Monetary Policy  
and the Natural Rate of Interest  

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Abstract  

It is most important for monetary policy to be able to track the (unobserved) natural rate of interest in environments in which that rate takes big and sustained swings away from its long run equilibrium. To establish this fact, we study two models: one is a standard New-Keynesian model; in the other, government bonds provide liquidity, and deficit financed spending shocks cause large and persistent movements in the natural rate because they create extra liquidity. Policy rules that cannot make the policy rate track the natural rate perform poorly in both of these models, but they are especially bad in the second model. Households would give up half of a percent of their consumption each period to have a rule that could track the natural rate. Even in the standard model, households would give up a quarter of a percent of consumption to obtain such a rule. First difference rules perform quite well in this environment, and they do not require any information about the natural rate of interest. Model uncertainty is perhaps the main reason why it is difficult to track the natural rate. When model uncertainty is taken into account, the dominance of the first difference rule is even more pronounced.

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1 Introduction

One can only say that if the bank policy succeeds in stabilizing prices, the bank rate must have been brought in line with the natural rate.

Orphanides and Williams[20]

Why is the natural rate of interest so important for inflation control? Consider a simple variant of the Taylor Rule

\[
i_t = i_t^n + 1.5(\pi_t - \bar{\pi})
\]

where \(\pi_t\) is the rate of inflation, \(\bar{\pi}\) is the inflation target, \(i_t\) is the policy instrument, and \(i_t^n\) is its natural rate – defined as the rate that would prevail if there were no nominal rigidities. Intuitively, when an increase in aggregate demand, or a decrease in productivity, pushes inflation above its target, the policy rate should be raised above its natural rate for a period of time, raising the real rate of interest to curb the rise in inflation.

The problem here is of course that the natural rate of interest is not observed directly, and estimating it is known to be difficult. In an early paper on the subject, Laubach and Williams[15] found considerable movement in their (imprecise) estimates of the natural rate; they concluded that "...this source of uncertainty needs to be

\footnote{The intercept term in the original Taylor Rule was the long run value of the natural rate; we have replaced it with the current period’s value, which of course will fluctuate over time.}

The original specification of the Taylor Rule also contained an output gap. We will suppress the gap in this paper. It is well known that (unless the natural rate of output can be measured accurately) smoothing the output gap is harmful in models where government spending shocks and productivity shocks play a prominent role. Here, we leave those well studied issues aside so that we can focus on the natural rate of interest.
taken account of in analyzing monetary policies that feature responses to the natural rate of interest." Nevertheless, Curdia, Ferrero, Ng and Tambalotti[7] argue that the Federal Reserve has a history of trying to track the natural rate. They quote Alan Greenspan as saying “... In assessing real rates, the central issue is their relationship to an equilibrium interest rate, specifically, the real rate level that, if maintained, would keep the economy at its production potential over time. Rates persisting above that level, history tells us, tend to be associated with slack, disinflation, and economic stagnation – below that level with eventual resource bottlenecks and rising inflation ... ." And the quote from Orphanides and Williams is an indication that the view is widespread.

In this paper, we show that tracking the natural rate is also important for household welfare. And it is most important in an environment where interest rates take large and persistent swings around their long run equilibrium values, making it difficult for the policy rate to catch up with its natural rate. We will use two models to demonstrate this fact: In one – which we call the Standard (New-Keynesian) Model – swings in the natural rate are substantial, but they are short lived. In the other model – which we call the Liquid Bonds Model – the swings can be larger and much more persistent. The reason is that government bonds provide liquidity in this model. So, an increase in public spending that is initially financed by selling bonds will provide extra liquidity in the Liquid Bonds Model, and this in turn generates protracted movements in consumer demand and equilibrium interest rates. We will show that tracking the natural rate is much more important for welfare in the Liquid Bonds Model.

In either of the models, the deviations of the policy rate from its natural rate will depend upon the monetary policy that is in place; we will consider four generic rules:

Rule 1 (the basic Taylor rule): \[ i_t = \bar{i} + 1.5(\pi_t - \bar{\pi}) \]
Rule 2 (the smoothed Taylor rule):  
\[ i_t = 0.8i_{t-1} + (1 - 0.8)[\bar{i} + 1.5(\pi_t - \bar{\pi})] \]

Rule 3 (the first difference rule):  
\[ i_t = i_{t-1} + 1.5(\pi_t - \bar{\pi}) \]

Rule 4 (the natural rate rule):  
\[ i_t = [r^n_t + E_t(\pi_{t+1})] + 1.5(\pi_t - \bar{\pi}) \]

where \( r^n_t \) is the real natural rate of interest and \( \bar{i} \) is the steady state value of the policy rate.

Rule 4 is the variant of the Taylor rule discussed above. Rule 4 implicitly assumes that the natural rate is observed; since it cannot be observed in practice, Rule 4 should be considered a benchmark case. Rule 1 is often used in the literature because it only assumes that the steady state value of the natural rate is known. Rule 2 is also used in the literature since interest rate smoothing is found in estimates of the central bank’s policy rule; the coefficients – 0.8 and 1.5 – are typical of what is found in those estimates. Rule 3 is known in the literature as the first difference rule; Rule A in Levin, Wieland and Williams[18][19] specified an inflation coefficient of 1.3; we simply use 1.5 across all of the models.

In the Standard Model, the difference in household utility between Rule 1 and Rule 4 is worth a quarter of a percent of consumption each period, which is a substantial number in the New Keynesian literature. In our preferred parameterization of the Liquid Bonds Model, the difference rises to half a percent of consumption each period. But in other parameterizations, we will see that the welfare gain is closer to that in the Standard Model. In all cases, the first difference rule provides virtually the same utility as Rule 4, and it does not require any knowledge of the natural rate of interest. Model uncertainty is perhaps the most important reason why central banks find it so difficult to track the natural rate of interest. When we go on to consider model uncertainty – utilizing Hansen and Sargent’s[12] notion of robustness – the dominance of the first difference rule is even greater.

The first difference rule has been much lauded in the past. In all of the papers
we are aware of, an ad-hoc welfare criterion is used instead of household utility – the central bank minimizes a weighted average of the variances of inflation, output, and the policy rate.\textsuperscript{2} The variance of the policy rate was a concern because of instrument instability. Orphanides and Williams\cite{22} and Laxton and Pesenti\cite{17} are perhaps the papers that are closest to ours. They discussed central bank misperceptions of a constant long run natural rate of interest, and they found that a first difference rule worked quite well in that context. Levin, Wieland and Williams\cite{18}\cite{19} discussed model uncertainty, and found a first difference rule worked quite well in that context; they made no mention of a fluctuating natural rate. Taylor and Williams\cite{27} provide a nice review of a number of papers in this vein, and provide an extensive bibliography.

Before proceeding to the analysis, we should address two questions. Do bonds have liquidity value? And if so, are they good substitutes for money? The basic premise should not be controversial. U.S. Treasuries facilitate transactions in a number of ways: they serve as collateral in many financial markets, banks hold them to manage the liquidity of their portfolios, and individuals hold them in money market accounts that offer checking services. However, the substitutability between money and bonds is less clear. We believe that substitutability is quite low, or that money and bonds are complements: cash is not used as collateral, and government bonds are not used to pay the barber. However, in what follows, we will consider various values for the elasticity of substitution, and we will see that this is an important parameter in our results.

We are of course not the first to study the transactions services of bonds. Early contributions to the literature include: Patinkin\cite{24}, who put both money and bonds in the household utility function; and Friedman\cite{9}, who discussed the optimum quantity of money and (private) bonds. More recent theoretical contributions include:

\textsuperscript{2}In some cases, there is just a limit placed on the variance of the interest rate.
Bansal and Coleman[1], who used the approach to study the equity premium puzzle and related issues; Holmstrom and Tirole[13], who argued that the private sector cannot satisfy its own liquidity needs; Calvo and Vegh[16], who studied the policy implications of liquid bonds; and Linnemann and Schabert[21], who used a model similar to ours to study macroeconomic policy.

Empirical contributions to this literature include: Friedman and Kuttner[8], who studied the imperfect substitutability of commercial paper and U.S. Treasuries; Greenwood and Vayanos[11], who find that the supply of long-term relative to short-term bonds is positively related to — and predicts — the term spread; Krishnamurthy and Vissing-Jorgensen[14], who find that the spread between liquid treasury securities and less liquid AAA debt moves systematically with the quantity of government debt; Bohn[4], who presents results similar to those of Krishnamurthy and Vissing-Jorgensen; and Pflueger and Viceira[25], who find a systematic liquidity premium.

The rest of the paper proceeds as follows: In Section 2, we outline the two models. In section 3, we compare the dynamic properties of the two models under alternative parameterizations. In Section 4, we perform the welfare analysis. And, in Section 5, we take model uncertainty into account, considering both model misperceptions and Hansen and Sargent’s approach to robustness.

2 The Liquid Bonds Model and a Standard Model

The Liquid Bonds Model was developed by Canzoneri, Cumby, Diba and Lopez-Salido[6]. The model extends a standard New Keynesian environment to reflect the fact that government bonds provide liquidity and are imperfect substitutes for money. The model uses Schmitt-Grohe and Uribe’s[26] specification of transactions costs, but it replaces money with a CES aggregate of money and bonds in their
definition of velocity.\(^3\) Imbedded in the Liquid Bonds Model is a Standard Model in which bonds have no liquidity value; it just returns to the original Schmitt-Grohe and Uribe definition of velocity. In the Standard Model, bonds have no liquidity value and are obviously not a substitute for money. The substitutability of money and bonds in the Liquid Bonds model plays a strong role in the sections that follow. We begin with the households and their transactions costs. That is where the action is.

\subsection{2.1 Households and Transactions Costs}

There is a continuum of households of measure one. Each household supplies labor to every firm; so, in a symmetric equilibrium the households’ behavior will be identical and we can dispense with household indices. Households maximize

\[ E_t \sum_{j=t}^{\infty} \beta^{j-t}[\log(c_j) - (1 + \chi)^{-1}n_j^{1+\chi}] \quad (2) \]

subject to a sequence of budget constraints,

\[ b_j + m_j + (1 + \tau_j)c_j + t_j = w_j n_j + \frac{(I_{j-1}b_{j-1} + m_{j-1})}{\Pi_j} + div_j \quad (3) \]

where \(c_t\) is household consumption of a composite final good in period \(t\), \(n_t\) is hours worked, \(w_t n_t\) is real labor income, and \(\tau_t c_t\) are household transactions costs. \(m_t\) and \(b_t\) are real money and bond holdings, \(I_t\) is the gross nominal return on a riskless, one-period government bond, \(\Pi_t \equiv \frac{p_t}{p_{t-1}}\) is the gross rate of inflation, \(t\) represents real lump sum taxes, and \(div_t\) represents dividends. \(I_t\) is a money market rate, and

\(^3\)Canzoneri, Cumby, Diba and Lopez-Salido\(^5\) developed a more structural model in which banks hold money and bonds to manage their deposits. The model we use here makes it easier to build a quantitative model because we can pin down a number of important parameters by matching the sample averages of some key monetary and fiscal variables in U.S. data.
we take it to be the central bank’s policy rate.\footnote{In the U.S. context, the Fed Funds rate is the policy rate. In normal times, the effective Fed Funds rate is very close to the T-bill rate. We do not model the money markets explicitly. Implicitly, we are assuming the Fed Funds rate and the T-bill rate are one and the same.}

We have followed Schmitt-Grohé and Uribe\cite{26} in assuming that transaction costs are proportional to consumption. The factor of proportionality, $\tau_t$, is an increasing function of velocity, $v$. Letting $v^*$ be the satiation level of velocity, we set

$$\tau_t = \frac{A}{v_t} (v_t - v^*)^2$$

where $A > 0$. Schmitt-Grohé and Uribe defined velocity as $v_t = c_t/m_t$, but in the Liquid Bonds Model, we broaden the notion of transactions balances and let

$$v_t = \frac{c_t}{m_t}$$

where $\tilde{m}_t$ is a CES bundle of money and bonds,

$$\tilde{m}_t^\rho = a^{1-\rho} m_t^\rho + (1-a)^{1-\rho} b_t^\rho$$

In the Standard Model, we revert to Schmitt-Grohé and Uribe’s definition of velocity; that is, $\tilde{m}_t = m_t$.

The elasticity of substitution in the Liquid Bonds Model is $\xi \equiv 1/(1-\rho)$. When $\xi < 1$, money and bonds are complements; and when $\xi > 1$, they are substitutes. As stated previously, this is an important parameter in the model.

The household’s first order conditions include:

$$w_t \lambda_t = n_t^\chi$$

and

$$1/c_t = \lambda_t [1 + 2A(v_t - v^*)]$$
where $\lambda_t$ is the real marginal utility of wealth. When real resources are depleted in
the purchase of consumption goods, the marginal utility of wealth is less than the
marginal utility of consumption. The first order conditions for money and bond
holdings are:

\[
\left\{1 - A[v_t^2 - (\nu^*)^2] \left( \frac{\alpha \tilde{m}_t}{m_t} \right)^{1-\rho} \right\} = (I_t^*)^{-1} \tag{9}
\]

\[
\left\{1 - A[v_t^2 - (\nu^*)^2] \left( \frac{1-a}{b_t} \tilde{m}_t \right)^{1-\rho} \right\} = \frac{I_t}{I_t^*} \tag{10}
\]

where $I_t^*$ is the gross nominal CCAPM interest rate; that is $(I_t^*)^{-1} \equiv \beta E_t \left\{ \frac{P_{t+1}}{P_t} \frac{\lambda_{t+1}}{\lambda_t} \right\}$. We will think of $I_t^*$ as the rate of return on a bond, $b_t^*$, that does not provide liquidity
services.$^{5}$

Equations (9) and (10) imply

\[
\frac{I_t^* - I_t}{I_t^* - 1} \equiv \left( \frac{1-a}{a} \right)^{1-\rho} \left( \frac{m_t}{b_t} \right)^{1-\rho} \tag{11}
\]

Since $b_t$ provides transactions services, it will be held at a lower rate of return than $b_t^*$; the spread, $I_t^* - I_t$, will be non-negative in equilibrium. The spread is the pecuniary
opportunity cost of holding the bond that does provide transactions services, and
$I^* - 1$ is the opportunity cost of holding money. So, equation (11) says that in the
optimal portfolio, the relative price of $m$ and $b$ is equated to the marginal rate of
substitution between $m$ and $b$.

When, for example, the central bank conducts an expansionary open market op-
eration, $\frac{m_t}{b_t}$ will rise; the marginal liquidity value of bonds (relative to money) will
rise. This will cause the relative price to rise, and the spread, $I_t^* - I_t$, will increase.

$^{5}$Households are identical in our models, and $b_t^*$ would be zero in equilibrium. For simplicity, we
have simply priced these bonds, rather than discuss them further.
Note also that when money and bonds are complements \((\rho < 0)\), a given movement in \(\frac{m_t}{b_t}\) will produce a larger change in the relative price, and in the spread, than when money and bonds are substitutes \((\rho > 0)\). This fact will play a role in what follows.

2.2 Intermediate Goods and the Final Consumption Good

The modeling of the production side of the economy is quite standard. Our description of it can be brief.

A continuum of monopolistically competitive firms, indexed by \(j\), produce intermediate goods using a common technology,

\[
y_{j,t} = z_t \bar{k} n_{j,t}^\alpha
\]

where \(\bar{k}\) is the firm’s fixed capital stock, \(0 < \alpha < 1\), and \(z_t\) is a common productivity shock that follows an AR(1) process:

\[
\ln(z_t) = (1 - \rho_z) \ln \bar{z} + \rho_z \ln(z_{t-1}) + \varepsilon_t^z
\]

where \(0 \leq \rho_z < 1\) and \(\bar{z} (= 1)\) is the steady state value of \(z_t\). Competitive retailers buy the intermediate goods and bundle them into the final good, \(y_t\), using a CES aggregator with elasticity \(\eta\).

Intermediate good firms engage in Calvo price setting. Each period, with probability \(1 - \theta\), a firm \(j\) gets to set an optimal new price; if the firm does not get to re-optimize, its price goes up automatically by the steady state rate of inflation, \(\bar{\Pi}\). Bars over variables indicate a steady state value.
2.3 Goods Market Clearing

The households transactions costs are a drain on resources. The market clearing condition for output is

\[ y_t = (1 + \tau_t)c_t + g_t \] (13)

where \( g_t \) is real government spending on the final good.

2.4 Fiscal Policy

Government spending follows an exogenous AR(1) process

\[ \ln(g_t) = (1 - \rho_g) \ln \bar{g} + \rho_g \ln(g_{t-1}) + \epsilon_t^g \] (14)

where \( 0 < \rho_g < 1 \) and \( \epsilon_t^g \) is a spending shock.

The government uses a lump sum tax, \( t \), to stabilize its debt. The tax rule is

\[ t_t = \bar{t} + \phi(b_{t-1} - \bar{b}) \] (15)

When \( \phi > \bar{I}/\bar{\Pi} - 1 \) (as we will assume), fiscal policy is stabilizing since tax increases are more than sufficient to pay the interest on any increase in the debt.

2.5 Steady State and Model Calibration

Canzoneri et al (REF) describes the steady state of the Liquid Bonds Model and its calibration in some detail. Our discussion here can be brief, and once again, the interested reader is referred to that paper for a more complete description.

The model’s calibration can usefully be divided into three sets of parameters. The first set is standard: \( \beta \) is set equal to 0.99, \( \alpha = 0.75 \) implies that the average duration of prices is four quarters; \( \eta = 7 \) implies that the steady state markup is 1.17; \( \chi = 1 \) implies that the disutility of work is quadratic; and \( \alpha = 0.66 \) implies that capital’s
share is one third. The second set uses sample averages during the Volker-Greenspan era (we do not include data from the financial crisis) to set the steady state values of $\bar{\Pi}$, $\bar{I}$, $\bar{b}/\bar{c}$, $\bar{b}/\bar{m}$, and $\bar{g}/\bar{y}$. Along with our choice of $\beta$, this sets $\bar{I}^* = \bar{\Pi}/\beta$. We use these sample averages to calculate $\bar{I}^* - \bar{I}$ and then compute $a$, $\nu^*$, and $A$ using,

$$a = \frac{\bar{m}/\bar{b}}{[\bar{m}/\bar{b} + \beta (\bar{I}^* - \bar{I}) / (\bar{\Pi} - \beta)]^{1/(1-\rho)}}$$ (16)

$$\bar{v} = \left[a^{(1-\rho)} (\bar{m}/\bar{c})^\rho + (1-a)^{(1-\rho)} (\bar{b}/\bar{c})^\rho\right]^{-1/\rho}$$ (17)

$$\frac{\nu^*}{\bar{v}} = \left(\frac{1-\beta/\bar{\Pi}}{a(\bar{m}/\bar{m})^{1-\rho}\bar{v} - \bar{\tau}} - \frac{1-\beta/\bar{\Pi}}{a(\bar{m}/\bar{m})^{1-\rho}\bar{v} + \bar{\tau}}\right)$$ (18)

$$A = \frac{1 - \beta/\bar{\Pi}}{a(\bar{m}/\bar{m})^{1-\rho}[ar{v}^2 - (\nu^*)^2]}$$ (19)

where $\bar{\tau}$ is set to 0.1 percent of steady state consumption.

The third set of parameters are determined using estimated values. Parameters in the stochastic processes for government spending and productivity are estimated using detrended quarterly U.S. data. We set $\rho_g = 0.95$, SD($\varepsilon^g$) = 0.01; $\rho_z = 0.93$, SD($\varepsilon^z$) = 0.01. The response of taxes to debt is set to 0.018. Bohn’s [2], [3], [4] estimates suggest a response in the range of 0.013 to 0.030.

## 3 Interest Rate Dynamics

Observing the natural rate of interest is presumably more important for policy making when the natural rate is moving a lot and the policy rate cannot keep up with it. Movements in the natural rate will of course depend on the source of the shock, but they will also depend on the economic environment: do bonds have liquidity, and
if so, are they good substitutes for money, or are they poor substitutes? In what follows, we will consider three cases for the elasticity of substitution: *complements* ($\xi = 0.75$), *unit elasticity* ($\xi = 1.00$), and *substitutes* ($\xi = 1.25$). Another parameter that is potentially important is the response of taxes to debt (or $\phi$); this parameter dictates how much extra liquidity is injected when there is an increase in government spending.

Equally important is presumably whether the policy rate can be made to follow closely the movements in the natural rate. This will of course depend on the policy rule that is in place. We will consider the Rules 1 through 4 that were defined in the introduction.

### 3.1 Importance of the Economic Environment

First, we look at the responses of the natural rate, and the deviation of the policy rate from its natural rate, to shocks in government purchases and productivity. And we see how these responses depend on the economic environment. For this exercise we assume that monetary policy is guided by Rule 1, the basic Taylor rule.

Figure 1A shows impulse response functions for an increase in government purchases. The basic Taylor rule has a constant intercept term; there is no attempt to follow unobserved fluctuations in the natural rate. The top panel shows responses of the real natural rate of interest, and the bottom panel shows the gap between the policy rate and its natural rate.

An increase in government purchases crowds out consumer spending, thus raising the natural rate of interest. In the Standard Model, the increase in the natural rate is moderate, and it dies out relatively quickly. The policy rate rises in response to the inflation caused by the spending increase, but by slightly less than the natural
rate. A small gap is created but it too closes rather quickly.

There is a very different response in the Liquid Bonds Model, and that response depends upon whether money and bonds are complements ($\xi = 0.75$) or substitutes ($\xi = 1.25$). In either case, the initial rise in the natural rate is moderate, not unlike the increase in the Standard Model. But here, the natural rate does not return to its steady state value for a very long time. The gap term shows a similar pattern. And these responses are much more pronounced when money and bonds are complements.

Why is this so? In both the Standard Model and the Liquid Bonds Model, the increase in government spending crowds out consumption and the natural rate rises. But when bonds provide liquidity, there is more: as seen in the bottom left panel of Figure 2, a bond financed increase in government spending increases the supply of bonds and that process continues for a very long time. In response to an innovation in government purchases of one percent of quarterly GDP, debt continues to rise for more than 11 years, and at its peak, the increase is nearly 20 percent of quarterly GDP.

As the debt rises and $\mu_{\tau} \beta_{\tau}$ falls, the marginal liquidity value of bond holdings declines, which raises the natural rate and reduces the spread, $I^* - I$. To see this, let $\Delta x$ be the difference in the response of variable $x$ between any two of the three models (the Standard Model, the Liquid Bonds Model where money and bonds are complements, and the Liquid Bonds Model where the two are substitutes). We can then decompose the difference in the natural rates into,

$$\Delta I^N = \Delta I^{*,N} - \Delta(I^{*,N} - I^N)$$

The upper left panel of Figure 2 shows that there is very little difference between the real natural CCAPM rates in the three models. So, the first term on the RHS of equation (20) is close to zero, and the differences in the natural rate of interest
are due essentially to differences in the marginal liquidity value of government debt, which is reflected in the smaller spread. Thus, the natural rate rises by more in the Liquid Bond Models than in the Standard Model. And because the supply of bonds grows over time, the difference in the real natural rates of interest rises over time as well.

The next question is, why is the response of the natural rate so much larger when money and bonds are complements? A fiscal expansion makes \( m/b \) fall.\(^6\) This leads to a larger decline in the marginal liquidity value of bonds when money and bonds are complements. From equation (11), when money and bonds are complements (and \( \rho \) is low), the decrease in \( m/b \) results in a larger change in the marginal liquidity value of bonds and a larger change in the spread, \( I^* - I \). As seen in Figure 2, the ratio of money to bonds is virtually the same in both cases so that essentially all the difference in the natural rates arises from the greater response in the spread, \( I^* - I \), to a given change in \( m/b \) when money and bonds are complements.

The increase in the natural rate in the Liquid Bonds Model is relatively moderate: an innovation in government purchases of one percent of GDP raises the annualized natural rate by about 35-50 basis points after 5 years (depending on the version of the model) and about 40 - 60 basis points after 10 years. But the increase in the natural rate is extremely persistent, reflecting debt dynamics. As seen in Figure 2, debt continues to grow, and \( m/b \) continues to decline, for more than 10 years following a persistent shock to government purchases. As a result, the marginal liquidity value of debt continues to decline, and the natural rate continues to rise, for more than 10 years before these changes begin to reverse. As we will see in the next section, failing to adjust monetary policy in response to these moderate but extremely persistent shocks can have significant long-term consequences.

\(^6\)A deficit financed increase in government purchases increases \( m + b \). Monetary policy responds to the increasing inflation with an open market sale of bonds. The net effect is a fall in \( m/b \).
persistent changes in the natural rate of interest has significant welfare implications.

These differences across models in the marginal liquidity value of government bonds explain the differences in the behavior in the gap between the policy rate and the natural rate as well. In all three cases, the policy rate, which follows the basic Taylor rule, rises in response to inflation. Initially, the gaps between the policy rates and the natural rates are all similar and small. Over time, however, as the policy rates respond only to inflation, and the natural rates move with the accumulating bond supplies, the gaps in either one of the Liquid Bonds Models grow. And because the rise in the natural rate is greater when money and bonds are complements, the gap between the policy rate and the natural rate is greater as well.⁷

Another important parameter in the Liquid Bonds Model is $\phi$, the tax response to a change in the level of the debt. In Figure 1B, we assume that money and bonds are complements, and we consider two values of $\phi$. The solid lines in Figure 1B are the same as the dashed lines in Figure 1A. The benchmark value of $\phi$ is 0.018; when we double that value, the persistence in the natural rate’s response to a spending shock goes down considerably. The reason for this is straightforward. The increase in spending is initially deficit financed, increasing the amount of liquid bonds; a stronger tax response closes the deficit more quickly and limits the bond provision. The persistence of the effect on the natural rate and the interest rate gap is much diminished. We should note that the benchmark response is strong to begin with: $\phi = 0.018$ is three times the steady state value of $R$. We have therefore been conservative in our benchmark specification of persistence. We will see in Section 4

⁷The gap between the policy rate and the natural rate is less than the rise in the natural rate. Because the policy rule fails to take into account the rise in the natural rate, the policy rate is set too low. As a result, inflation rises more in the Liquid Bonds Model and is persistent. The policy rule raises the real policy rate when inflation rises, thereby reducing the gap somewhat.
that this parameter is also important to welfare.

A productivity shock, unlike a shock to government purchases, does not have a direct effect on government liabilities, $m + b$. But, monetary policy changes their composition, and this has real effects through (11). Figure 3 shows that a productivity shock has sizable effects on the natural rate of interest; however, they are not long lasting. A positive productivity shock increases supply, and this drives the natural rate down. The effect is more pronounced in the Liquid Bonds Model, but the differences, while long lasting, are not large.

Summing up: In the Standard Model, government spending shocks cause the natural rate to rise moderately, and productivity shocks cause the natural rate to fall moderately. But in each case, movements in the natural rate die out relatively quickly. Moreover, the policy rate tracks movements in the natural rate rather closely. When bonds provide liquidity, roughly the same conclusions hold with respect to productivity shocks. However, bond financed government spending shocks induce sustained movements in the natural rate (because of sticky prices); the basic Taylor rule cannot make the policy rate keep up. In this case, observing and tracking the natural rate would seem to be most important. Finally, this last result is more pronounced when money and bonds are complements; when they are substitutes, a given change in the money to bonds ratio does not cause as large a swing in the relative prices of the two assets or the interest rate spread. And simulations show that movements in the spread are closely related to movements in the natural rate.

3.2 Importance of the Interest Rate Rule

Some policy rules are better than others in making the policy rate track its natural rate. Rule 1 has a constant intercept term, set at the steady state value
of the nominal natural rate of interest; this rule makes no direct attempt to track short run fluctuations in the natural rate. Neither does Rule 3, the first difference rule. But there is no intercept term in it, and it may track the natural rate better than the basic Taylor rule. Why? A unit root process may be better at picking up prolonged movements in the natural rate. Rule 4 assumes that the natural rate is actually observed; this rule must be considered a benchmark case, since it probably cannot be implemented in practice.

In this section, we examine the performance of these three rules in the Standard Model and in our Liquid Bonds Model. We begin with the Liquid Bonds Model; for this exercise, we set the elasticity of substitution between money and bonds equal to one ($\xi = 1$); this case is right in between the complements ($\xi = 0.75$) case and the substitutes ($\xi = 1.25$) case.

Figure 4 shows responses in the interest rate gap to government spending and productivity shocks. As might be expected, the basic Taylor rule does a very poor job of tracking the natural rate. For an increase in government purchases, the interest rate gap is still widening after five years. This is, once again, because a bond financed expansion reduces the marginal liquidity of bonds and induces a protracted increase in the natural rate. The rule fares somewhat better for an increase in productivity: the interest rate gap has a half life of four years and a quarter. The first difference rule does a much better job of tracking the natural rate: the half life is just two quarters for either of the shocks. And, not surprisingly, Rule 4 tracks the natural rate perfectly.

Figure 5 gives the analogous results for the Standard Model. Since bond financed fiscal expansions do not directly affect liquidity, the basic Taylor rule fares much better

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8Rule 2 tracks the natural rate better than Rule 1, and worse than Rule 3. For simplicity of exposition, we do not include it in this section.
in the case of an increase of government spending: the interest rate gap has a half life of just four years. Rule 1 also does somewhat better in the case of a positive productivity shock: the gap has a half life of less than three years. The first difference rule appears to fare a little better than in the Liquid Bonds model, though the half life of the gap is still two quarters for either shock. And once again, Rule 4 tracks the natural rate perfectly.

*Summing up:* Rules that quickly bring the policy rate in line with its natural rate are presumably better policies, a normative question we pursue in Section 4. The basic Taylor rule makes no explicit attempt to track the natural rate, and in fact, it does a bad job of it. This is especially true for bond financed spending increases in the Liquid Bonds Model. The first difference rule does a much better job, in both models and for both government spending shocks and productivity shocks. This may be because a unit root process is better at picking up prolonged movements in the natural rate. Rule 4 assumes that the natural rate of interest is directly observable, and as a consequence it tracks the natural rate perfectly; however, it is probably not implementable in practice.

4 Is Observing the Natural Rate of Interest Important in Terms of Household Utility?

The previous section conducted a positive analysis of three of the policy rules; here we study the normative performance of Rules 1 though 4. The notion guiding the positive analysis was that good policy rules make the policy rate track its natural rate closely. If that notion is correct, then Rule 2 is better than Rule 1, Rule 3 is better than Rule 2 and Rule 4 is better than Rule 3. Here, we confirm that notion,
and we ask whether the gain from going from one rule to the next is quantitatively important. We also compare different economic environments in this regard; that is, we study the Standard Model and the Liquid Bonds Model. In the latter we consider cases where money and bonds are complements ($\xi = 0.75$) case and where they are substitutes ($\xi = 1.25$).

The metric we use is the expected discounted value of household utility, conditional on the economy starting in its steady state. But since this metric has no clear meaning, we follow the custom of comparing the policy rules in terms of consumption units: what percent of consumption would households be willing to give up each period – *assuming that the work effort is held constant* – to move from one rule to the next.

The top panel of Table 1 presents the welfare results. In it, the comparisons are all with respect to Rule 1, the basic Taylor rule, with a constant intercept term.

Rule 2 is the smoothed Taylor rule that is used in much of the New-Keynesian literature. Households would give up a little over a tenth of a percent of consumption to have interest rate smoothing in either the Standard Model or the Liquid Bonds Model with complements; in the Liquid Bonds Model with substitutes, it is only half as much.

Rule 3, the first difference rule, does much better when compared with the basic Taylor rule: a quarter of a percent in the Standard Model, and a half a percent in the Liquid Bonds Model with complements. These are large numbers by the standards of the New-Keynesian literature. And the fact that the first difference rule does particularly well in the Liquid Bonds Model with complements is perhaps not surprising; in that environment, government spending shocks cause a sustained movements in the natural rate that are hard for either of the Taylor rules to track. The gain is not as large in the Liquid Bonds Model with substitutes: about a third
of a percent of consumption.

It is interesting to note that the first difference rule performs virtually as well as the natural rate rule. To implement the first difference rule, there is no need to observe the natural rate. It may be somewhat surprising that the first difference rule does so well, but it is consistent with earlier results discussed in the Introduction.

The second panel in Table 1 shows that the relative performances of the various rules depend on the parameter $\phi$, the response of taxes to the level of debt. When we double $\phi$, the welfare gains are smaller. This is not surprising in light of Figure 1B. A quicker response of taxes to debt closes the deficit faster and limits the provision of liquid bonds. We should note, however, that our benchmark value of $\phi$ is three times the steady state value of the rate of return on government bonds; the benchmark response is already very strong. In this sense, we have been conservative in our estimates of the welfare gains.

*Summing up:* In the Standard Model, households would be willing to give up a quarter of a percent of consumption to replace the basic Taylor rule with the natural rate rule. In our preferred Liquid Bonds Model, where money and bonds are complements, the household would be willing to give up a half of a percent of consumption to get the natural rate rule. Finally, the first difference rule performs virtually as well as the natural rate rule in all of these cases. Presumably that is because a unit root process can better pick up sustained movements in the natural rate.

## 5 Accounting for Model Uncertainty

Using our preferred parameterization of the Liquid Bonds Model, we have shown that it is very important for the central bank to track the natural rate of interest,
or if it cannot do that, to follow a first difference rule. But is the Liquid Bonds Model an accurate representation of the "true" model of the economy? Most central bankers would admit to considerable uncertainty about their models, and indeed, model uncertainty is one of the reasons why it is so difficult for them to track the natural rate in practice.

In this section, we take model uncertainty into account in two separate ways. In the first exercise, we consider a particular example of model misspecification.9 We assume that the central bank can actually compute the natural rate of interest, but that it uses the wrong model in doing so. Can the first difference rule do better than the natural rate rule in this environment? In the second exercise, we consider a much more general characterization of model uncertainty in the spirit of Hansen and Sargent.10 Can the first difference rule perform well in this environment?

5.1 Central Bank Uses the Wrong Model

Suppose the Liquid Bonds Model is the true model, but the central bank does not recognize that government bonds provide liquidity; it believes the Standard Model to be the true model of the economy. Suppose further (and counter to our previous presumptions) that the central bank is able to compute the natural rate of interest using the Standard Model.

The middle panel of Table 1 gives the welfare comparisons for this particular misspecification. As might be expected, the natural rate rule does worse than the first difference rule in this environment, but the difference in welfare is very small. Households would be willing to give up two one-hundredths of a percent of steady

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9 This exercise is in the spirit of a literature that looks for policies that do well in a variety of model. See for example McCallum[23] and Levin et.al.[20].

10 This exercise is in the spirit of Hansen and Sargent[12] and Giordani and Soderlind [10].
state consumption (or even less in the substitutes case) to have the central bank use
the first difference rule rather than the natural rate rule.

While the natural rate rule is inferior to the first difference rule, it is still superior
to the simple Taylor rule. Movements of the natural rate in the Standard Model
and Liquid Bonds Models are correlated; so, in this exercise, the economy performs
better if the central bank tracks the incorrect estimates of the natural rate than if it
follows the basic Taylor rule.

While the welfare difference here is small, it does tell a cautionary tale. If the
natural rate is imprecisely estimated, the far simpler first difference rule may be the
better part of valor.

5.2 Robustness

There can be little doubt that government bonds provide liquidity. But, does
that liquidity come precisely in the manner prescribed by our Liquid Bonds Model?
We have already admitted to uncertainty about the elasticity of substitution between
money and bonds, and we have seen that a reduction in that elasticity significantly
increases the natural rate’s persistence in response to government spending shocks.
Variations in other model parameters may also affect the natural rate’s dynamics –
the tax response to debt, \( \phi \), comes readily to mind. Figure 1B showed that increasing
\( \phi \) significantly lowered the natural rate’s persistence in response to a spending shock.
More generally, bond liquidity may also matter to firms, affecting marginal costs
in addition to marginal utilities; the "true" transactions technology may look very
different than the one we have modeled.

The range of possible misspecification is broad and rather ill defined. We will
therefore not even attempt to specify a set of competing models with corresponding
probabilities. Instead, we will consider model uncertainty in the spirit of Hansen and Sargent[12].

More specifically, agents within the economy believe that the true model lies in a well specified neighborhood around the reference model, which we assume to be the benchmark Liquid Bond Model. Following Giordani and Soderlind[10], we linearize the reference model, and represent it by the law of motion:

\[
\begin{bmatrix}
  x_{1,t+1} \\
  E_t(x_{2,t+1})
\end{bmatrix}
= A
\begin{bmatrix}
  x_{1,t} \\
  x_{2,t}
\end{bmatrix}
+ BR_t
+ \begin{bmatrix}
  C \\
  O
\end{bmatrix}
(\varepsilon_{t+1} + \zeta_{t+1})
\]  

(21)

where \(O\) is a matrix of zeros, \(x_{1,t}\) is a vector of state variables, \(x_{2,t}\) is a vector of forward looking (or jump) variables, \(\varepsilon_t\) is a vector of primitive shocks and \(\zeta_t\) is a vector that generates model misspecifications in a manner that will be described shortly. The linearization of the Liquid Bonds Model is tedious; it is relegated to an appendix.

In the Hansen-Sargent approach to robustness, the potential for model misspecification is represented by additional additive errors, \(\zeta_t\). It is standard in this literature to use the metaphor of an evil agent who chooses these \(\zeta_t\) to maximize the resulting cost to households. Here we want to evaluate our four generic policy rules; so, we follow Giordani and Soderlind’s approach.

More precisely, the evil agent chooses \(\zeta_{t+1}\) (as a linear function of \(x_{1,t}\)) to maximize \(E_0 \sum_{t=0}^{\infty} \beta^t(\Pi_t - \bar{\Pi})^2\), subject to the law of motion, one of our interest rate rules, and what might be called a neighborhood constraint:

\[E_0 \sum_{t=0}^{\infty} \beta^t \zeta_{t+1} \zeta_{t+1} \leq \Theta\]

A single parameter, \(\Theta\), defines the neighborhood around the reference model that is

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11 Later, we will use the software provided by these authors to calculate the economy’s dynamics in the "worst case model" – that is under the worst possible misspecification in the neighborhood of the reference model.
thought to contain the set of plausible misspecifications, and this is the set of possible misspecifications that the evil agent is allowed to choose from. Since $\zeta_{t+1}$ can be a linear function of the state variables, $x_{1,t}$, this approach allows for parameter uncertainty. $\zeta_{t+1}$ cannot be a function of expectations of the forward looking variables, $x_{2,t}$. In this formulation, only the state variables have shocks, and the evil agent’s actions have to be conflated with these primitive shocks.

Critics of the Hansen-Sargent approach have argued that if $\Theta$ is very big, the agents’ behavior may be determined by rather improbable misspecifications, and this might make them look overly cautious. The trick is to choose a value of $\Theta$ that is big enough to reflect a reasonable degree of caution, but not so big as to make agents look foolish.\(^{12}\)

Since there is no set of specified alternative models to provide the context for evaluating the policy rules, we compare the welfare of the representative agent under the worst case model for the alternative policy rules. Welfare results are presented in the lower panel of Table 1, and they show that the first difference rule outperforms the Taylor rules. Letting the benchmark Liquid Bonds Model be the reference model, households would be willing to give up three quarters of a percent of steady state consumption to have the central bank use the first difference rule instead of the basic Taylor rule, Rule 1. If instead the substitutes case is chosen as the reference model, the welfare gain falls to two fifths of a percent, which however is still a large number.

*Summing up:* When model uncertainty is taken into account, the case for

\(^{12}\text{Hansen and Sargent suggest a way to do this. The bigger is }\Theta\text{, the greater is the probability of statistically distinguishing between the reference model and the worst case model. The }\Theta\text{ we use here sets the probability of detection at one third, which is somewhat higher than the probabilities recommended by Hansen and Sargent. We are being conservative in our specification of uncertainty, therefore the welfare gains we report below may understate our basic result.}*
first difference rule is much bolstered. If the central bank uses the wrong model to estimate the natural rate, then the result can easily be worse than the simple first difference rule that requires no estimation. If the central bank cannot estimate the natural rate precisely, then discretion may be the better part of valor. And when model uncertainty is profound, Hansen and Sargent’s approach to robustness suggests that the first difference rule’s dominance is clear.

6 Conclusion

In Section 2, we showed that – in some economic environments and with some standard monetary policy rules – shocks to the economy can make the natural rate of interest deviate substantially from its steady state value for a very long time. In Section 3, we showed that in this kind of environment, household utility improves significantly when the central bank makes the policy rate track its natural rate precisely. This may be very difficult to do in practice. Fortunately, a first difference rule – a rule that does not require any information about the natural rate – performs virtually as well as the natural rate rule. In Section 4, we took model uncertainty into account in several ways, and in each case the dominance of the first difference rule was only enhanced. More detailed results have already been summarized at the ends of the preceding sections.

There is a timely lesson to be learned from our analysis. As of this writing, many OECD countries are undertaking, or contemplating, large cuts in government spending to stabilize their sovereign debts. If bonds do provide liquidity services, then our results suggest that the natural rate of interest will be on the move and hard to track. The first difference rule seems to be made for just this situation.
References


7 Appendix: linearization & model dynamics

In this section we linearize the model and write it in the form

$$
\begin{bmatrix}
x_{1,t+1} \\
E_t(x_{2,t+1})
\end{bmatrix} = A \begin{bmatrix} x_{1,t} \\
x_{2,t} \end{bmatrix} + B I_t + \begin{bmatrix} C \\
O \end{bmatrix} \varepsilon_t
$$

where $x_{1,t}$ is a vector of predetermined variables and $x_{2,t}$ is a vector of forward looking (jump) variables, and $\varepsilon_t$ is a vector of exogenous shocks.

7.1 Government

The government’s flow budget constraint can be written as follows

$$
b_t + m_t = \Pi_t^{-1}(I_{t-1}b_{t-1} + m_{t-1}) + g_t - t_t
$$

We let $d_t = b_t + m_t$ and write (22) as

$$
d_{t+1} = \Pi_{t+1}^{-1}[I_t d_t - (I_t - 1)m_t] + g_{t+1} - t_{t+1}
$$

to get the log-linear version

$$
\hat{d}_{t+1} = - \left( \frac{I d - (I - 1)m}{\Pi d} \right) \hat{\Pi}_{t+1} + \left( \frac{I}{\Pi} \right) \hat{d}_t - \left[ \frac{(I - 1)m}{\Pi d} \right] \hat{m}_t \\
+ I \left( \frac{d - m}{\Pi d} \right) \hat{I}_t + \left( \frac{g}{d} \right) \hat{g}_{t+1} - \left( \frac{t}{d} \right) \hat{t}_{t+1}
$$

We then define the predetermined state variable

$$
q_t = \hat{d}_t + \left( \frac{I d - (I - 1)m}{\Pi d} \right) \hat{\Pi}_t = \hat{d}_t + \delta_0 \hat{\Pi}_t
$$

and write the government budget constraint as

$$
q_{t+1} = \left( \frac{I}{\Pi} \right) \left[ q_t - \delta_0 \hat{\Pi}_t \right] - \left[ \frac{(I - 1)m}{\Pi d} \right] \hat{m}_t \\
+ I \left( \frac{d - m}{\Pi d} \right) \hat{I}_t + \left( \frac{g}{d} \right) \hat{g}_{t+1} - \left( \frac{t}{d} \right) \hat{t}_{t+1}
$$
Later, we will replace $\hat{g}_{t+1}$ and $\hat{t}_{t+1}$ by their transition functions, and we will eliminate $\hat{m}_t$. Although $\hat{d}_t$ will not appear in the final representation we will use, we keep it as a variable for now, to be eliminated using (23) later.

### 7.2 Consumers

The log-linearized versions of expressions (4)-(6) are

$$\hat{\tau}_t = \gamma_\tau \hat{\nu}_t$$  \hspace{1cm} (24)

$$\hat{\theta}_t = \gamma_\theta \hat{\nu}_t$$  \hspace{1cm} (25)

where

$$\gamma_\tau \equiv \frac{v + v^*}{v - v^*}$$

and

$$\gamma_\theta \equiv \frac{v}{v - v^*}$$

The log-linear version of the CES aggregator

$$\tilde{m}_t^\rho = a^{1-\rho} m_t^\rho + (1-a)^{1-\rho} (d_t - m_t)^\rho$$

is

$$\hat{\tilde{m}}_t = (\tilde{m})^{-\rho} \left\{ [a^{1-\rho}m^\rho - (1 - a)^{1-\rho} m (d - m)^{\rho-1}] \hat{m}_t + [(1 - a)^{1-\rho} d (d - m)^{\rho-1}] \hat{d}_t \right\}$$

which gives

$$\hat{\nu}_t = \hat{c}_t - \gamma_m \hat{m}_t - \gamma_d \hat{d}_t$$  \hspace{1cm} (26)

with

$$\gamma_m \equiv \left( \frac{v}{c} \right)^\rho \left[ a^{1-\rho}m^\rho - (1 - a)^{1-\rho} m (d - m)^{\rho-1} \right]$$

and

$$\gamma_d \equiv \left( \frac{v}{c} \right)^\rho \left[ (1 - a)^{1-\rho} d (d - m)^{\rho-1} \right]$$
Log-linearizing the households’ optimality condition (10)

\[
\left\{ 1 - A[v_t^2 - (v^*)^2] \left( \frac{(1-a) \tilde{m}_t}{b_t} \right)^{1-\rho} \right\} = \frac{I_t}{I^*_t}
\]

we get

\[
\frac{I_t}{I^*_t} (\tilde{I}_t^* - \tilde{I}_t) = 2A v^2 \left( \frac{(1-a) \bar{m}}{d-m} \right)^{1-\rho} \tilde{v}_t
\]

\[
(1-\rho) \left( \frac{(1-a) \bar{m}}{d-m} \right)^{1-\rho} A[v^2 - (v^*)^2] \left\{ \frac{1}{\bar{m}} \left( \gamma_m \tilde{m}_t + \gamma_d \tilde{d}_t \right) - \left( \frac{dd_t}{d-m} - \frac{\bar{m} \tilde{m}_t}{d-m} \right) \right\}
\]

\[
\tilde{I}_t^* - \tilde{I}_t = \delta_1 \tilde{v}_t + \delta_2 \left[ \gamma_m + \left( \frac{m}{d-m} \right) \right] \tilde{m}_t + \delta_2 \left[ \gamma_d - \left( \frac{d}{d-m} \right) \right] \tilde{d}_t
\]

(27)

where

\[
\delta_1 = 2A \left( \frac{I^*_t}{I_t} \right) (v)^2 \left[ \frac{(1-a) \bar{m}}{d-m} \right]^{1-\rho}
\]

and

\[
\delta_2 = A \left( \frac{I^*_t}{I_t} \right) (1-\rho) \left[ v^2 - (v^*)^2 \right] \left[ \frac{(1-a) \bar{m}}{d-m} \right]^{1-\rho}
\]

Log-linearizing (9),

\[
1 - (I_t^*)^{-1} = A[v_t^2 - (v^*)^2] \left( \frac{a \tilde{m}_t}{m_t} \right)^{1-\rho}
\]

we get

\[
\tilde{I}_t^* = I \left[ \frac{(d-m)a}{(1-a)\bar{m}} \right]^{1-\rho} \left\{ \delta_1 \tilde{v}_t + \delta_2 \left[ (\gamma_m-1) \tilde{m}_t + \gamma_d \tilde{d}_t \right] \right\}
\]

(28)

The rest of the consumer block is:

\[
\tilde{I}_t^* - E_t \tilde{I}_{t+1} = \sigma(E_t \tilde{c}_{t+1} - \tilde{c}_t) + \frac{2Av}{1+\vartheta} E_t (\tilde{v}_{t+1} - \tilde{v}_t)
\]

(29)

\[
\tilde{w}_t = \chi \tilde{n}_t + \sigma \tilde{c}_t + \frac{2Av}{1+\vartheta} \tilde{v}_t
\]

(30)
7.3 Firms and the supply side

The supply side equations associated with Calvo pricing need not be reproduced here. Their log-linearization yields

\[ \hat{\Pi}_t = \beta E_t \left( \hat{\Pi}_{t+1} \right) + \lambda_p \hat{mc}_t \]  
(31)

\[ \hat{mc}_t = \hat{w}_t - \hat{y}_t + \hat{n}_t \]  
(32)

\[ \hat{y}_t = \alpha \hat{n}_t + \hat{\gamma}_t \]  
(33)

\[ \hat{\gamma}_t = \tau \gamma_c \hat{r}_t + (1 + \tau) \gamma_c \hat{c}_t + \gamma_g \hat{g}_t \]  
(34)

where \( \gamma_c = \frac{\sigma}{y}, \gamma_g = \frac{\sigma}{y} \), and the slope coefficient, \( \lambda_p \equiv \frac{(1-\theta)(1-\beta)}{\theta (1+\theta)} \).

7.4 System Reduction

We define

\[ x_t = \sigma \hat{c}_t + \left( \frac{2Av}{1 + \vartheta} \right) \hat{\gamma}_t \]

and write (29) as

\[ \hat{I}_t^e = E_t \hat{\Pi}_{t+1} + E_t x_{t+1} - x_t \]  
(35)

Substituting this in (28), we get

\[ E_t \hat{\Pi}_{t+1} + E_t x_{t+1} = x_t + I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1-\rho} \left\{ \delta_1 \left( \frac{1 + \vartheta}{2Av} \right) (x_t - \sigma \hat{c}_t) + \delta_2 \left[ (\gamma_m - 1) \hat{m}_t + \gamma_d \hat{d}_t \right] \right\} \]

We then use (26), to get

\[ \hat{m}_t = \frac{1}{\gamma_m} \left\{ 1 + \sigma \left( \frac{1 + \vartheta}{2Av} \right) \right\} \hat{c}_t - \left( 1 + \vartheta \right)(x_t - \hat{\gamma}_t) \]  
(36)

and use this to eliminate \( \hat{m}_t \) and get

\[ E_t \hat{\Pi}_{t+1} + E_t x_{t+1} = \delta_3 x_t + \delta_4 \hat{d}_t + \delta_5 \hat{c}_t \]  
(37)
with

\[
\delta_3 = 1 + I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1 - \rho} \left( \frac{1 + \vartheta}{2Av} \right) \left\{ \delta_1 - \left[ \frac{\delta_2 (\gamma_m - 1)}{\gamma_m} \right] \right\}
\]

\[
\delta_4 = I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1 - \rho} \left( \frac{\gamma_d \delta_2}{\gamma_m} \right)
\]

\[
\delta_5 = I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1 - \rho} \left\{ \frac{\delta_2 (\gamma_m - 1)}{\gamma_m} \left[ \frac{\sigma (1 + \vartheta)}{2Av} + 1 \right] - \sigma \delta_1 \left( \frac{1 + \vartheta}{2Av} \right) \right\}
\]

Next, we make the same substitutions in (27), to get

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = x_t + \tilde{I}_t + \delta_1 \left( \frac{1 + \vartheta}{2Av} \right) (x_t - \sigma \tilde{c}_t) + \delta_2 \left[ \gamma_d - \left( \frac{d}{d - m} \right) \right] \tilde{d}_t
\]

\[
+ \frac{\delta_2}{\gamma_m} \left[ \gamma_m + \left( \frac{m}{d - m} \right) \right] \left\{ 1 + \sigma \left( \frac{1 + \vartheta}{2Av} \right) \right\} \tilde{c}_t - \left( \frac{1 + \vartheta}{2Av} \right) x_t - \gamma_d \tilde{d}_t
\]

which simplifies to

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = \tilde{I}_t + \delta_6 x_t + \delta_7 \tilde{d}_t + \delta_8 \tilde{c}_t
\]

(38)

with

\[
\delta_6 = 1 + \left( \frac{1 + \vartheta}{2Av} \right) \left\{ \delta_1 - \frac{\delta_2}{\gamma_m} \left[ \gamma_m + \left( \frac{m}{d - m} \right) \right] \right\}
\]

\[
\delta_7 = \delta_2 \left[ \gamma_d - \left( \frac{d}{d - m} \right) \right] - \frac{\delta_2 \gamma_d}{\gamma_m} \left[ \gamma_m + \left( \frac{m}{d - m} \right) \right]
\]

\[
\delta_8 = \frac{\delta_2}{\gamma_m} \left[ \gamma_m + \left( \frac{m}{d - m} \right) \right] \left[ \frac{\sigma (1 + \vartheta)}{2Av} + 1 \right] - \sigma \delta_1 \left( \frac{1 + \vartheta}{2Av} \right)
\]

Then, we use (37) to eliminate \( \tilde{c}_t \) from (38) and get

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = \tilde{I}_t + \delta_6 x_t + \delta_7 \tilde{d}_t + \frac{\delta_8}{\delta_5} \left[ E_t \tilde{\pi}_{t+1} + E_t x_{t+1} - \delta_3 x_t - \delta_4 \tilde{d}_t \right]
\]

which, using (23), gives

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = \left( \frac{\delta_5}{\delta_5 - \delta_8} \right) \left[ \tilde{I}_t + \delta_9 x_t + \delta_{10} \tilde{q}_t - \delta_{10} \delta_0 \tilde{\Pi}_t \right]
\]

(39)

with

\[
\delta_9 = \delta_6 - \delta_3 \left( \frac{\delta_8}{\delta_5} \right)
\]

35
\[ \delta_{10} = \delta_7 - \delta_4 \left( \frac{\delta_8}{\delta_5} \right) \]

Next, we write the Phillips curve with no indexation as

\[ \ddot{\Pi}_t = \beta E_t \ddot{\Pi}_{t+1} + \lambda_p \ddot{m}_c_t \]

We use (37) and (38) to calculate

\[ \hat{c}_t = \left( \frac{1}{\delta_5 - \delta_8} \right) \left\{ \ddot{I}_t + (\delta_6 - \delta_3) x_t + (\delta_7 - \delta_4) \left[ q_t - \delta_0 \ddot{\Pi}_t \right] \right\} \]  \hspace{1cm} (40)

and we put this and

\[ \hat{v}_t = \left( \frac{1 + \vartheta}{2 A v} \right) (x_t - \sigma \ddot{c}_t) \]

in the market clearing condition,

\[ \hat{y}_t = \tau \gamma_c \gamma_r \ddot{v}_t + (1 + \tau) \gamma_c \ddot{c}_t + \gamma_g \ddot{g}_t \]

to get

\[ \hat{y}_t = \tau \gamma_c \gamma_r \left( \frac{1 + \vartheta}{2 A v} \right) x_t + \left[ (1 + \tau) \gamma_c - \tau \gamma_c \gamma_r \sigma \left( \frac{1 + \vartheta}{2 A v} \right) \right] \left( \frac{1}{\delta_5 - \delta_8} \right) \left\{ \ddot{I}_t + (\delta_6 - \delta_3) x_t + (\delta_7 - \delta_4) \left[ q_t - \delta_0 \ddot{\Pi}_t \right] \right\} + \gamma_g \ddot{g}_t \]

which gives

\[ \hat{y}_t = \gamma_g \ddot{g}_t + \delta_{11} \ddot{I}_t + \left[ \tau \gamma_c \gamma_r \left( \frac{1 + \vartheta}{2 A v} \right) + \delta_{11} (\delta_6 - \delta_3) \right] x_t + \delta_{11} (\delta_7 - \delta_4) \left[ q_t - \delta_0 \ddot{\Pi}_t \right] \]

with

\[ \delta_{11} = \left[ (1 + \tau) \gamma_c - \tau \gamma_c \gamma_r \sigma \left( \frac{1 + \vartheta}{2 A v} \right) \right] \left( \frac{1}{\delta_5 - \delta_8} \right) \]

We can write the marginal-cost deviation in terms of the output gap as usual

\[ \ddot{m}_c_t = \ddot{w}_t - \ddot{y}_t + \frac{1}{\alpha} (\ddot{y}_t - \ddot{a}_t) = \chi \ddot{n}_t + \sigma \ddot{c}_t + \frac{2 A v}{1 + \vartheta} \ddot{v}_t - \ddot{y}_t + \frac{1}{\alpha} (\ddot{y}_t - \ddot{z}_t) \]

\[ = \left( \frac{1 + \chi - \alpha}{\alpha} \right) \ddot{y}_t - \left( \frac{1 + \chi}{\alpha} \right) \ddot{z}_t + x_t \]

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and use the solution for the output gap to get

\[ \hat{m}_t = \left( \frac{1 + \chi - \alpha}{\alpha} \right) (\gamma \hat{g}_t + \delta_{11} \hat{I}_t) + \left( \frac{1 + \chi - \alpha}{\alpha} \right) \delta_{11} (\delta_7 - \delta_4) \left( q_t - \delta_0 \hat{\Pi}_t \right) + \left\{ 1 + \left( \frac{1 + \chi - \alpha}{\alpha} \right) \left[ \tau \gamma_c \gamma_r \left( \frac{1 + \vartheta}{2Av} \right) + \delta_{11} (\delta_6 - \delta_3) \right] \right\} x_t - \left( \frac{1 + \chi}{\alpha} \right) \hat{z}_t \]

This leads to the Phillips curve

\[ \beta E_t \hat{\Pi}_{t+1} = \delta_{12} \hat{\Pi}_t - \lambda_p \left( \frac{1 + \chi - \alpha}{\alpha} \right) (\gamma \hat{g}_t + \delta_{11} \hat{I}_t) - \lambda_p \left[ \delta_{13} x_t + \delta_{14} q_t - \left( \frac{1 + \chi}{\alpha} \right) \hat{z}_t \right] \]

with

\[ \delta_{12} = 1 + \lambda_p \delta_6 \delta_{11} \left( \frac{1 + \chi - \alpha}{\alpha} \right) (\delta_7 - \delta_4) \]
\[ \delta_{13} = 1 + \left( \frac{1 + \chi - \alpha}{\alpha} \right) \left[ \tau \gamma_c \gamma_r \left( \frac{1 + \vartheta}{2Av} \right) + \delta_{11} (\delta_6 - \delta_3) \right] \]

and

\[ \delta_{14} = \left( \frac{1 + \chi - \alpha}{\alpha} \right) \delta_{11} (\delta_7 - \delta_4) \]

Next, we eliminate \( \hat{m}_t \) from the government budget constraint,

\[ q_{t+1} - \left( \frac{g}{d} \right) \hat{g}_{t+1} + \left( \frac{t}{d} \right) \hat{t}_{t+1} = \left( \frac{I}{\Pi} \right) [q_t - \delta_0 \hat{\Pi}_t] - \left[ \frac{(I - 1) m}{\Pi d} \right] \hat{m}_t + I \left( \frac{d - m}{\Pi d} \right) \hat{I}_t \]

Substituting (40) in (36), we have

\[ \hat{m}_t = \frac{1}{\gamma_m} \left\{ \delta_{15} \left[ \hat{I}_t + (\delta_6 - \delta_3) x_t + (\delta_7 - \delta_4) \left( q_t - \delta_0 \hat{\Pi}_t \right) \right] - \left( \frac{1 + \vartheta}{2Av} \right) x_t - \gamma_d \left( q_t - \delta_0 \hat{\Pi}_t \right) \right\} \]

where

\[ \delta_{15} = \left[ 1 + \sigma \left( \frac{1 + \vartheta}{2Av} \right) \right] \left( \frac{1}{\delta_5 - \delta_8} \right) \]

which simplifies to

\[ \hat{m}_t = \left( \frac{\delta_{15}}{\gamma_m} \right) \hat{I}_t + \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) (\delta_6 - \delta_3) - \left( \frac{1 + \vartheta}{2 Av \gamma_m} \right) \right] x_t + \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) (\delta_7 - \delta_4) - \left( \frac{\gamma_d}{\gamma_m} \right) \right] q_t - \left( \frac{\delta_0}{\gamma_m} \right) (\delta_{15} (\delta_7 - \delta_4) - \gamma_d) \hat{\Pi}_t \]
We specify the tax rule

\[
\left( \frac{t}{d} \right) \hat{t}_{t+1} = \phi_d \left( \hat{a}_t \right) + u_{t+1} = \phi_d \left( q_t - \delta_0 \Pi_t \right) + u_{t+1}
\]

where \( u \) is generated by an AR(1) process, and we write the budget constraint as

\[
q_{t+1} - \left( \frac{g}{d} \right) \hat{g}_{t+1} + u_{t+1} = \left[ \frac{I}{\Pi - \frac{I m}{\Pi d}} \left( 1 + \frac{\delta_{15}}{\gamma_m} \right) + \left( \frac{m \delta_{15}}{\Pi d \gamma_m} \right) \right] \hat{I}_t + \delta_{16} \hat{\Pi}_t - \delta_{17} x_t + \delta_{18} q_t
\]

with

\[
\delta_{16} = \left[ \frac{(I - 1) m}{\Pi d} \right] \left( \frac{\delta_0}{\gamma_m} \right) [\delta_{15} (\delta_7 - \delta_4) - \gamma_d] - \delta_0 \left( \frac{I}{\Pi} - \phi_d \right)
\]

\[
\delta_{17} = \left[ \frac{(I - 1) m}{\Pi d} \right] \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) (\delta_6 - \delta_3) - \left( \frac{1 + \vartheta}{2 \delta} \right) \right]
\]

and

\[
\delta_{18} = \left( \frac{I}{\Pi} - \phi_d \right) - \left[ \frac{(I - 1) m}{\Pi d} \right] \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) (\delta_7 - \delta_4) - \left( \frac{\gamma_d}{\gamma_m} \right) \right]
\]

The first two equations of the dynamic system are then the exogenous processes for productivity and government purchases, \( \hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon^z_t \) and \( \hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon^g_t \). When we include a tax shock, the third equation is \( u_t = \rho u_{t-1} + \varepsilon^u_t \). The fourth equation is,

\[
q_{t+1} - \left( \frac{g}{d} \right) \hat{g}_{t+1} + u_{t+1} = \left[ \frac{I}{\Pi - \frac{I m}{\Pi d}} \left( 1 + \frac{\delta_{15}}{\gamma_m} \right) + \left( \frac{m \delta_{15}}{\Pi d \gamma_m} \right) \right] \hat{I}_t + \delta_{16} \hat{\Pi}_t - \delta_{17} x_t + \delta_{18} q_t
\]

and the remaining two equations are,

\[
E_t \hat{\Pi}_{t+1} + E_t x_{t+1} = \left( \frac{\delta_5}{\delta_5 - \delta_8} \right) \left[ \hat{I}_t + \delta_9 x_t + \delta_{10} q_t - \delta_{10} \delta_0 \hat{\Pi}_t \right]
\]

and

\[
\beta E_t \hat{\Pi}_{t+1} = \delta_{12} \hat{\Pi}_t - \lambda_p \left( \frac{1 + \chi - \alpha}{\alpha} \right) \left( \gamma g \hat{g}_t + \delta_{11} \hat{I}_t \right)
\]

\[
- \lambda_p \left[ \delta_{13} x_t + \delta_{14} q_t - \left( \frac{1 + \chi}{\alpha} \right) \hat{z}_t \right]
\]

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Table 1: Welfare Comparison

<table>
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<th>substitutes</th>
<th>compliments</th>
<th>Standard Model</th>
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<th>( \phi = 0.036 )</th>
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Figure 1A: Government Spending Shock, Rule 1

Standard Model: solid line; Liquid Bonds model, compliments: dashed line; substitutes: dotted line
Figure 1E: Government Spending Shock, Rule 1

Liquid Bonds Model, money and bonds are complements.

$\phi = 0.018$ (benchmark case) solid line; $\phi = 0.036$ dashed line
Figure 2: Government Spending Shock, Rule 1

- Real natural CCAPM rate
- Real CCAPM natural rate - real natural rate
- Debt to (steady state) GDP ratio
- Money to bonds ratio

Standard Model: solid line; Liquid Bonds model, compliments: dashed line; substitutes: dotted line
Figure 3: Productivity Shock, Rule 1

- The figure illustrates the relationship between the real natural interest rate and the gap, defined as the difference between the real interest rate and the real natural interest rate.

- The graphs display the dynamics for different models:
  - Standard Model: solid line
  - Liquid Bonds model, compliments: dashed line
  - Substitutes: dotted line

- The y-axis shows the gap values, with the x-axis representing time in quarters.

- The scale on the y-axis is 10^-3 for the real natural interest rate and 10^-4 for the gap.
Figure 4: Liquid Bonds Model, unit elasticity, Policy Rules

$\times 10^{-5}$ gap = real interest rate - real natural interest rate, government spending shock

Rule 1: solid line; Rule 3: dashed line; Rule 4: dotted line

Figure 4: Liquid Bonds Model, unit elasticity, Policy Rules

$\times 10^{-4}$ gap = real interest rate - real natural interest rate, productivity shock

Rule 1: solid line; Rule 3: dashed line; Rule 4: dotted line
Figure 5: Standard Model, Policy Rules

\[ \text{gap} = \text{real interest rate} - \text{real natural interest rate}, \text{ Government Spending Shock} \]

Rule 1: solid line; Rule 3: dashed line; Rule 4: dotted line

\[ \text{gap} = \text{real interest rate} - \text{real natural interest rate}, \text{ Productivity Shock} \]

Rule 1: solid line; Rule 3: dashed line; Rule 4: dotted line