( (T + 1) \)-period economy are sufficiently close to each other.

An important difference relative to Kubler and Schmedders (2003) and Cao (2010) is that entrepreneurs are only subject to the natural borrowing limit, therefore when \( \omega_t \) is exactly at the natural borrowing limit \( \omega_t = \omega_t = 0, c_t = 0 \) and the first-order conditions (8) and (9) are not well-defined.\(^20\) To deal with this issue, we solve separately the equilibrium of the economy at \( \omega_t \). At \( \omega_t \), we know \( c_t = 0 \), so we solve for \( \{h_t, b_t, c_t', h_t', b_t', L_t', \omega_{t+1}\} \) given the same set of equations described above.\(^21\) Given that the behavior of the econom-

\(^20\)When the entrepreneurs have some labor endowment, \( \omega_t \) can actually be slightly negative.

\(^21\)We can show that \( \omega_{t+1} = 0 \). In Appendix C, we provide an analytical example that it is possible (even in finite horizon) to have \( \omega_{t+1} = 0 \) and \( c_{t+1} = 0 \) but \( h_{t+1}, q_{t+1} > 0 \).
Table 2: Baseline Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{A}$</td>
<td>1</td>
</tr>
<tr>
<td>mean technology</td>
<td></td>
</tr>
<tr>
<td>$j$</td>
<td>0.1</td>
</tr>
<tr>
<td>$H$</td>
<td>1</td>
</tr>
<tr>
<td>$\beta$</td>
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<tr>
<td>$\gamma$</td>
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<tr>
<td>$\sigma_1$</td>
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</tr>
<tr>
<td>$\sigma_2$</td>
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</tr>
<tr>
<td>$\sigma_h$</td>
<td>1.00</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.00</td>
</tr>
<tr>
<td>$\upsilon$</td>
<td>0.03</td>
</tr>
</tbody>
</table>

2.4 Parameter Values

We use parameter values from Iacoviello (2005), given in Table 2. In particular, the value of $\upsilon$ is chosen to make sure that the value of land holding for entrepreneurs (commercial real estate in the data) in steady state is around 20%.

In order to use the global nonlinear method in Subsection 2.3.2, we need to discretize the process for $A_t$ by a finite number of points. For $A_t$ we use a three point process, $A_t \in \{A - \Delta, A, A + \Delta\}$, which corresponds to booms, $s_t = G$, normal times, $s_t = N$, and recessions, $s_t = B$.

We assume the following form of transition matrix

$$
\Pi = \begin{bmatrix}
\pi & 1 - \pi & 0 \\
\frac{1 - \pi_0}{2} & \pi_0 & \frac{1 - \pi_0}{2} \\
0 & 1 - \pi & \pi
\end{bmatrix}.
$$

The exogenous stochastic process of productivity is totally symmetric. However, due to incomplete markets, the resulting dynamics of the economy become asymmetric as shown in Subsection 2.5 below.

The values of $\pi_0$ and $\pi$ are calibrated using historical U.S. data. According to the definitions of business cycle expansions and recessions by NBER, there are 12 recessions in

---

22Given that the households are allowed to borrow from the entrepreneurs, the upper bound for $\omega_t$ should exceed 1. However, $\omega_t$ tends to fall below 1, because of the households’ tendency to lend given their higher discount factor.

23A two-state process is enough to illustrate the amplification and asymmetric effects, but in order to match several moments of the productivity process in the U.S. we need at least three states.
total from post-WWII (1945) to December 2013, with an average length of each recession of 3.6 quarters. The share of months spent in recessions is 15.7%. Given the transition matrix $\Pi$ above, the average length of each recession is $\frac{1}{1-\pi}$ so $\pi = 72.04\%$. $\pi_0$ is chosen such that the probability of a recession in the stationary distribution for $A_t$, i.e., $A_t = A - \Delta$ matches the share of months spent in recession which implies $\pi_0 = 87.2\%$.\textsuperscript{24} As we normalize $A$ to 1, in numerical simulations, we vary $\Delta$ from 1% to 6%, in order to study the non-linear effects of shocks with different sizes. \textsuperscript{25} In the benchmark set of parameters, we choose an intermediate value for $\Delta, \Delta = 3\%$.

### 2.5 Numerical Results

In this subsection, we present the numerical results for the benchmark incomplete markets economy with the parameters in Subsection 2.4.

One important ingredient of our solution method is that for each $(s_t, \omega_t)$, we have to solve for the future wealth distribution $\omega_{t+1}$ for each realization of $s_{t+1}$. The left panel of Figure 1 shows $\omega_{t+1}$ as functions of $\omega_t$ and $s_{t+1}$ given that $s_t$ is in the normal state, i.e. $s_t = N$. Given that the entrepreneurs are more exposed to the productivity shock because they are the only agents who can produce, their wealth increases (relative to the households’) as the good shock hits next period (solid blue line), and decreases as the bad shock hits next period (dotted red line). If $s_{t+1}$ stays at the normal state, then the wealth distribution remains almost unchanged as the future wealth function (dashed purple line) stays close to the 45\textdegree line (dashed black line). The transition functions for the wealth distribution combined with the transition matrix of the exogenous states determine the stationary wealth distribution in the right panel (we plot the density of the distribution). Given the small share of land in the aggregate production function, the wealth share of the entrepreneurs always stays below 10\% in the stationary distribution.

Figure 2 shows the policy functions (consumption of the entrepreneurs and the households and aggregate output) and pricing function for land conditional on the exogenous state $s_t$ and the endogenous state $\omega_t$.\textsuperscript{26} Even though the global nonlinear method solves...

\textsuperscript{24}From the transition matrix for $A_t$, the probability of recession in the stationary distribution is $\frac{1}{1-\pi_0}$.\textsuperscript{25}Given the exogenous process for productivity, the standard deviation of productivity is given by $\sqrt{\frac{1-\pi_0}{\pi-\pi_0}}\Delta$. From our own estimates using quarterly TFP data from St Louis Fed (see empirical appendix D), $\Delta$ is approximately 1.8\%. This low value of $\Delta$ is consistent with the the estimates from Liu, Wang, and Zha (2013), which yield $\Delta$ slightly lower than 1\%. However, to generate the annual standard deviation of productivity in the U.S. economy of about 15\% in Khan and Thomas (2013), $\Delta$ is approximately 6\%. Given the wide range of the possible values of $\Delta$, we choose the range [1\%, 6\%] for $\Delta$.\textsuperscript{26}A more precise measure of output should include imputed rental value of land consumed by the
for the policy and pricing functions for the whole range of $\omega_t$, Figure 2 is restricted to the values of $\omega_t$ in the support of the stationary distribution of $\omega_t$. We observe that, despite the absence of a collateral constraint, land price and output functions are nonlinear in the wealth distribution. In particular they are more sensitive to changes in the wealth distribution when the wealth of the entrepreneurs is low.\footnote{Another way to see the nonlinearity is to look at $\frac{dx}{d\omega_t}$, $x = q$ or $Y_t$, as functions of $\omega_t$, i.e. the marginal effect of redistributing wealth from the households to the entrepreneurs to the households on land price and output. Figure 2 suggests that this function is decreasing in $\omega_t$ and is much higher at lower $\omega_t$.}

Nonlinearity implies asymmetric responses of the equilibrium land price and output with respect to productivity shocks. Starting from $s_t = N$, the good shock, i.e., $s_{t+1} = G$ increases the entrepreneurs’ wealth and the bad shock $s_{t+1} = B$ decreases the entrepreneurs’ wealth as shown in Figure 1. But conditional on the same change in entrepreneurs’ wealth, land price and output increase after a good shock by less than they decrease after a bad shock due to nonlinearity. This leads to the asymmetric responses of equilibrium land price and output to symmetric shocks, as shown quantitatively in Table 1.

The asymmetric responses come from the \textit{net worth effect} as follows. An initial negative shock decreases land productivity and consequently land price. Because of market households, i.e.

\[
\dot{Y}_t = Y_t + \frac{h'_t}{c'_t - \sigma h'_t} h'_t.
\]

But the results are very similar to the results for the current output measure $Y_t$.\footnote{Another way to see the nonlinearity is to look at $\frac{dx}{d\omega_t}$, $x = q$ or $Y_t$, as functions of $\omega_t$, i.e. the marginal effect of redistributing wealth from the households to the entrepreneurs to the households on land price and output. Figure 2 suggests that this function is decreasing in $\omega_t$ and is much higher at lower $\omega_t$.}
Figure 2: Policy and Pricing Functions under Incomplete Markets
incompleteness, the value of debt of the entrepreneurs, which is determined from the last period, is independent of the shock. Therefore, the decrease in land price leads to a decrease in the net worth of the entrepreneurs, which reduces their demand for land. Because only the entrepreneurs can use land to produce, the reduction in their land demand further depresses the price of land, and further lowering their net worth, setting off the vicious cycle of falling land price and falling net worth. Thus the effect of a negative productivity shock on land price and production is amplified. Moreover, because land price is a concave function of the net worth of the entrepreneurs, a negative productivity shock has larger effect than a positive productivity shock through the net worth channel. In Appendix C, we study a two-period version of our model with log utilities to derive analytically the net worth effect.\footnote{To illustrate this effect, Figure 3 plots the portfolio choice (land and bond holdings) of the entrepreneurs as functions of wealth distribution given that the current state is normal (solid blue lines). The figure also plots the portfolio choice next period if the economy stays in the normal state (dashed purple lines) or if the economy enters a recession (dotted red lines). The figure shows that after a bad shock, the entrepreneurs reduce their land holding, as well as borrowing.\footnote{Quantitatively, Row 1 in Table 1 shows the average (over the stationary distribution) of the changes in land price and output given the current normal state, $s_t = N: \frac{x_{t+1} - x_t}{x_t}$, where $x = q$ or $Y$. We observe significant amplification and asymmetric effects under incomplete markets, even though these effects are smaller compared to the responses in the model with collateral constraint below. The last row of Table 1 shows the changes of land prices in response to shocks in the long run in the complete markets equilibrium presented in Appendix A. Compared to the complete markets outcomes, the model with incomplete markets exhibits both amplification and asymmetric effects. Lastly, Table 1 only shows the average responses. Due to the nonlinearity of the solution shown in Figure 2, the amplification and asymmetric effects are also much larger conditional on the lower values of $\omega_t$.}}

To illustrate this effect, Figure 3 plots the portfolio choice (land and bond holdings) of the entrepreneurs as functions of wealth distribution given that the current state is normal (solid blue lines). The figure also plots the portfolio choice next period if the economy stays in the normal state (dashed purple lines) or if the economy enters a recession (dotted red lines). The figure shows that after a bad shock, the entrepreneurs reduce their land holding, as well as borrowing.\footnote{We also show that, under complete markets, this effect disappears because the entrepreneurs can perfectly insure net worth against productivity shocks.}

\footnote{The entrepreneurs can also smooth consumption and land holding by borrowing more from the households. But given that all entrepreneurs try to do the same, each entrepreneur does not take into account the effect of her increased borrowing demand in increasing interest rate for other entrepreneurs. That is, there is a pecuniary externality a la Geanakoplos and Polemarchakis (1986) in interest rate. Indeed, interest rate increases so much that the entrepreneurs actually reduce their borrowing.}
Figure 3: Portfolio Choice for Incomplete Markets Model
3  Incomplete Markets with Collateral Constraint

While the benchmark model with incomplete markets delivers amplified and asymmetric responses of the economy to exogenous shocks, adding a collateral constraint as in Kiyotaki and Moore (1997) will a priori exacerbate these responses. To quantitatively examine the significance of this constraint in addition to the incomplete markets channel presented in the previous section, we impose a collateral constraint on the borrowing decision of the entrepreneurs. We use the same global solution method presented in Subsection 2.3.2 to solve for the dynamic stochastic general equilibrium in this model. In addition, with the collateral constraint, the model has a steady state, so we can log-linearize around the steady state (assuming the collateral constraint is always binding) as in Iacoviello (2005).\footnote{In a recent paper, Liu, Wang, and Zha (2013) use the same log-linearization method to solve for the equilibrium in an elaborate version of the model in Iacoviello (2005).}

We can then compare the accuracy of the two solution methods.

As in Kiyotaki and Moore (1997) and Iacoviello (2005), we assume a limit on the obligations of the entrepreneurs. Suppose that, if borrowers repudiate their debt obligations, the lenders can repossess the borrower’s assets by paying a proportional transaction cost \( (1 - m) \mathbb{E}_t [q_{t+1}] h_t \). In this case the maximum amount that a creditor can borrow is bounded by \( m \mathbb{E}_t [q_{t+1}] h_t \), i.e.

\[
\begin{align*}
  b_t + m \mathbb{E}_t [q_{t+1}] h_t &\geq 0, \\
\end{align*}
\]

where \( b_t \) is the saving (\( -b_t \) is borrowing).\footnote{Given their higher discount factor, the households tend to lend to the entrepreneurs so we do not need to impose any borrowing constraint on the households.}

Let \( \mu_t \) denote the Lagrangian multiplier on the entrepreneurs’ collateral constraint. The F.O.C in the entrepreneurs’ land holding is

\[
\begin{align*}
  (\pi_t - q_t) c_t^{-\sigma_1} + \mu_t m \mathbb{E}_t [q_{t+1}] + \gamma \mathbb{E}_t \left[ q_{t+1} c_t^{-\sigma_1} \right] &= 0, \\
\end{align*}
\]

and the complementary-slackness condition is

\[
\mu_t (b_t + mh_t \mathbb{E}_t [q_{t+1}]) = 0. 
\]

The F.O.C for the entrepreneurs with respect to bond holding is

\[
\begin{align*}
  -p_t c_t^{-\sigma_1} + \mu_t + \gamma \mathbb{E}_t \left[ c_t^{-\sigma_1} \right] &= 0. \\
\end{align*}
\]
We rewrite (14) as

\[ q_t = \pi_t + \gamma E_t \left[ q_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma_1} \right] + \mu_t m E_t [q_{t+1}] c_t^{\sigma_1}. \]

The first two terms on the right hand side of this equation show that, from the entrepreneurs point of view, land price is the net present value of present and future profit from production using land. In addition, the last term on the right hand side shows the collateral value of land in the land valuation of the entrepreneurs. Iterating this equation forward, we obtain the expression for land price as the present discounted value of profit, with the discount factor depending on the marginal utility of the entrepreneurs as well as on the multiplier on the collateral constraint:

\[ q_t = \pi_t + E_t \left[ q_{t+1} \left\{ \gamma \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma_1} \right\} + \mu_t m c_t^{\sigma_1} \right] \]

\[ = E_t \left[ \sum_{s=0}^{\infty} \prod_{r=0}^{s-1} \left\{ \gamma \left( \frac{c_{t+r+1}}{c_{t+r}} \right)^{-\sigma_1} + \mu_{t+r} m c_{t+r}^{\sigma_1} \right\} \pi_{t+s} \right]. \]

Other conditions are the same as in the benchmark incomplete markets model in Section 2. Similar to Lemma 1 in Section 2, we can easily show that \( \omega_t \geq 0 \) in any competitive equilibrium under collateral constraints.\(^{33}\)

### 3.1 Numerical Results

We can apply the nonlinear global solution method for the benchmark model in the previous section to solve for the dynamic equilibrium of this model. In addition, in this model, a steady state exists, we can also log-linearize around the steady-state and examine the accuracy of the log-linearized solution. The equations that determine the steady state and the log-linearized dynamics are presented in Online Appendix F. In this subsection, we report numerical results for the collateral constraint model with the size of the productivity shock chosen at 3\% and other parameters taken from Table 2.4. Moreover, we set the margin \( m = 0.89 \) as in Iacoviello (2005). The most important properties of the numerical solutions are the following.

First of all, the fully nonlinear solution for Markov equilibrium features an occasion-\(^{32}\)

\(^{32}\)This formula is similar to the one in Mendoza (2010). In particular, when the entrepreneurs cannot borrow against their land holding, i.e., \( m = 0 \), there will not be any collateral premium in the pricing of land.

\(^{33}\)We can also show that at the lower bound \( \omega_t = 0, q_t = p_t = 0 \) and \( h_t = c_t = Y_t = 0 \).
ally binding collateral constraint. Collateral constraint (13) binds when the entrepreneurs’ wealth is sufficiently low. In this binding region, endogenous variables including land price and output are more sensitive to changes in wealth distribution. Second, the equilibrium is asymmetric with respect to bad shocks versus good shocks despite the fact that the stochastic structure of the shocks is totally symmetric. Third, the log-linearization solution, by assuming that the collateral constraint always binds, over-estimates the effect of the shocks. Lastly, the probability that the collateral constraint binds decreases rapidly with the size of the shocks due to the precautionary saving motive of the entrepreneurs. Indeed, in the stationary distribution, the binding probability is 83.76% when the size of the shock is 1% and 5.43% when the size of the shock is 5%.

Figure 4 shows the equilibrium price, output, and consumption of the consumers and entrepreneurs for 3% shocks as functions of $\omega$ in recession (thick blue lines) compared against the same functions in the benchmark model (thin-dashed red lines). With the collateral constraint, the functions exhibit significantly more nonlinearity when $\omega_t$ is close to zero. This is because the collateral constraint is binding when $\omega_t$ is close to 0. And when the collateral constraint binds, the standard feedback effect in Kiyotaki and Moore (1997) kicks in: after a negative productivity shock, even temporary, the entrepreneurs have to cut back their land holding, $h_t$, in order to smooth consumption. This reduction in land demand depresses land price, $q_t$, which through the collateral constraint, forces the entrepreneurs to reduce their debt, partly by reducing consumption, and further cut back their land holding. This vicious cycle results in a significant decline in land price, as well as in the entrepreneurs’ land holding and total output. There is also an inter-temporal feedback: lower current wealth of the entrepreneurs leads to lower future wealth and lower future land prices. Given that the current land price is a weighted average of current per unit profit and future land price, as shown in the asset pricing equations, lower future land price in turn leads to lower current land price. The entrepreneurs use land to produce; so a significant decrease in land holding leads to a significant decrease in output.

The advantage of the nonlinear solution method used in this paper is that, we can see clearly two regions of the state space in which the economy follows different dynamics. In one region, when the collateral constraint is binding (or nearly binding), the feedback effect is important. In the other region when the collateral constraint is far from binding, asset price and output are less sensitive to changes in the wealth distribution.\footnote{As noticed in footnote 27, we can see different dynamics between the two regions by looking at $\frac{dx_t}{d\omega_t}$, $x = q_tY$, as functions of $\omega_t$, i.e. the marginal effect of redistributing wealth from the households to the entrepreneurs to the households on land price and output. Figure 4 shows that this function is decreasing in $\omega_t$ and is much higher at lower $\omega_t$ when the collateral constraint is binding.}

Figure 5 corresponds to Figure 1 for this case with the collateral constraint. Definition
2 of Markov equilibrium and the algorithm in Subsection 2.3.2 provide the evolution of wealth distribution over time. To illustrate the method, the left panel of Figure 5 shows $\omega_{t+1}$ as functions of $\omega_t$ when the current state is $s_t = N$. Next period’s wealth distribution depends on the realization of the exogenous state $s_{t+1}$. By assumption on the stochastic structure of shocks described in Subsection 2.4, $s_{t+1}$ can be $G$, $N$ or $B$. As shown in the figure, when $\omega_t$ is close to $\underline{\omega}_t$, $\omega_{t+1} > \omega_t$, thus the lowest levels of wealth are never reached. This result is in contrast to the one in the benchmark model (or in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013)). Using these transition functions for the wealth distribution, we can compute the stationary distribution for the wealth distribution, $\omega_t$, over the business cycles. The right panel of Figure 5 shows the stationary distribution (density) for $\omega_t$ conditional on the exogenous state $s_t$. Different from the benchmark model, the density becomes zero before the lowest thresholds $\underline{\omega}_t$.

The most important difference between the model with a collateral constraint and the benchmark model is the possibility that the collateral constraint is binding. Figure 6 illustrates this point. The upper panel shows the portfolio choice of the entrepreneurs as a function of $\omega_t$ and $s_t = N$. On the left of the vertical green line (in both panels), the collateral constraint is binding. A 3% shock yields a (conditional) binding probability in normal
Figure 5: Wealth Distribution Transition in Normal Times
times of 23.49%. The binding probabilities for other states are given in Table 3. Because of the important nonlinearity when the collateral constraint is binding, dynamically the entrepreneurs try to avoid this region by precautionary saving. Precautionary saving decreases the likelihood of a binding collateral constraint, and significantly so when shocks are large.

Figure 7 compares the impulse-responses of the log-linearization versus the global nonlinear method. Starting from the long run mean level of wealth, we assume the economy is hit by a sequence of 4 good shocks (red dotted line) and 4 bad shocks (blue solid line) respectively and returns to the normal state afterwards.\textsuperscript{35,36} In the case of bad shocks, we plot the minus of relative changes in land price. This figure illustrates the asymmetric effect of collateral constraint. Positive shocks increase the land price by only

\textsuperscript{35}As documented in Subsection 2.4, the average length of recessions is 3.6 quarters, so we use 4 shocks for the impulse responses.

\textsuperscript{36}Another way to plot the impulse-responses is to simulate the economy using the transition matrix $\Pi$ in Subsection 2.4 starting from a good shock or a bad shock, and take the average dynamics of the economy across simulations.
3.2% at impact, while negative shocks decrease land price by 3.8% at impact. Negative shocks also have a more persistent effect on land price. The black dotted line shows the IRFs under log-linearization. Under log-linearization, the responses to shocks are perfectly symmetric so we only plot the IRFs under positive shocks. As shown in the figure, by assuming that the collateral constraint is always binding, log-linearization overstates the effects of shocks. Land price changes by close to 5% at impact and the changes are also more persistent.

Another way to capture the asymmetric effect of the collateral constraint is to calculate the average (weighted by the stationary distribution) percentage change in land price and output in normal times, i.e., $s_t = N$, when the shocks hit the economy. The third row of Table 1 shows that good shock changes price by 3.30% and output by 3.15% on average while bad shock changes price by $-3.76\%$ and output by $-3.5\%$. Compared to the complete markets outcomes (the last row), the collateral constraint model exhibits both amplification and asymmetric effects.\footnote{Even though the amplification effect here is relatively small - land price declines by 3.76\% after bad shock, on average, compared to 3\% under complete markets - the effect will be significantly larger if we increase the share of land, $v$, in the production function as in \textit{Kocherlakota (2000)}. Moreover, land price also declines much more after bad shocks when the collateral constraint is binding.}

The difference between the log-linearization solution and fully nonlinear solution de-
Table 3: Probabilities of Binding Collateral Constraint

<table>
<thead>
<tr>
<th>ΔA</th>
<th>Expansion</th>
<th>Normal</th>
<th>Recession</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>5.28%</td>
<td>98.04%</td>
<td>99.83%</td>
<td>83.76%</td>
</tr>
<tr>
<td>2%</td>
<td>1.51%</td>
<td>33.37%</td>
<td>86.06%</td>
<td>36.64%</td>
</tr>
<tr>
<td>3%</td>
<td>0%</td>
<td>23.49%</td>
<td>70.22%</td>
<td>27.14%</td>
</tr>
<tr>
<td>4%</td>
<td>0%</td>
<td>4.89%</td>
<td>40.40%</td>
<td>9.70%</td>
</tr>
<tr>
<td>5%</td>
<td>0%</td>
<td>0.53%</td>
<td>32.30%</td>
<td>5.43%</td>
</tr>
<tr>
<td>6%</td>
<td>0%</td>
<td>0%</td>
<td>16.33%</td>
<td>2.56%</td>
</tr>
</tbody>
</table>

Table 4: Average land price and output changes after a 6% increase or 6% decrease in aggregate productivity

<table>
<thead>
<tr>
<th>Type of Friction</th>
<th>Land price</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expansion</td>
<td>Recession</td>
</tr>
<tr>
<td>incomplete markets (Model 1)</td>
<td>6.32%</td>
<td>-6.66%</td>
</tr>
<tr>
<td>collateral constraint (IM - Model 2)</td>
<td>6.34%</td>
<td>-6.72%</td>
</tr>
</tbody>
</table>

The result in Figure 7 comes from the assumption that the collateral constraint always binds around the steady state using log-linearization method. This assumption becomes less accurate as the size of shock increases because the entrepreneurs would engage more in precautionary saving. As a result the collateral constraint will not bind all the time. Table 3 shows the binding probability in the long run stationary distribution for wealth distributions as a function of the size of the shock. Column 2-4 shows the binding probabilities conditional on the realization of the exogenous shocks, and column 5 shows the unconditional binding probabilities. The binding probabilities decrease very fast in the size of the shock. This result also suggests that when Δ is larger than 3%, the gap between the amplification and asymmetric effects in the collateral constraint model and the benchmark model without collateral constraint in Table 1 will be smaller because the collateral constraint binds less often. For example, Table 4 shows the average responses of land price and output to TFP shocks at Δ = 6% in the collateral constraint model and in the benchmark model. The responses are very close to each other.

We end this subsection by noting that our fully nonlinear solution does not exhibit the volatility paradox presented in Brunnermeier and Sannikov (2014), i.e., lower exogenous risk can lead to higher endogenous risk. As shown in Table 3, as we decrease the size of the exogenous shocks, the binding probability goes to 1 and the nonlinear solu-

---

38 In Appendix C of Iacoviello (2005), the author shows that the binding probability is close to 1 when the size of the shocks is calibrated to the U.S. data. The binding probability is much smaller in this paper because of our different stochastic structure of the productivity shocks. For example, our stochastic structure implies more persistent shocks (which leads to more precautionary saving) given the same standard deviation of the shocks.
tion becomes closer to the log-linear solution, and both converge to the steady state with no endogenous risk. The difference between our solution and Brunnermeier and Sannikov (2014)’s comes from the fact that we do not allow the households to start producing when the entrepreneurs’ wealth goes to zero. If we assume, as in Brunnermeier and Sannikov (2014), that the households can use an alternative, inefficient production technology, $Ah^\nu L^{1-\nu}$, where $A < \min(A_t)$ to produce, then we should recover the volatility paradox. In the Online Appendix, we present such a model.

### 3.2 Alternative Collateral Constraints

In the collateral constraint (13), we use the expected future land price. This constraint can be micro-founded under limited commitment and has been used in a large number of papers including Kiyotaki and Moore (1997), Iacoviello (2005), and Cao (2011).\(^{39}\) However, in practice collateralized contracts are often written using current asset prices and this is also assumed in a large number of papers, for example Mendoza (2010). Figure 5 shows that the wealth distribution, $\omega_t$, moves very slowly over the business cycles, as a result land price also moves slowly. Therefore using current or expected future land price should not imply quantitatively significant differences between the two models. We can show this result rigorously by solving an alternative model in which the collateral constraint (13) is replaced by the following alternative collateral constraint:

$$b_t + mq_t h_t \geq 0.$$  

Fortunately this case is a special case of the general Markov equilibrium definition and the solution method in Cao (2010). We solve for the Markov equilibrium under this alternative collateral constraint for the same parameters used in Subsection 3.1. The solution is quantitatively similar to the solution of the collateral constraint model in Subsection 3.1. For example, Row 3 in Table 1, shows that the amplification and asymmetric effects are only slightly higher than the ones in Subsection 3.1 (Row 2).\(^{40}\)

---

\(^{39}\)Cao (2011) shows that when the lender can seize a fraction of the asset upon default, the collateral constraint arises endogenously and has the form

$$b_t + mh_t \min_{\bar{s}, \bar{q}} q_{t+1} \left(s^t + 1\right) \geq 0.$$  

\(^{40}\)However, the binding probability is significantly higher at 46.54% compared to 27.14% in the model in Subsection 3.1.
4 Complete Markets with Collateral Constraints

The comparison between the benchmark incomplete markets model in Section 2 and the collateral constraint model in Section 3 suggests that one of the main ingredients for the amplification and asymmetric effects is market incompleteness beside the collateral constraint. To further demonstrate this point, we study a variation of the collateral constraint model, which we call the collateral constraint with complete markets model. In this model, we maintain the collateral constraint, however we allow the agents to trade a complete set of Arrow securities, subject to the collateral constraint. In history $s_t^l$, let $p_t(s_{t+1})$ denote the price of the Arrow security that pays off one unit of consumption good if $s_{t+1}$ happens and nothing otherwise. Let $\phi_t(s_{t+1})$ and $\phi_t'(s_{t+1})$ denote the holdings of the entrepreneurs and the households, respectively, of the Arrow security. The definition of competitive equilibrium as well as Markov equilibrium are exactly the same as in the benchmark model, except now we need to impose the condition that the markets for the Arrow securities clear, i.e. $\phi_t(s_{t+1}) + \phi_t'(s_{t+1}) = 0$.

In this model, the budget constraint of the entrepreneurs becomes:

$$c_t + q_t(h_t - h_{t-1}) + \sum_{s_{t+1}|s^l} p_t(s_{t+1}) \phi_t(s_{t+1}) \leq \phi_{t-1}(s_t) + Y_t - w_t L_t. \quad (17)$$

and the collateral constraint (13) is now

$$\phi_t(s_{t+1}) + mq_{t+1} h_t \geq 0 \forall s_{t+1}|s^l. \quad (18)$$

Let $\tilde{p}_t(s_{t+1}) = \frac{p_t(s_{t+1})}{\text{Pr}(s_{t+1}|s_t)}$ and $\mu_t(s_{t+1}) \text{Pr}(s_{t+1}|s_t)$ denote the multiplier on the constraint (18) for each $s_{t+1}|s^l$. The first-order condition in $\phi_t(s_{t+1})$ implies

$$-\tilde{p}_t(s_{t+1}) c_t^{-\sigma_1} + \mu_t(s_{t+1}) + \gamma (c_{t+1}(s^l, s_{t+1}))^{-\sigma_1} = 0 \quad (19)$$

and the first-order condition in $h_t$ implies

$$(\pi_t - q_t) c_t^{-\sigma_1} + m \mathbb{E}_t [\mu_t(s_{t+1}) q_{t+1}] + \gamma \mathbb{E}_t [q_{t+1} c_t^{-\sigma_1}] = 0. \quad (20)$$

Similarly, the budget constraint of the households changes to:

$$c'_t + q_t(h'_t - h'_{t-1}) + \sum_{s_{t+1}|s^l} p_t(s_{t+1}) \phi'_t(s_{t+1}) \leq \phi'_{t-1}(s_t) + w_t L'_t.$$
Figure 8: Price and Policy Functions under Collateral Constraints with Complete Markets versus Collateral Constraint with Incomplete Markets, $s_t = B$.

From the optimal decision of the households, we have

$$-\tilde{p}_t (s_{t+1}) c'_{t}^{-\sigma_2} + \beta (c'_{t+1} (s^t, s_{t+1}) )^{-\sigma_2} = 0. \quad (21)$$

Other conditions, including the first-order condition with respect to $h'_t$ of the households stay the same as in the collateral constraint collateral constraint and the benchmark models in Sections 2 and 3.

Figure 8 shows the differences between the price and policy functions in this current model with collateral constraints and complete markets and in the model with a collateral constraint and incomplete markets in Section 3 (when $s_t = B$). In contrast to the collateral constraint model in Section 3, land price in the current model differs from land price in the collateral constraint with incomplete markets model only for intermediate levels of wealth of the entrepreneurs.

More importantly, numerical simulations show that, unlike the cases with incomplete markets (with or without collateral constraint), the long run stationary distribution of wealth is degenerate and concentrates on $\omega^*$, regardless of the exogenous state. At
\( \omega_t = \omega^* \), the collateral constraint is binding for all future states, and land demand of the entrepreneurs is given by \( h^* \). Therefore: 
\[
\omega^* = \frac{q_{t+1}^* - m q_{t+1} h^*}{q_{t+1} H} = (1 - m) \frac{h^*}{H} .
\] In Online Appendix G, we present the other equations that determine the equilibrium at \( \omega^* \). The parameters in Subsection 2.4 imply that \( \omega^* \) is around 0.0235.

At \( \omega^* \), Table 1 shows that both the amplification and asymmetric effects disappear. In order to understand how the complete set of Arrow securities help the entrepreneurs to insure against negative shocks, Figure 9, lower panel, shows the entrepreneurs’ optimal choices of \( \phi_t(s_{t+1}) \) as functions of the wealth distribution and the future exogenous state \( s_{t+1} \), given the current state \( s_t = N \). For clarity we also plot the choice of \( \phi_t \) for \( s_{t+1} = G \) and \( s_{t+1} = B \). Below \( \omega^* \), the collateral constraints are binding for all future states. However, above but close to \( \omega^* \), the collateral constraints are not binding, and the entrepreneurs borrow relatively more from the future good state compared to the future bad state.

The solution of this current model with collateral constraint but with complete markets shows that when the agents have access to a complete set of securities, amplification and asymmetric effects disappear. This point is also made by Krishnamurthy (2003) in a
simple 3-period economy and DiTella (2014) in a continuous-time model.\footnote{However, in DiTella (2014), collateral constraint arises endogenously because of moral hazard problems that constrains the entrepreneurs’ borrowing capacity, as derived in He and Krishnamurthy (2011).}

5 Some Empirical Evidence

The results from our model show that productivity (TFP) shocks have asymmetric effects on the output and land prices. First, negative TFP shocks have larger effects than positive TFP shocks as indicated in Table 1. Second, the effects also depend on the wealth distribution: they are larger when the wealth share of the entrepreneurs is low. Lastly, combining the previous two points, the responses should be the strongest when we have both low TFP shocks and low wealth share of the entrepreneurs. We offer some empirical evidence of these implications in this section.\footnote{Following Liu, Wang, and Zha (2013), we interpret the entrepreneurs in the model as the non-financial business sector in the data.} In particular, we use four quarterly U.S. time series: the TFP shock, the real price of land, the real output and the real estate leverage ratio of the non-financial business (corporate and non-corporate) sector. The real estate leverage ratio is defined and constructed as the ratio of the credit market debt to the value of real estate which is used as a proxy for the wealth share of the entrepreneurs. As can be seen from Figure 10, the real estate leverage ratio of the entrepreneurs is strictly decreasing in their wealth share in incomplete markets model with or without collateral constraints. The sample covers the period from 1951:Q4 to 2010:Q4. Appendix D provides more detailed descriptions of the data.

Our specifications take the following three forms:

\[ \Delta \log x_t = \alpha_1 + \beta_{tfp,high} I_t \Delta \log \text{TFP}_t + \beta_{tfp,low} (1 - I_t) \Delta \log \text{TFP}_t + \gamma_1 Z_t + \epsilon_{1t} \]  

\[ \Delta \log x_t = \alpha_2 + \beta_{lev,high} I_{t-1} \Delta \log \text{TFP}_t + \beta_{lev,low} (1 - I_{t-1}) \Delta \log \text{TFP}_t + \gamma_2 Z_t + \epsilon_{2t} \]  

\[ \Delta \log x_t = \alpha_3 + \beta_{lev,high,tfp,high} I_{t-1} \Delta \log \text{TFP}_t + \beta_{lev,high,tfp,low} I_{t-1} (1 - I_t) \Delta \log \text{TFP}_t + \beta_{lev,low,tfp,high} (1 - I_{t-1}) \Delta \log \text{TFP}_t + \beta_{lev,low,tfp,low} (1 - I_{t-1}) (1 - I_t) \Delta \log \text{TFP}_t + \gamma_3 Z_{t-1} + \epsilon_{2t} \]  

where \( x_t \) is either real output or real land price, \( I_t \) is an indicator for TFP shock which equals value 1 when TFP shocks are high, and value 0 when they are low\footnote{We use the first quintile of the TFP shocks as the threshold value. As indicated by our model, a negative TFP shock generates the amplified effects only when it is low enough.}. Similarly, \( \mathcal{I}_t \)
Figure 10: Real Estate Leverage Ratio and Wealth Distributions

Note: The y-axis is the ratio of the entrepreneurs’ bond holding to the value of their land holding, which is the negative of the real estate leverage ratio.

is an indicator for the real estate leverage ratio which equals value 1 when the leverage ratio is high, and value 0 when the leverage ratio is low. We classify the leverage ratio as high when they are above the linear trend over the sample period\(^{44}\). As a state variable, we use the lagged value for \( I_t \) in the regressions. \( Z_t \) is a vector of control variables.

In equation (22), due to the asymmetric effect on TFP shocks, we expect \( \beta_{\text{tfp, low}} \) larger than \( \beta_{\text{tfp, high}} \). In equation (23), the responses to TFP shocks depend on the real estate leverage ratio in the business sector. Our theoretical results predict that TFP shocks should have larger effects when the wealth share of the entrepreneurs are low, and thus \( \beta_{\text{lev, high}} \) should be larger than \( \beta_{\text{lev, low}} \). In equation (24), the responses to TFP shocks depend both on whether the real estate leverage ratio is high or low and whether the TFP shock is high or low, so we have four different values of \( \beta \) for four different situations. We expect \( \beta_{\text{lev, high, tfp, low}} \) to be the largest among the four.

Table 6 presents the regression results about the effect of TFP shocks on changes in land price. Specification (1) does not differentiate whether the TFP shocks or the real estate leverage ratio is high or low. We find that on average 1 percent of TFP shock in-

\(^{44}\) As robustness checks, we also use other two definitions for \( I_t \) including comparing the leverage ratio with its quadratic time trend and with the sample mean. The results obtained are similar.
creases land price by 0.59 percent. This result is significant at the 10% level. Specification (2), which corresponds to equation (22), depends on whether the TFP shock is high or low, and shows that the asymmetric effect on this dimension is strong. The effect of a high TFP shock is small and not significant, while a low TFP shock has large and significant effect such that a 1% TFP shock has strong amplified effect of 1.83% on housing price. Specification (3), which corresponds to equation (23) shows that when the real estate leverage ratio is high, the responses are large and significant at 5% level. On the contrary, when the real estate leverage ratio is low, the responses to TFP shocks are insignificant. Specification (4), which corresponds to equation (24), combines the previous two cases of high or low TFP shock and high or low real estate leverage ratio, resulting in four possible combinations.

We find that TFP shocks have significant effects on land price only when TFP shocks are low, and the effect is the largest when we have both a high real estate leverage ratio and a low TFP shock. The coefficients are only significant at 10% level, possibly because we add many structures in one regression. All the results shown here are consistent with the predictions of our model.

Table 7 presents the regression results about the effect of TFP shocks on changes in real GDP. Similarly, the coefficients show high asymmetric effects depending on whether the TFP shock or the real estate leverage ratio is high or low. Compared to the results of land price in Table 6, the responses of output are smaller, but the asymmetric effects are very robust.

The empirical findings in Tables 6 and 7 complement the recent results in Guerrieri and Iacoviello (2015a), which emphasizes the asymmetric responses of economic activities, including consumption and employment, to changes in house prices. Using state and MSA-level data, they find that changes in economic activities correlate more with house prices when house prices are low. This is consistent with our results because in our model land price is strictly increasing in the entrepreneurs’ wealth share, and thus is strictly decreasing in real estate leverage ratio.

6 Conclusion

In this paper, we have shown that market incompleteness, independently of the collateral constraint, plays a quantitatively significant role in the amplified and asymmetric responses of the economy to exogenous shocks. For simplicity, we only consider one type of shock - productivity shocks. However, it is easy to extend the paper to incorporate other shocks such as housing preference shocks. It would also be interesting to incorporate money into the model to consider the effect of monetary shocks as in Iacoviello.
The current model does not have capital. We can consider adding capital into the model to examine how the amplification and asymmetric effects affect the capital accumulation and the aggregate production processes. Cao (2010) offers a way to introduce capital into this kind of model which enables the use of a similar global nonlinear solution method to the one in Subsection 2.3.2. However, in this case we need to keep track of two endogenous state variables: wealth distribution and aggregate capital.

The importance of market incompleteness shown in this paper also suggests that state-contingent debt can be an important macro-prudential policy tool. Using a model with collateral constraints, Geanakoplos (2010) argues that leverage should be restricted in booms to avoid the fire-sale externality and financial crises in subsequent recessions. Given that market incompleteness plays an important role, designing debts with some insurance for downturns should also be effective in reducing the magnitude of the subsequent recessions. Theoretically, Section 4 shows that complete state-contingent assets, even being subject to collateral constraints, can nullify the amplification and asymmetric effects. In practice, for example, Mian and Sufi (2014) argue that share-responsible mortgages, i.e., mortgages that reduce principal and mortgage payments upon significant declines in housing prices, might have significantly reduced the size of the U.S. financial and economic crisis in 2007-2008.

References


Appendix

A Complete Markets

When markets are complete, because the entrepreneurs are less patient than the households, they tend to consume all their future net worth in the present. If they are risk-neutral as in Brunnermeier and Sannikov (2014), they will consume their entire net worth at time 0, but in our model, they are risk-averse, so they consume their net worth over-time, and their wealth relative to the households’ wealth goes to zero as time goes to infinity. The entrepreneurs can do so by issuing equity to households thanks to frictionless financial markets. We consider the long run limit, when the wealth of the entrepreneurs approaches zero. The economy converges to a representative household economy, with the production technology of the entrepreneurs. Due to the Markovian feature of uncertainty, the endogenous variables depend only on the aggregate state $s_t$: $x_t = x(s_t)$.

Given that the households own the whole production sector, the marginal utility from the marginal profit per unit of land should be equal to the marginal utility of one unit of land consumption, i.e., $(c_t')^{-\sigma} \pi_t = j (h_t')^{-\sigma_h}$. To evaluate the marginal utility of consumption, we observe that the households consume the whole output in the long run so

$$c_t' = Y_t = A_t h_t^v L_t^{1-v}. \tag{25}$$

From the equalization of the marginal utility from the marginal profit and marginal utility from land consumption, and using the expression for wage and profit in Subsection 2.3, we have

$$j (H - h_t)^{-\sigma_h} = \left( A_t h_t^v L_t^{1-v} \right)^{-\sigma_2} v A_t h_t^{v-1} L_t^{1-v}. \tag{26}$$

The consumption and labor trade-off equation (12) implies

$$(1 - v) A_t h_t^v L_t^{-v} \left( A_t h_t^v L_t^{1-v} \right)^{-\sigma_2} = L_t^{v-1}. \tag{26}$$

The two equations (25) and (26) help us solve for the two unknowns $(h_t, L_t)$ for each $A_t$.

Now, given $\pi_t$ as a function of $A_t, h_t$, and $L_t$, land price is determined by the pricing kernel using the marginal utility of the representative household:

$$q(s_t) - \pi(s_t) = \sum_{s_{t+1}|s_t} \beta \frac{u'(c_{t+1}(s_{t+1}))}{u'(c_t(s_t))} q(s_{t+1}) \Pr(s_t, s_{t+1}).$$
Given the solution, we can compute the changes of land price and output after the shock \( s_t \) changes from normal to expansion or recession. The result is shown in the last row of Table 1.

### B Incomplete Markets with Exogenous Borrowing Constraint

In this appendix, we examine an alternative model with exogenous borrowing constraint. The model has the same ingredients as the one in Section 3 except for the following borrowing constraint instead of the collateral constraint (13):

\[
b_t \geq -B. \tag{27}
\]

The borrowing constraint \( B \) is chosen exogenously and we study the behavior of the model when we vary \( B \). Let \( \mu_t \) denote the Lagrangian multiplier associated to this borrowing constraint. The first-order condition with respect to \( h_t \) and \( b_t \) in the maximization problem of the entrepreneurs implies

\[
(\pi_t - q_t)c_t^{1-\sigma_1} + \gamma E_t[q_{t+1}c_{t+1}^{1-\sigma_1}] = 0 \tag{28}
\]

and

\[
- p_t c_t^{1-\sigma_1} + \mu_t + \gamma E_t \left[ c_t^{1-\sigma_1} \right] = 0, \tag{29}
\]

and the complementary-slackness condition is satisfied:

\[
\mu_t (b_t + B) = 0. \tag{30}
\]

Other conditions are the same as in the model with endogenous borrowing constraint in Section 3.

We first solve for the steady state of this model. As in Section 3, from the first-order condition of the households, we have \( p = \beta \). From the first-order condition (29), we have

\[
\mu = (\beta - \gamma) c^{1-\sigma_1}.
\]

From the first-order condition (28), we have

\[
q = \frac{1}{1-\gamma} v A h^{\nu-1} L^{1-\nu}.
\]

This expression of price is different from the one in Section 3 in the discount factor \( \gamma \).
Table 5: Average land price and output changes in normal state, 3% shock

<table>
<thead>
<tr>
<th>Type of Friction</th>
<th>Land price</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expansion</td>
<td>Recession</td>
</tr>
<tr>
<td>incomplete markets ((\bar{B} = \infty))</td>
<td>3.21%</td>
<td>-3.45%</td>
</tr>
<tr>
<td>incomplete markets ((\bar{B} = 2))</td>
<td>3.21%</td>
<td>-3.23%</td>
</tr>
<tr>
<td>incomplete markets ((\bar{B} = 1))</td>
<td>3.14%</td>
<td>-3.13%</td>
</tr>
<tr>
<td>complete markets</td>
<td>3.00%</td>
<td>-3.00%</td>
</tr>
</tbody>
</table>

Instead of \(\gamma^e\). Given that \(\gamma < \gamma^e\), given the same steady state level of \(h\) and \(L\), the land price is lower under the exogenous borrowing constraint than under the endogenous collateral constraint. Consequently, this model with the exogenous borrowing constraint and the collateral constraint model do not share the same steady state.

Outside the steady state, we can use the global nonlinear solution method presented in Subsection 2.3.2 to solve for a Markov equilibrium in this economy. Table 5 is the counterpart of Table 1 for this model with the exogenous borrowing constraint. In particular, Row 1 of Table 5 corresponds to Row 1 in Table 1, in which there is no upper bound on the borrowing of the entrepreneurs. When we tighten the exogenous constraint, the amplification and asymmetric effects are actually reduced. At first sight, this result seems counter-intuitive. However, this result is in line with the discussions in Mendoza (2010) and Kocherlakota (2000). The exogenous borrowing constraint reduces the borrowing of the entrepreneurs, and thus reduces the net worth effect in the benchmark incomplete markets model. An important difference here compared to Kocherlakota (2000), is that under uncertainty, it is possible to have infinite exogenous borrowing constraint in the incomplete markets model (the entrepreneurs limit themselves from borrowing too much because of the precautionary saving motive). Infinite exogenous borrowing constraint leads to the maximal net worth effect, thus significant amplification and asymmetric effects.

C Two-Period Economy

In this appendix, we present a simple version of the general model with two periods and log-linear utility in consumption, land, and leisure.

Consider a two-period version of the benchmark model in Section 2, with \(\sigma_1 = \sigma_2 = \sigma_h = \eta = 1\). That corresponds to the utility of the households,

\[
\mathbb{E}_0 \sum_{t=0}^{1} \beta^t \left\{ \log c_t' + j \log h_t' - L_t' \right\}.
\]
and the utility of the entrepreneurs

$$E_0 \sum_{t=0}^{1} \gamma^t \log c_t.$$  

At time $t = 0$, the entrepreneurs are endowed with a fraction $\omega_0$ of total land supply, and the households are endowed with the remaining fraction, $1 - \omega_0$. The sequential budget constraints, production function, and competitive and Markov equilibria under different financial markets structure are defined as in the main body of the paper.

In order to solve for the Markov equilibrium of this economy, we first solve for the equilibrium in the last period $t = T = 1$ given the financial wealth share of the entrepreneur, $\omega_T$.\(^{45}\)

The entrepreneurs maximize consumption,

$$\max_{c_T, h_T, L_T} \log c_T$$

subject to the budget constraint

$$c_T + q_T h_T = q_T H \omega_T + A_T h_T L_T^{1-v} - w_T L_T.$$  

From the F.O.Cs

$$w_T = (1 - v) A_T \left( \frac{h_T}{L_T} \right)^v$$

$$q_T = v A_T \left( \frac{h_T}{L_T} \right)^{v-1}.$$  

Substituting these expressions into the entrepreneurs’ budget constraint implies

$$c_T = q_T H \omega_T.$$  

The households maximize

$$\max_{c_T', h_T', L_T'} \{ \ln c_T' + j \ln h_T' - L_T' \}$$

subject to

$$c_T' + q_T h_T' = q_T H (1 - \omega_T) + w_T L_T'.$$  

\(^{45}\)This financial wealth share is determined endogenously from equilibrium in period $t = T - 1$, but at period $T$, the agents take it as given.
From the F.O.Cs in $c'_T, L'_T, h'_T$:

\[
\begin{align*}
  c'_T &= w_T \\
  q_T &= j. \\
  \frac{c'_T}{q_T} &= \frac{h'_T}{j}.
\end{align*}
\]

Therefore

\[
\frac{H - h_T}{j} = \frac{w_T}{q_T} = \frac{(1 - v) h_T}{v L'_T},
\]

where the second equality comes from the F.O.Cs of the entrepreneurs. So

\[
L_T = \frac{j(1 - v)}{v} \frac{h_T}{H - h_T}
\]

and

\[
q_T = v A_T \left[ \frac{v}{j(1 - v)(H - h_T)} \right]^{v-1}.
\]

Combining these results with the market clearing condition in the good markets: $c_T + c'_T = Y_T$, we obtain

\[
\begin{align*}
  h_T &= \frac{1 + j \omega_T}{j/v + 1} H \\
  h'_T &= \frac{j/v - j \omega_T}{j/v + 1} H,
\end{align*}
\]

and

\[
q_T = v A_T \left[ \frac{v(1 - \omega_T v)}{(1 - v)(j + v)} H \right]^{v-1}. \tag{31}
\]

We can also solve for $c_T$ and $c'_T$ explicitly:

\[
\begin{align*}
  c_T &= q_T \omega_T = v A_T \left[ \frac{v(1 - v \omega_T)}{(1 - v)(j + v)} H \right]^{v-1} H \omega_T \\
  c'_T &= v A_T \left[ \frac{v (1 - v \omega_T) H}{(1 - v)(j + v)} \right]^{v-1} \frac{(1 - v \omega_T) H}{j + v}.
\end{align*}
\]

These expressions also imply that

\[
\frac{c_T}{c'_T} = (j + v) \frac{\omega_T}{1 - v \omega_T}, \tag{32}
\]

which is strictly increasing in $\omega_T$. We will use this result later when we analyze the two
period economy.

The following propositions summarize the key properties of the equilibrium in the last period:

**Proposition 1.** \( \frac{q_T}{A_T} \) is strictly increasing in \( \omega_T \) and \( q_T > 0 \) when \( \omega_T = 0 \).

*Proof.* From (31),

\[
q_T = v \left[ \frac{v(1 - \omega_T v)}{(1 - v)(j + v)} H \right]^{v-1},
\]

which is strictly increasing in \( \omega_T \) given that \( v < 1 \). Moreover, when \( \theta_T = 0 \), \( q_T = vA_T \left[ \frac{v}{(1-v)(j+v)} H \right]^{v-1} > 0 \). \( \square \)

**Proposition 2.** \( \frac{Y_T}{A_T} \) and \( \frac{\bar{Y}_T}{A_T} \) are strictly increasing in \( \omega_T \), where \( \bar{Y}_T \) is the output taking into account the imputed value of housing services to the household defined in footnote 26:

\[
\bar{Y}_T = Y_T + \frac{j (h_T')^{-\sigma_h}}{(c_T')^{-\sigma_z} h_T'}.
\]

Moreover, \( Y_T \) and \( \bar{Y}_T > 0 \) when \( \omega_T = 0 \).

*Proof.* After lengthy algebra manipulations, we arrive at

\[
\frac{Y_T}{A_T} = H^v v^v (1 - v)^{1-v} \frac{1}{(j + v)^{1 - \omega_T}} \frac{1 + j\omega_T}{(1 - v\omega_T)^{1 - \omega_T}},
\]

and

\[
\frac{\bar{Y}_T}{A_T} = H^v v^v (1 - v)^{1-v} \frac{1}{(j + v)^{1 - \omega_T}} \frac{1 + j + j(1 - v)\omega_T}{(1 - v\omega_T)^{1 - \omega_T}}.
\]

It is easy to see that both functions are strictly increasing in \( \omega_T \) and strictly positive even when \( \omega_T = 0 \). \( \square \)

Propositions 1 and 2 show that, because the entrepreneurs are the only producers in the economy, the more wealth (or land share) they own, the more they can use leverage to purchase more land for production. That leads to higher land price as well as aggregate output.

Given the equilibrium at \( t = T = 1 \), we consider the equilibrium of the economy at \( t = T - 1 = 0 \) under two different financial markets structure. In the first case, financial markets are complete, i.e. from the first period \( t = T - 1 = 0 \), the entrepreneurs and households have access to a complete set of securities contingent on the realizations of aggregate productivity \( A_T \) in the second period. In the second case, financial markets are
incomplete, i.e. from the first period \( t = T - 1 = 0 \), the entrepreneurs and households can only trade bonds with payoff independent of the realizations of the aggregate productivity \( A_T \) in the second period. Under complete markets, we show that \( \frac{q_T}{A_T} \) and \( \frac{Y_T}{A_T}, \frac{\tilde{Y}_T}{A_T} \) are independent of \( A_T \). While under incomplete markets, we show that \( \frac{q_T}{A_T} \) and \( \frac{Y_T}{A_T}, \frac{\tilde{Y}_T}{A_T} \) are strictly increasing in \( A_T \).

**Complete markets:**

At time \( t = 0 \), the entrepreneurs and households can trade state contingent securities - \( \phi_0(s_1) \) and \( \phi'_0(s_1) \) denote the holdings of the entrepreneurs and the households respectively - which pays off one unit of consumption good if state \( s_1 \) is realized at \( t = 1 \) and 0 otherwise. The prices of these securities are \( p_0(s_1) \) and are determined such that the markets for these securities clear, i.e. \( \phi_0(s_1) + \phi'_0(s_1) = 0 \), for each \( s_1 \). The entrepreneurs and households can also trade in the land market at price \( q_0 \), and land market clears, i.e. \( h_0 + h'_0 = H \). The entrepreneurs are initially endowed with a fraction \( \omega_0 \) of total stock of land.

At time \( t = 0 \), the entrepreneurs solve

\[
\max_{c_0,c_1(\cdot),h_0,\phi_0(\cdot),L_0,h_1(\cdot),L_1(\cdot)} \log c_0 + \gamma \mathbb{E} [\log c_1]
\]

subject to:

\[
c_0 + q_0 h_0 + \sum_{s_1} p_0(s_1) \phi_0(s_1) = q_0 H \omega_0 + A_0 h_0^v L_0^{1-v} - w_0 L_0,
\]

and

\[
c_1 + q_1 h_1 = q_1 H \omega_1 + A_1 h_1^v L_1^{1-v} - w_1 L_1,
\]

where \( \omega_1 = \frac{q_1 h_1 + \phi_0(s_1)}{q_1 H} \). The households maximize their inter-temporal expected utility subject to similar constraints.

Proposition 3 below shows that, with complete markets, land price and output at time \( t = 1 \) are proportional to the realization of productivity \( A_1 \).

**Proposition 3.** In any competitive equilibrium, the ratios \( \frac{q_1}{A_1}, \frac{Y_1}{A_1}, \frac{\tilde{Y}_1}{A_1} \) are independent of the realization of \( s_1 \).
**Proof.** From the F.O.Cs of the entrepreneurs:

\[
\frac{q_0 - vA_0 h_0^{v-1} L_0^{1-v}}{c_0} = \gamma \mathbb{E} \left[ \frac{q_1}{c_T} \right] \\
\frac{p_0(s_1)}{c_0} = \gamma \pi (s_1 | s_0) \frac{1}{c_T(s_1)} \\
w_0 = (1 - v) A_0 h_0^{v} L_0^{-v}
\]

Similarly, from the F.O.Cs of the households

\[
\frac{q_0}{c_0} = \frac{j}{h_0} + \beta \mathbb{E} \left[ \frac{q_1}{c_T} \right] \\
\frac{p_0(s_1)}{c_0} = \beta \pi (s_1 | s_0) \frac{1}{c_T(s_1)} \\
c'_0 = w_0.
\]

From the FOCs in state-contingent security holding of the entrepreneurs and the households,

\[
\frac{c_0}{c_T} = \frac{\beta}{\gamma} \frac{c_1(s_1)}{c_T(s_1)}.
\]

From the equilibrium in the last period, in particular, the equation (32), \( \frac{c_1}{c_T} \) is strictly increasing in \( \omega_1(s_1) \). Thus, the last equation implies that \( \omega_1(s_1) \) is independent of \( s_1 \). Also from the analysis of the equilibrium in the last period, as \( \frac{q_T}{A_T}, \frac{\gamma_T}{A_T}, \frac{\tilde{Y}_T}{A_T} \) are strictly increasing functions of \( \omega_1(s_1) \), these ratios are independent of \( s_1 \).

This proposition implies that symmetric shocks to productivity lead to symmetric responses of land price and output.\(^{46}\)

**Incomplete markets:**

In contrast to complete markets, under incomplete markets, at time \( t = 0 \), in addition to trading in the market for land, the entrepreneurs and households can only trade in a state non-contingent bond - \( b_0 \) and \( b'_0 \) denote the bond holdings for the entrepreneurs and the households respectively - which pays off one unit of consumption good independent of the realization of state \( s_1 \) at \( t = 1 \). The price of each of unit of the bond is \( p_0 \) and is determined such that the market for the bond clears, i.e. \( b_0 + b'_0 = 0 \). Similar to the model with complete markets, the entrepreneurs and households can also trade in the market for land at price \( q_0 \), and land market clears, i.e. \( h_0 + h'_0 = H \). The entrepreneurs are initially

\(^{46}\)The magnitude of the response of course depends on the initial wealth share of the entrepreneurs, \( \omega_0 \), as it determines the equilibrium at time 0, \( q_0, p_0(.) \) and \( h_0, \phi_0(.) \).
endowed with a fraction $\omega_0$ of total land supply.

At time $t = 0$, the entrepreneurs solve

$$\max_{c_0, c_1, h_0, b_0, L_0, h_1, L_1} \log c_0 + \gamma E[\log c_1]$$

subject to:

$$c_0 + q_0 h_0 + p_0 b_0 = q_0 H \omega_0 + A_0 h_0^v L_0^{1-v} - w_0 L_0,$$

and

$$c_1 + q_1 h_1 = q_1 H \omega_1 + A_1 h_1^v L_1^{1-v} - w_1 L_1,$$

where $\omega_1 = \frac{q_1 h_1 + b_0}{q_1 H}$.

Proposition 4 below shows that, under some conditions on the initial wealth distribution, because of market incompleteness, the ratio of land price and output to productivity at time 1 to productivity depends on the realization of productivity itself.

**Proposition 4.** In a competitive equilibrium with $b_0 < 0$, in the second period, $\frac{q_1}{A_1}, \frac{Y_1}{A_1}, \frac{\tilde{Y}_1}{A_1}$ are all strictly increasing in $A_1$.

**Proof.** From the analysis of the equilibrium in the last period,

$$q_1 = v A_1 \frac{1}{(1 - \omega_1 v)^{1-v}} \left[ \frac{v}{(1 - v)(j + v)} H \right]^{v-1}, \quad (33)$$

and from the definition of $\omega_1$,

$$\omega_1 = \frac{h_0}{H} + \frac{b_0}{q_1 H}. \quad (34)$$

These two equations help determine $q_1$ and $\omega_1$ as functions of $h_0, b_0$ and $A_1$.\footnote{Given the solution for $q_1$ and $\omega_1$ from equations (33) and (34), $h_0, b_0$ are determined in equilibrium at time 0.} In Figure 11, the equilibrium values of $q_1$ and $\omega_1$ are determined by the intersection of the two curves that relate entrepreneurs’ wealth share and land price at $t = 1$ - BS curve (green) which corresponds to balance sheet equation (34) and Q curve (red) that corresponds to pricing equation (33). When $b_0 < 0$, both curves are upward sloping. As depicted in the figure, a decrease in $A_1$ shifts the Q curve to the left (red curve to dotted red curve), while the BS curve remains unchanged. Thus this shift decreases the equilibrium $\omega_1$, as well as equilibrium price $q_1$. Because the pricing equation is strictly concave in $\omega_1$ and the balance-sheet function is strictly increasing and concave in $\omega_1$, a decrease in $A_1$ leads to a larger decrease in $\omega_1$ than an increase in $A_1$ does. \qed
Figure 11: Two-Period Equilibrium
This amplification effect is similar to the balance sheet effect described in Krishna-murthy (2010). However, here we do not impose the additional collateral constraint as in the paper. First observe that land price is increasing in the wealth share (relative net worth) of the entrepreneurs (equation (33)). A negative shock to aggregate productivity, keeping the wealth share as given, decreases land price. This decreases wealth share decreases housing price (equation (34)), further decreases relative net worth, setting off the vicious circle of falling land price and relative net worth.

D Data Description and Empirical Results

Our empirical results are based on four U.S. aggregate series: the TFP shock, the real price of land, the real output and the real estate leverage ratio in the non-financial business sector. The data of real price of land covers the period 1975:Q1 to 2010:Q4. For the other three series, the sample period is from 1951:Q4 to 2014:Q4. All series are seasonally adjusted using the X-13ARIMA-SEATS program provided by the U.S. Census Bureau.

The TFP shock is retrieved from Federal Reserve Bank of St. Louis. We use utilization-adjusted quarterly-TFP series for the U.S. Business Sector, produced by John Fernald (Fernald, 2014). We divide the original series by four to cancel his annualization adjustment.

The real price of land is from Liu, Wang, and Zha (2013). As claimed in their paper, fluctuations in real estate values are mostly driven by changes in land prices. They use liquidity-adjusted price index for residential land, which they construct from the FHFA Home Price Index using the method provided in Davis and Heathcote (2007). The series is adjusted using the price index of nondurables consumption and services from Bureau of Economic Analysis. See Appendix A of their paper for more details on this series.

The real output data is the real GDP from Federal Reserve Bank of St. Louis using chained dollars method (2009=100). We divide the original series by four to cancel their annualization adjustment.

The real estate leverage ratio is the ratio of the net value of credit market debt (credit market debt - credit market instruments) to the market value of real estate in the non-financial business sector. All the data used to construct this ratio is from the Financial Accounts of United States provided by the Federal Reserve Board.
Table 6: Regression of Housing Prices on TFP Shocks

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<td>$\Delta TFP_t$, lev high</td>
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<td>$\Delta TFP_t$ high, lev low</td>
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<td>$\Delta TFP_t$ low, lev low</td>
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<td>$\Delta hp t_{t-1}$</td>
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</table>

Note: Regressions of housing price changes on TFP shocks using quarterly U.S. aggregate data from 1975Q1 to 2010Q4. Standard errors in parenthesis. ***, **, * mean coefficients significant at 1%, 5% and 10% levels respectively. The regressand is the percentage change in land prices. $\Delta TFP_t$ represents TFP shocks. $\Delta TFP_t$ high means the current TFP shock is high, and $\Delta TFP_t$ low means the current TFP shock is low. $\Delta TFP_t$, lev high is the TFP shock when the leverage ratio in the previous quarter is high, and $\Delta TFP_t$, lev low is the TFP shock when the leverage ratio in the previous quarter is low. $\Delta TFP_t$ high, lev high means both the current TFP shock and the leverage ratio from the previous period are high. The other three cases are defined similarly. $\Delta hp t_{t-1}$ is the percentage change in housing price in the previous quarter. See Appendix D for more details for the data description.
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<td>(0.03)</td>
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<td>$\Delta TFP_t$, lev high</td>
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<td>$\Delta TFP_t$, lev low</td>
<td>0.08*</td>
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<td>$\Delta TFP_t$, high, lev high</td>
<td>0.10**</td>
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<tr>
<td>$\Delta TFP_t$, low, lev high</td>
<td>0.19**</td>
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<td>$\Delta TFP_t$, low, lev low</td>
<td>0.22**</td>
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<tr>
<td>$\Delta output_{t-1}$</td>
<td>0.38***</td>
<td>0.37***</td>
<td>0.38***</td>
<td>0.38***</td>
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<td></td>
<td>(0.06)</td>
<td>(0.06)</td>
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<td>Observations</td>
<td>252</td>
<td>252</td>
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</table>

Note: Regressions of real GDP changes on TFP shocks using quarterly U.S. aggregate data from 1951Q4 to 2014Q4. Standard errors in parenthesis. ***, **, * mean coefficients significant at 1%, 5% and 10% levels respectively. The regressand is the percentage change in land prices. $\Delta TFP_t$ represents TFP shocks. $\Delta TFP_t$, high means the current TFP shock is high, and $\Delta TFP_t$, low means the current TFP shock is low. $\Delta TFP_t$, lev high is the TFP shock when the leverage ratio in the previous quarter is high, and $\Delta TFP_t$, lev low is the TFP shock when the leverage ratio in the previous quarter is low. $\Delta TFP_t$, high, lev high means both the current TFP shock and the leverage ratio from the previous period are high. The other three cases are defined similarly. $\Delta output_{t-1}$ is the percentage change in real GDP in the previous quarter. See Appendix D for more details for the data description.
Amplification and Asymmetric Effects without Collateral Constraints: Online Appendix

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August 2015

Abstract

In this online appendix, beside the additional derivations for the main paper (Appendices E and F), we present two variants (Appendices G and H) of the models presented in the main paper. The first variant is a continuous time version of the benchmark model in the main paper. The Markov equilibrium in this continuous time model is similar to the Markov equilibria in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013). Thus, we can use the algorithms in their papers to solve for the Markov equilibrium in our model. The second variant of model is in discrete time, and we allow the households to produce using an inefficient technology. In this model, the households will start producing when the wealth of the entrepreneurs reaches its lower bound.
1 Appendix E: Steady State and Log-Linearization for Collateral Constraint Model

Becker (1980) shows that in a neoclassical growth model with heterogeneous discount factors, long run wealth concentrates on the most patient agents, in this case the households. However, in our model, due to the collateral constraint, the entrepreneurs can only pledge a fraction of their future wealth to borrow. Therefore, despite their lower discount factor, their wealth does not disappear in the long run. In particular, the model admits a long run steady state in the absence of uncertainty. In this subsection, we solve for the steady state in our model.

Suppose that there is no uncertainty, i.e., $A_t(s_t) = A$. In steady state, all variables are constant, so we can omit the subscript $t$. For the ease of notation, denote

$$\gamma_e = m\beta + (1 - m)\gamma,$$

is the average discount factor for the entrepreneurs’ investment in land as in Iacoviello (2005). The first order condition in $b_t'$,

$$b_t' : -p_t c_t'^{-\sigma_1} + \beta E_t \left[ c_{t+1}'^{-\sigma_1} \right] = 0,$$

in the steady state, $c_t' = c_{t+1}' = c'$, implies that $p = \beta$. Because $\gamma < \beta$, the entrepreneur wants to borrow as much as possible up to the collateral constraint. Indeed, the first order condition for $b$ implies that the collateral constraint is strictly binding and the Lagrange multiplier $\mu$ on the constraint is strictly positive:

$$\mu = (\beta - \gamma) c^{-\sigma_1} > 0.$$

Given that the collateral constraint is binding, we have $b = -mqh.$

From the first-order condition in $h$,

$$(\pi_t - q_t)c_t'^{-\sigma_1} + \mu_t m E_t [q_{t+1}] + \gamma E_t \left[ q_{t+1} c_{t+1}'^{-\sigma_1} \right] = 0,$$

in the steady state, we have

$$q = \frac{1}{1 - \gamma_e} v Ah^{v-1} L^{1-v}.$$

The steady state version of the first order condition in $L_t$,

$$w_t = (1 - v) A_i h_i^v L_i^{-v},$$

is the average discount factor for the entrepreneurs’ investment in land as in Iacoviello (2005). The first order condition in $b_t'$,
\[ w = (1 - v)A h^v L^{-v}. \] (1)

From the budget constraint of the entrepreneurs, we obtain
\[ c = \frac{(1 - \gamma)(1 - m)v}{1 - \gamma e} A h^v L^{1-v}. \]

Combining with the market clearing condition in the market for consumption good, we have \( c' = A h^v L^{1-v} - c \). The market clearing conditions in the housing market and labor market imply, \( h' = H - h \) and \( L' = L \).

So in the steady-state all variables can be expressed as functions of two unknowns, \( h \) and \( L \). The first-order conditions on \( h' \) and \( L' \) of the households provide two equations that help determine the two unknowns:
\[-q (c')^{-\sigma_2} + j \left( h' \right)^{-\sigma_h} + \beta q (c')^{-\sigma_2} = 0 \]

and
\[ w \left( c' \right)^{-\sigma_2} = \left( L' \right)^{\eta-1}. \]

For example, when \( \sigma_2 = 1 \) and \( \sigma_h = 1 \) as in Iacoviello (2005), the second equation, combined with the labor choice equation at the steady state (1) implies
\[ L = \left[ \frac{1 - v}{1 - \frac{(1-\gamma)(1-m)}{1-\gamma c} v} \right]^\frac{1}{\eta}, \]

From the first equation, \( h \) is determined as
\[ \frac{h}{H} = \frac{v (1 - \beta)}{v (1 - \beta) + j \left[ (1 - \gamma e) - (1 - \gamma) (1 - m) v \right]}. \]

Given the determination of the steady state level of \( h \) and \( L \), the steady state level of wealth distribution defined by
\[ \omega_t = \frac{q_t h_{t-1} + b_{t-1}}{q_t H}, \]
is \[ \omega = \frac{(1-m)h}{H}. \]

Following Iacoviello (2005), we assume that the collateral constraint always binds around the steady state. Relative to the standard log-linearization technique, we need
to solve for the shadow value of the collateral constraint, i.e., the multiplier \( \mu_t \), in addition to prices and allocation. Given a variable \( x_t \), let \( \hat{x}_t \) denote the percentage deviation of \( x_t \) from its steady state value, i.e., \( \hat{x}_t = \frac{x_t - \hat{x}}{\hat{x}} \). Given the exogenous processes for the productivity shock \( \hat{A}_t \), we solve for the endogenous variables \( \hat{c}_t, \hat{c}'_t, \hat{h}_t, \hat{b}'_t, \hat{\theta}_t, \hat{\omega}_t, \hat{p}_t, \hat{L}_t, \hat{\mu}_t \) using the method of undetermined coefficients.\(^1\) The following linear system characterizes the dynamics of the economy around the steady state:

\[
\begin{align*}
(q_t - \sigma_2 \hat{c}'_t) &= -\sigma_h (1 - \beta) \hat{h}_t + \beta E_t [(q_{t+1} - \sigma_2 \hat{c}'_{t+1})] \\
\hat{p}_t &= \sigma_2 (\hat{c}'_t - E_t \hat{c}'_{t+1}) \\
\hat{\omega}_t - \sigma_2 \hat{c}'_t &= (\eta - 1) \hat{L}_t \\
(1 - \gamma_e) [\hat{A}_t + (v - 1) \hat{h}_t + (1 - v) \hat{L}_t - \sigma_1 \hat{c}_t] &= (q_t - \sigma_1 \hat{c}_t) \\
+ m(\beta - \gamma) (\hat{\mu}_t + E_t \hat{q}_{t+1}) + \gamma E_t (q_{t+1} - \sigma_1 \hat{c}_{t+1}) &= 0 \\
\beta (p_t - \sigma_1 \hat{c}_t) &= (\beta - \gamma) \hat{\mu}_t - \gamma \sigma_1 E_t (\hat{c}_{t+1}) \\
\hat{\omega}_t &= \hat{A}_t + \nu \hat{h}_t - \nu \hat{L}_t \\
\hat{b}'_t &= \hat{h}_t + E_t \hat{q}_{t+1} \\
c^* \hat{c}_t + c^{*'} \hat{c}'_t &= Y^* [\hat{A}_t + \nu \hat{h}_t + (1 - v) \hat{L}_t] \\
h^* \hat{h}_t + h^* \hat{h}'_t &= 0 \\
c' \hat{c}'_t + qh' (\hat{h}'_t - \hat{h}'_{t-1}) &= \beta b^* (\hat{p}_t + \hat{b}'_t) \\
= b^* \hat{b}'_{t-1} + wL (\hat{\omega}_t + \hat{L}_t).
\end{align*}
\]

2 Appendix F: Stationary State under Collateral Constraint with Complete Markets

In the Markov equilibrium, we look for a stationary level of wealth distribution, i.e., starting from this level, wealth distribution does not change over time. Therefore, prices and allocations depend only on the exogenous state \( s_t \). At this stationary state, we simplify

\(^1\)Given the special 3-state structure of the stochastic shocks assumed in the main paper, we cannot directly use Dynare to solve for the log-linearized version of the model.
the notation of state contingent prices and bond holdings by \( \tilde{p}_t(s_{t+1}) = \tilde{p}(s_t, s_{t+1}) \) and \( \phi_t(s_{t+1}) = \phi(s_t, s_{t+1}) \). Moreover, numerically, we find that at this stationary state, collateral constraints

\[
\phi_t(s_{t+1}) + mq_{t+1} h_t \geq 0
\]

are all binding, i.e., \( \phi^*(s_t, s_{t+1}) = -mq(s_{t+1}) h(s_t) \). This implies \( \omega_{t+1} = (1 - m) \frac{h}{H} \). In order for \( \omega_{t+1} \) not to depend on \( s_{t+1} \), we must have then \( h_t = h^* \) independent of the exogenous state.

From the budget constraint of the entrepreneurs,

\[
c(s_t) = \sum_{s_{t+1} | s_t} p(s_t, s_{t+1}) mq(s_{t+1}) h^*
\]

\[
- mq(s_t) h^* + A(s_t) (h^*)^\nu (L_t)^{1-\nu} - w(s_t) L_t
\]

and by the market clearing condition for consumption good,

\[
c'(s_t) = A(s_t) (h^*)^\nu (L(s_t))^{1-\nu} - c(s_t).
\]

Given \( h^* \) and \( L(s_t) \), \( w(s_t) \) is determined by the first-order condition from the entrepreneurs’ optimal choice of \( L_t \), i.e., \( w_t = (1 - \nu) A_t h_t^\nu L_t^{-\nu} \). Therefore, we only need to solve for

\[
\{ h^*, q(s_t), \tilde{p}(s_t, s_{t+1}), L(s_t) \}.
\]

The first-order condition with respect to \( \phi'_t(s_{t+1}) \) implies

\[
\tilde{p}(s_t, s_{t+1}) = \beta \left( \frac{c'(s_{t+1})}{c'(s_t)} \right)^{-\sigma_2}.
\]

We can choose \( \mu(s_t, s_{t+1}) \) so that the F.O.C. in \( \phi_t(s_{t+1}) \),

\[
-\tilde{p}_t(s_{t+1}) c_t^{-\sigma_1} + \mu_t(s_{t+1}) + \gamma (c_{t+1} (s_{t+1}))^{-\sigma_1} = 0
\]

is satisfied

\[
\mu(s_t, s_{t+1}) = \tilde{p}(s_t, s_{t+1}) c_t^{-\sigma_1} - \gamma (c_{t+1} (s_{t+1}))^{-\sigma_1}
\]

\[
= \beta \left( \frac{c'(s_{t+1})}{c'(s_t)} \right)^{-\sigma_2} c_t^{-\sigma_1} - \gamma (c_{t+1} (s_{t+1}))^{-\sigma_1}
\]

\[
> 0.
\]
Plugging this expression for $\mu(s_t, s_{t+1})$ into the F.O.C. in $h_t$
\[
(\pi_t - q_t)c_t^{\sigma_1} + m\mathbb{E}_t[\mu_t(s_{t+1})q_{t+1}] + \gamma\mathbb{E}_t[q_{t+1}c_{t+1}^{\sigma_1}] = 0.
\]
we obtain another set of equations that help determine $\{q(s_t)\}$. The equations that determine $\{L(s_t)\}$ comes from the optimal labor-consumption decision of the households,
\[
w_t^{c'-\sigma_2} = L_t^{\eta-1}.
\]
And lastly, $h^*$ must be determined so that the first-order conditions in the housing choice of the household are satisfied in all exogenous states:
\[
\{q(s_t) - \mathbb{E}_t[q(s_{t+1})\tilde{p}(s_t, s_{t+1})]\} (c'(s_t))^{-\sigma_2} = j(H - h^*)^{-\eta_0},
\]
given the expression of $\tilde{p}(s_t, s_{t+1})$ derived above.

3 Appendix G: Continuous Time Model

In this section, we present a continuous time version of the benchmark incomplete markets model in the main paper. The equilibrium in this model can be solved using the methods developed in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013).

3.1 Economic Environment

The environment is exactly the same as in the main paper, except that time is continuous. The aggregate productivity $A_t$ follows a diffusion process
\[
dA_t = \mu^A(A_t)dt + \sigma^A(A_t)dZ_t,
\]
where $Z_t$ is a standard Brownian motion.

The dynamics of land price $q_t$ is determined by
\[
dq_t = \mu^q_t dt + \sigma^q_t dZ_t,
\]
where $\mu^q_t$ and $\sigma^q_t$ are endogenously determined.

Given the process of land price $q_t$, interest rate $r_t$ and wage rate $w_t$, households maxi-
mize a lifetime utility function given by
\[
E_0 \int_0^{\infty} e^{-\rho t} \left\{ \frac{(c_t')^{1-\sigma_2} - 1}{1 - \sigma_2} + j \frac{(h_t')^{1-\sigma_h} - 1}{1 - \sigma_h} - \frac{1}{\eta} (L_t') \eta \right\},
\] (3)
where \( E_0 [.] \) is the expectation operator, \( \rho > 0 \) is the discount rate, \( c_t' \) is consumption at time \( t \), \( h_t' \) is the holding of land. \( L_t' \) denotes the hours of work. Households can trade in the market for land as well as a state-incontingent bond market that yield instantaneous rate of return \( r_t \). Let \( b_t' \) denote the holding of state-incontingent bond of the households. The households are subject to the following constraint on the dynamics of their net worth \( n_t' \):
\[
dn_t' = h_t' (\mu_t q_t dt + \sigma_t q_t dZ_t) + r_t b_t' dt - c_t' dt + w_t L_t' dt
\]
\[
n_t' = q_t h_t' + b_t'
\] (4)
Given land in the utility function of households, implicitly \( h_t' \geq 0 \). The households are also subject to the No-Ponzi scheme conditions
\[
\lim_{T \to \infty} E_t \left[ e^{-\int_0^T r_s ds} b_T \right] \geq 0.
\]
Entrepreneurs use a Cobb-Douglas constant-returns-to-scale technology that uses land and labor as inputs. They produce consumption good \( Y_t \) according to
\[
Y_t = A_t h_t^\nu L_t^{1-\nu},
\] (5)
where \( A_t \) is the aggregate productivity which depends on the aggregate state \( s_t \), \( h_t \) is real estate input, and \( L_t \) is labor input.

The entrepreneurs discount the future at the discount rate \( \gamma > \rho \). The entrepreneurs maximize
\[
E_0 \int_0^{\infty} e^{-\gamma t} (c_t)^{1-\sigma_1} - 1 \]
subject to the following constraint on the dynamic of their net worth, \( n_t \):
\[
dn_t = h_t (\mu_t q_t dt + \sigma_t q_t dZ_t) + \pi_t h_t + r_t b_t dt - c_t dt
\]
\[
n_t = q_t h_t + b_t.
\] (7)
Output \( Y_t \) is produced by combining land and labor using the production function (5). Given the production function of the entrepreneurs, we have implicitly \( h_t \geq 0 \). The en-
entrepreneurs are also subject to the Non-Ponzi condition:

$$\lim_{T \to \infty} E_t \left[ e^{-\int_t^T r_s ds} p_T \right] \geq 0.$$ 

### 3.2 Equilibrium

The definition of the sequential competitive equilibrium for this economy is standard and is a continuous version of the competitive equilibrium in the main paper.

**Definition 1.** A competitive equilibrium is sequences of prices \( \{q_t, r_t, w_t\}_{t=0}^{\infty} \) and allocations \( \{c_t, h_t, b_t, L_t, c'_t, h'_t, b'_t, L'_t\} \) such that (i) \( q_t \) follows the dynamics (2), (ii) the \( \{c'_t, h'_t, b'_t, L'_t\} \) maximize (3) subject to the dynamic net worth constraint (4) and the no-Ponzi condition and \( \{c_t, h_t, b_t, L_t\} \) maximize (6) subject to dynamic net worth constraint (7) and the no-Ponzi condition, and production technology (5) given \( \{q_t, r_t, w_t\} \) and initial asset holdings \( \{h_0, b_0, h'_0, b'_0\} \); (iii) land, bond, labor, and good markets clear: \( h_t + h'_t = H, b_t + b'_t = 0, \) \( L_t = L'_t, c_t + c'_t = Y_t. \)

Let \( \omega_t \) denote the normalized financial wealth of the entrepreneurs

$$\omega_t = \frac{n_t}{q_t H},$$

and \( \omega'_t \) denote the normalized financial wealth of the households:

$$\omega'_t = \frac{n'_t}{q_t H}.$$ 

By the land and bond market clearing conditions, we have \( \omega'_t = 1 - \omega_t \) in any competitive equilibrium. Therefore in order to keep track of the normalized financial wealth distribution between the entrepreneurs and the households, \( (\omega_t, \omega'_t) \), in equilibrium, we only need to keep track of \( \omega_t \). To simplify the language, we use the term wealth distribution for normalized financial wealth distribution.

Markov equilibrium is also a continuous version of Markov equilibrium in the main paper.

**Definition 2.** A Markov equilibrium is a competitive equilibrium in which prices and allocations at time \( t \), depend only on the wealth distribution at time \( t, \omega_t \) and the exogenous state \( A_t \).

This Markov equilibrium is the same as the Markov equilibria in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013). We can use the algorithms in their papers to solve for the Markov equilibrium in our paper. However, there are two important
differences. First, because of persistent TFP shocks (instead of I.I.D. depreciation shocks to capital stock as in Brunnermeier and Sannikov (2014) or I.I.D. return shocks on dividend as in He and Krishnamurthy (2013)), in our Markov equilibrium, we need to keep track of both wealth distribution and the current exogenous shock. Second, we allow for general utility functions, instead of linear or log utility functions as in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013).

4 Appendix H: Multiple Production Technologies

In this section, we simplify our model in the main paper in the spirit of Brunnermeier and Sannikov (2014) as well as Cordoba and Ripoll (2004) and Kiyotaki and Moore (1997). We assume that the households do not have a preference for housing but have access to an inefficient production function

\[ Y' = Ah^\nu L^{1-\nu} \]  

(8)

with \( A < \min(A_t) \). Households maximize a lifetime utility function given by

\[ \mathbb{E}_0 \sum_{t=0}^\infty \beta^t \left\{ \frac{(c'_t)^{1-\sigma_2} - 1}{1-\sigma_2} - \frac{1}{\eta} (\tilde{L}_t)^\eta \right\}, \]  

(9)

where \( \tilde{L}_t \) is the hours of work (instead of \( L'_t \) in the benchmark model). The budget constraint of the households is

\[ c'_t + q_t(h'_t - h'_{t-1}) + p_t b'_t \leq b'_{t-1} + w_t \tilde{L}_t + Y'_t - w_t L'_t. \]  

(10)

Housing is no longer in the utility function of households, so we have to impose explicitly

\[ h'_t \geq 0. \]

Given their land holding at time \( t \), \( h_t \), the households choose labor demand \( L'_t \) to maximize profit

\[ \max_{L'_t} \{ Y'_t - w_t L'_t \} \]

subject to their production technology (8) if they produce. The first order condition with respect to \( L'_t \) implies

\[ w_t = (1 - \nu') AH_t^\nu (L'_t)^{-\nu'}, \]
i.e. \( L'_t = \left( \frac{(1-\nu')A_t}{\omega_t} \right)^{\frac{1}{\nu}} h'_t \) and profit

\[
Y'_t - w_t L'_t = \pi'_t h'_t
\]

where \( \pi'_t = v' A \left( \frac{(1-\nu')A_t}{\omega_t} \right)^{\frac{1-\nu}{\nu}} \) is profit per unit of land for the households.

**Definition 3.** A competitive equilibrium is sequences of prices \( \{p_t, q_t, w_t\}_{t=0}^{\infty} \) and allocations \( \{c_t, h_t, b_t, L_t, c'_t, h'_t, b'_t, L'_t, \tilde{L}_t\} \) such that (i) \( \{c'_t, h'_t, b'_t, L'_t, \tilde{L}_t\} \) maximize (9) subject to budget constraint (10), the production technology (8), \( h'_t \geq 0 \) and the No-Ponzi condition, and \( \{c_t, h_t, b_t, L_t\} \) maximize (6) subject to the entrepreneurs’ budget constraint, production technology (5), and a collateral constraint, given \( \{p_t, q_t, w_t\} \) and initial asset holdings \( \{h_{-1}, b_{-1}, h'_{-1}, b'_{-1}\} \); (ii) land, bond, labor, and good markets clear: \( h_t + h'_t = H, b_t + b'_t = 0, L_t + L'_t = \tilde{L}_t, c_t + c'_t = Y_t \).

In the steady state, the entrepreneurs own the whole supply of land. Outside the steady state, we can use the definition of Markov equilibrium and the associated solution method as in the benchmark model in the main paper. The main difference between the solution of this model and the benchmark model is that at the natural borrowing limit for the entrepreneurs, i.e. \( \omega_t = 0 \), the households start producing using their inefficient production function. This puts a lower bound on the total output as well as land price.

**References**


