Monetary Policy
and the Natural Rate of Interest

Matthew Canzoneri* Robert Cumby† Behzad Diba‡

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Abstract

It is most important for monetary policy to be able to track the (unobserved) natural rate of interest in economic environments where interest rates take large and sustained swings away from their long run equilibrium values. Here, we study two models: one is a standard New-Keynesian model; in the other, government bonds provide liquidity, and deficit financed spending shocks cause large and persistent movements in the natural rate because they create extra liquidity. Policy rules that cannot make the policy rate track its natural rate perform poorly in both of these models, but they are especially bad in the second. Standard Taylor rules do not track the natural rate well, while first difference rules do surprisingly well, at least after the first few quarters. The welfare differences (measured in household consumption equivalences) under the various rules we study are greatest when money and bonds are complements in the provision of liquidity, and we provide empirical evidence suggesting that this is indeed the case. Model uncertainty is perhaps the main reason why central banks have such a difficult time tracking the natural rate directly. When model uncertainty is explicitly taken into account, the dominance of the first difference rule is even more pronounced. This suggest that highly inertial policies should be pursued during times of big fiscal adjustments.

*Georgetown University, canzonem@georgetown.edu
†Georgetown University, cumbyr@georgetown.edu
‡Georgetown University, dibab@georgetown.edu
One can only say that if the bank policy succeeds in stabilizing prices,
the bank rate must have been brought in line with the natural rate.

Orphanides and Williams[20]

1 Introduction

Why is the natural rate of interest so important for monetary policy? Consider a simple
variant of the Taylor Rule
\[ i_t = i^n_t + 1.5(\pi_t - \bar{\pi}) \] (1)
where \( \pi_t \) is the rate of inflation, \( \bar{\pi} \) is the inflation target, \( i_t \) is the policy instrument, and \( i^n_t \) is its natural rate – defined as the rate that would prevail if there were no nominal rigidities.

Intuitively, when an increase in aggregate demand, or a decrease in productivity, pushes
inflation above its target, the policy rate should be raised above its natural rate for a period
of time, raising the real rate of interest to curb the rise in inflation. This “natural rate rule”
is good for inflation control, and as we shall see, it is also good for household welfare.

The problem here is of course that the natural rate of interest is not observed directly,
and estimating it is known to be difficult. In an early paper on the subject, Laubach and
Williams[14] found considerable movement in their (imprecise) estimates of the natural rate;
they concluded that “…this source of uncertainty needs to be taken account of in analyzing

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1 The intercept term in the original Taylor Rule was the long run value of the natural rate; we have
replaced it with the current period’s value, which will of course fluctuate over time.

The original specification of the Taylor Rule also contained an output gap. We will suppress the gap term
in this paper. It is well known that (unless the natural rate of output can be measured accurately) smoothing
the output gap is harmful in models where productivity shocks play a prominent role. Here, we leave those
well studied issues aside so that we can focus on the natural rate of interest.
monetary policies that feature responses to the natural rate of interest.” And indeed, Curdia, Ferrero, Ng and Tambalotti[7] argue that the Federal Reserve has a history of trying to track the natural rate. They quote Alan Greenspan as saying “... In assessing real rates, the central issue is their relationship to an equilibrium interest rate, specifically, the real rate level that, if maintained, would keep the economy at its production potential over time. Rates persisting above that level, history tells us, tend to be associated with slack, disinflation, and economic stagnation – below that level with eventual resource bottlenecks and rising inflation ... .” And the quote from Orphanides and Williams is an indication that the view is widespread.

In this paper, we show that tracking the natural rate is also important for household welfare. Moreover, we show that it is most important in an environment where interest rates take large and persistent swings around their long run equilibrium values, and it is difficult for standard monetary policy rules to make the policy rate catch up with its natural rate. We use two models to illustrate this fact: In one – which we call the Standard (New-Keynesian) Model – swings in the natural rate are substantial, but they are short lived. In the other model – which we call the Liquid Bonds Model – the swings can be larger and much more persistent. The reason is that government bonds provide liquidity in this model. So, an increase in public spending that is (initially) debt financed will provide extra liquidity in this model, generating protracted movements in consumer demand and equilibrium interest rates. We will show that tracking the natural rate is much more important for household welfare in the Liquid Bonds Model.

In either of the models, deviations of the policy rate from its natural rate will depend upon the monetary policy that is in place; we will consider four generic rules:

Rule 1 (the basic Taylor rule): \[ i_t = \bar{i} + 1.5(\pi_t - \bar{\pi}) \]

Rule 2 (the smoothed Taylor rule): \[ i_t = 0.8 i_{t-1} + (1 - 0.8)[\bar{i} + 1.5(\pi_t - \bar{\pi})] \]

Rule 3 (the first difference rule): \[ i_t = i_{t-1} + 1.5(\pi_t - \bar{\pi}) \]

Rule 4 (the natural rate rule): \[ i_t = [r^n_t + E_t(\pi_{t+1})] + 1.5(\pi_t - \bar{\pi}) \]
where $r^a_t$ is the real natural rate of interest and $\bar{r}$ is the steady state value of the policy rate.

Rule 4 is a variant of the natural rate rule discussed above, and it performs very well in both of our models. Indeed, Rule 4 virtually eliminates the cost of nominal rigidities: if it were in place, households would be willing give up less than 0.00002 percent of consumption each period to live in an economy with flexible prices. However, Rule 4 assumes that the natural rate is known each and every period. Since the natural rate is not observed in practice, we view Rule 4 as the benchmark by which more operational rules are to be judged.

Rules 1 and 2 are conventional Taylor rules that have been studied extensively in the literature. They are thought to be operational since they only assume that the long run equilibrium value of the natural rate is known. These rules have been shown to provide a good empirical description of the monetary policies used by many central banks, and they are often used in policy evaluation exercises. The coefficients we have postulated are typical of estimates that are found in the empirical literature. Rule 3 does not fall in the class of conventional Taylor rules, and we are unaware of any empirical literature, or central bank statement, suggesting that it has been followed in practice. However, this first difference rule would clearly be implementable; in fact, it does not even require the long run equilibrium value of the natural rate.

In the Standard Model, the difference in household utility between Rule 1 and Rule 4 is worth a quarter of a percent of steady state consumption each period, which is a substantial number in the New Keynesian literature. In our benchmark calibration of the Liquid Bonds Model, the difference rises to half a percent of consumption each period. Money and bonds are complements in the benchmark calibration; we take our value for the elasticity of substitution between money and bonds from an estimation of one of the model’s equations. However, our estimates are not very precise, and we show that if instead money and bonds were substitutes, then the difference in utility between Rule 1 and Rule 4 could be closer to what we find in the Standard Model. Finally, we show that Rule 3 does the best job of
tracking the natural rate after the initial four or five quarters, and it performs almost as well as Rule 4 in terms of household utility. We think the reason for this is that, being free of a fixed intercept term, the policy rate is free to move widely over time.

Model uncertainty is perhaps the most important reason why central banks find it so difficult to track the natural rate of interest. When we go on to consider model uncertainty—utilizing Hansen and Sargent’s[11] notion of robustness—the dominance of the first difference rule is even greater. These results suggest that highly inertial rules should be given more attention, both in the academic literature and in practice, and especially during times of large fiscal adjustments.

The first difference rule has been much lauded in the past. In all of the papers of which we are aware, an ad-hoc welfare criterion is used instead of household utility—the central bank minimizes a weighted average of the variances of inflation, output, and the policy rate. The variance of the policy rate was thought to be a concern because of instrument instability. Orphanides and Williams[21] and Laxton and Pesenti[16] are perhaps the papers that are closest to ours. They discussed central bank misperceptions of a constant long run natural rate of interest, and they found that a first difference rule worked quite well in that context. Levin, Wieland and Williams[17][18] discussed model uncertainty, and found a first difference rule worked quite well in that context; they made no mention of a fluctuating natural rate. Taylor and Williams[26] provide a nice review of a number of papers in this vein, and they provide an extensive bibliography.

Before proceeding to the analysis, we should address a question that is at the heart of our paper: Do bonds actually have liquidity value? The basic premise should not be controversial. U.S. Treasuries facilitate transactions in a number of ways: they serve as collateral in many financial markets, banks hold them to manage the liquidity of their portfolios, and individuals hold them in money market accounts that offer checking services. There

\[2\]
\[In some cases, there is just a limit placed on the variance of the interest rate.]
is an empirical literature which is generally supportive of this notion; it includes Friedman and Kuttner[8], Greenwood and Vayanos[10], and Pflueger and Viceira[24]. Moreover, the liquidity premium has been shown to depend on the level of debt; see for example, Krishnamurthy and Vissing-Jorgensen[13], Bohn[4]. And of course, we are not the first to study the liquidity services of bonds; contributions to the theoretical literature include Patinkin[23], Bansal and Coleman[1], Holmstrom and Tirole[12], Calvo and Vegh[15] and Linnemann and Schabert[20].

The rest of the paper proceeds as follows: In Section 2, we outline the two models and describe their calibration, including our estimate of the elasticity of substitution between money and bonds. In section 3, we compare the dynamic properties of the two models under alternative parameterizations and policy rules; and in particular, we show how well the various rules track the natural rate of interest. In Section 4, we perform the welfare analysis. And, in Section 5, we take model uncertainty directly into account, considering both model misperceptions and Hansen and Sargent’s approach to robustness; the linearization of the model for this purpose is relegated to an appendix.

2 The “Liquid Bonds Model” and the “Standard Model”

The Liquid Bonds Model was developed by Canzoneri, Cumby, Diba and Lopez-Salido[6]. The model extends a standard New Keynesian environment to reflect the fact that government bonds provide liquidity. The model uses Schmitt-Grohe and Uribe’s[25] specification of transactions costs, but it replaces money with a CES aggregate of money and bonds in their definition of velocity. Imbedded in the Liquid Bonds Model is a Standard Model in which bonds have no liquidity value; it just returns to the original Schmitt-Grohe and

Canzoneri, Cumby, Diba and Lopez-Salido[5] developed a more structural model in which banks hold money and bonds to manage their deposits. The model we use here makes it easier to build a quantitative model because we can pin down a number of important parameters by matching the sample averages of some key monetary and fiscal variables in U.S. data
Uribe definition of velocity. In the Standard Model, bonds have no liquidity value; they are not a substitute for money. The substitutability of money and bonds in the Liquid Bonds model plays a strong role in the sections that follow; we will therefore provide an estimate of the elasticity of substitution as part of our calibration of that model. We begin with the households and their transactions costs; that is where the action is.

2.1 Households and Transactions Costs

There is a continuum of households of measure one. Each household supplies labor to every firm; so, in a symmetric equilibrium the households’ behavior will be identical and we can dispense with household indices. Households maximize

$$E_t \sum_{j=t}^{\infty} \beta^{j-t} \left[ \log(c_j) - (1 + \chi)^{-1} n_j^{1+\chi} \right]$$

subject to a sequence of budget constraints,

$$b_j + m_j + (1 + \tau_j) c_j + t_j = w_j n_j + \frac{(I_{j-1} b_{j-1} + m_{j-1})}{\Pi_j} + div_j$$

where $c_t$ is household consumption of a composite final good in period $t$, $n_t$ is hours worked, $w_t n_t$ is real labor income, and $\tau_t c_t$ are household transactions costs. $m_t$ and $b_t$ are real money and bond holdings, $I_t$ is the gross nominal return on a riskless, one-period government bond, $\Pi_t \equiv \frac{p_t}{p_{t-1}}$ is the gross rate of inflation, $t_t$ are real lump sum taxes, and $div_t$ represents dividends. $I_t$ is a money market rate, and we take it to be the central bank’s policy rate.\footnote{In the U.S. context, the Fed Funds rate is the policy rate. In normal times, the effective Fed Funds rate is very close to the T-bill rate. We do not model the money markets explicitly. Implicitly, we are assuming the Fed Funds rate and the T-bill rate are one and the same.}

We follow Schmitt-Grohé and Uribe\cite{SchmittGroheUribe} in assuming that transaction costs are proportional to consumption. The factor of proportionality, $\tau_t$, is an increasing function of velocity,
Letting $\nu^*$ be the satiation level of velocity, we set

$$
\tau_t = \begin{cases} 
(A/v_t) (v_t - \nu^*)^2 & \text{for } v > \nu^* \\
0 & \text{otherwise}
\end{cases}
$$

where $A$ is a positive parameter. Schmitt-Grohé and Uribe defined velocity as $v_t = c_t/m_t$, but in the Liquid Bonds Model, we broaden the notion of transactions balances; more specifically, we let

$$
v_t = \frac{c_t}{\tilde{m}_t}
$$

where $\tilde{m}_t$ is a CES bundle of money and bonds,

$$
\tilde{m}_t^\rho = a^{1-\rho} m_t^\rho + (1 - a)^{1-\rho} b_t^\rho
$$

In the Standard Model, we revert to Schmitt-Grohé and Uribe’s definition of velocity; or alternatively, we set $\tilde{m}_t = m_t$.

The household’s first order conditions include:

$$
w_t \lambda_t = n_t^x
$$

and

$$
1/c_t = \lambda_t [1 + 2A(v_t - \nu^*)]
$$

where $\lambda_t$ is the real marginal utility of wealth. When real resources are depleted in the purchase of consumption goods, the marginal utility of wealth is less than the marginal utility of consumption. The first order conditions for money and bond holdings are:

$$
\left\{ 1 - A[v_t^2 - (\nu^*)^2] \left( \frac{a \tilde{m}_t}{m_t} \right)^{1-\rho} \right\} = (I_t^*)^{-1}
$$
\[
\left\{ 1 - A[v_i^2 - (\nu^*)^2] \left( \frac{(1-a) \tilde{m}_t}{b_t} \right)^{1-\rho} \right\} = \frac{I_t}{I^*_t}
\]  \hspace{1cm} (10)

where \( I^*_t \) is the gross nominal CCAPM interest rate; that is, \( (I^*_t)^{-1} = \beta E_t \left\{ \frac{P_t}{P_{t+1}} \frac{\lambda_{t+1}}{\lambda_t} \right\} \). We will think of \( I^*_t \) as the rate of return on a bond, \( b^*_t \), that does not provide liquidity services.\(^5\)

Equations (9) and (10) imply

\[
\frac{I^*_t - I_t}{I^*_t - 1} = \left( \frac{1-a}{a} \right)^{1-\rho} \left( \frac{m_t}{b_t} \right)^{1-\rho}
\]  \hspace{1cm} (11)

Since \( b_t \) provides transactions services, it will be held at a lower rate of return than \( b^*_t \); the spread, \( I^*_t - I_t \), will be non-negative in equilibrium. This spread measures the pecuniary opportunity cost of holding the bond that does provide transactions services; \( I^*_t - 1 \) is the opportunity cost of holding money. So, equation (11) says that in the optimal portfolio, the relative price of \( m \) and \( b \) is equated to the marginal rate of substitution between \( m \) and \( b \).\(^6\)

When, for example, the central bank conducts an expansionary open market operation, \( \frac{m_t}{b_t} \) will rise; the marginal liquidity value of bonds (relative to money) will rise. This will cause the relative price to rise, and the spread, \( I^*_t - I_t \), will increase.

\[\text{2.2 Intermediate Goods and the Final Consumption Good}\]

The modeling of the production side of the economy is quite standard. Our description of it can be brief.

A continuum of monopolistically competitive firms, indexed by \( j \), produce intermediate goods using a common technology,

\[
y_{j,t} = z_t \tilde{k} n_{j,t}^\alpha
\]  \hspace{1cm} (12)

\(^5\)Households are identical in our models, and \( b^*_t \) would be zero in equilibrium. For simplicity, we have simply priced these bonds, rather than discuss them further.

\(^6\)Later, we will use this equation to estimate the elasticity of substitution between money and bonds, \( \xi \equiv 1/(1-\rho) \).
where $\bar{k}$ is the firm’s fixed capital stock, $0 < \alpha < 1$, and $z_t$ is a common productivity shock that follows an AR(1) process:

$$\ln(z_t) = (1 - \rho_z) \ln \bar{z} + \rho_z \ln(z_{t-1}) + \varepsilon_{t}^{z}$$

where $0 \leq \rho_z < 1$ and $\bar{z} (= 1)$ is the steady state value of $z_t$. Competitive retailers buy the intermediate goods and bundle them into the final good, $y_t$, using a CES aggregator with elasticity $\eta$.

Intermediate good firms engage in Calvo price setting. Each period, with probability $1 - \theta$, a firm $j$ gets to set an optimal new price; if the firm does not get to re-optimize, its price goes up automatically by the steady state rate of inflation, $\bar{\Pi}$. Bars over variables indicate a steady state value.

### 2.3 Goods Market Clearing

The households transactions costs are a drain on resources. The market clearing condition for output is

$$y_t = (1 + \tau_t) c_t + g_t$$  \quad (13)

where $g_t$ is real government spending on the final good.

### 2.4 Fiscal Policy

Government spending follows an exogenous AR(1) process

$$\ln(g_t) = (1 - \rho_g) \ln \bar{g} + \rho_g \ln(g_{t-1}) + \varepsilon_{t}^{g}$$  \quad (14)

where $0 < \rho_g < 1$ and $\varepsilon_{t}^{g}$ is a spending shock.

The government uses a lump sum tax, $t$, to stabilize its debt. The tax rule is
\[ t_t = \ddot{t} + \phi (b_{t-1} - \ddot{b}) \quad (15) \]

We assume \( \phi > I/\bar{\Pi} - 1 \); so, tax increases are more than sufficient (on average) to pay the interest on any increase in the debt. This stabilizes the debt.

### 2.5 Model Calibration

The elasticity of substitution between money and bonds in the Liquid Bonds Model is \( \xi \equiv 1/(1 - \rho) \). When \( \xi < 1 \), money and bonds are complements; when \( \xi > 1 \), they are substitutes. Equation (11) implies that when money and bonds are complements, a given movement in portfolio shares, or \( \frac{m_t}{b_t} \), will produce a larger change in the interest rate spread than when they are substitutes. This fact will play an important role in what follows.

When money and bonds are complements, it is very important for monetary policy to be able to track the natural rate; when they are substitutes, the importance is diminished, but somewhat greater than in the Standard Model.

Because of the importance of this parameter, and because of the paucity of estimates to be found elsewhere, we will attempt to estimate it. The rest of the model’s calibration is standard, with many of the parameters taken from the literature or other papers of our own; we can therefore be brief in its description. We begin with our estimation of \( 1 - \rho \).

#### 2.5.1 Estimating the Elasticity of Substitution Between Money and Bonds

Taking logs of both sides of equation (11), we obtain

\[
\log \left( \frac{I_t^* - I_t}{I_t^* - 1} \right) = \text{constant} + (1 - \rho) \log \left( \frac{m_t}{b_t} \right) \quad (16)
\]

where \( \text{constant} = (1 - \rho) \log \left( \frac{1 - a}{a} \right) \). We will use this equation to estimate \( 1 - \rho \), which we can then invert to get an estimate of the elasticity, \( \xi \).
We do not observe the CCAPM interest rate, $I^*_t$; following Krishnamurthy and Vissing-Jorgensen[13], we use the yield on AAA-rated corporate bonds as a proxy. We use the yield on 10-year U.S. Treasury bonds for $I_t$. Then, we use the adjusted monetary base from the Federal Reserve Bank of St. Louis for $m_t$, and debt held by the public (excluding Federal Reserve System holdings) for $b_t$. All variables are measured at the end of the fiscal year, and our sample period begins in 1980 to coincide with the sample used to estimate other parameters and ratios in the model’s calibration. We consider two samples: one ends in 2007 (so that it excludes the financial crisis and the Federal Reserve’s unconventional monetary policy), and the other ends in 2012 (for this sample, we included a dummy variable for 2007-2012).

(11) is an exact equation; it has no error term. There is, however, good reason to think that we have measurement error when we use the AAA-rated corporate bond rate as a proxy for the CCAPM rate: among other things, the corporate bond rate includes a default risk premium, while the CCAPM rate does not. So, there will be an error term in our estimation of equation (11). A complicating factor here is that the default risk premium, and thus the measurement error tacked onto (11), may be correlated with the business cycle.

Ordinary least squares (OLS) may produce inconsistent estimates of $1 - \rho$ since the money to bonds ratio may also be correlated with the business cycle, and therefore the measurement error. To control for this possibility, we use both OLS and instrumental variables (IV) estimation procedures. The instrumental variables should be correlated with the money to bonds ratio, but uncorrelated with the measurement error, which as we have suggested may be correlated with the business cycle. The ratio of money to bonds is determined by the Federal Reserve through its open market operations, so we use as instruments one of the two variables found in the Taylor rule – the inflation rate – along with lagged values of the regressors and the dependent variable. We omit the output gap, since it may be correlated with the measurement error.
Our estimates of (11) are reported in Table 1. The point estimates of $1 - \rho$ from the instrumental variables regressions, and two of the three least squares regressions, are greater than one, suggesting that money and bonds are complements (or that $\xi = \frac{1}{1-\rho} < 1$). But the point estimates of $1 - \rho$ are imprecise. In the instrumental variables regressions, they are barely a standard error above one; so, a 95% confidence interval would included our other two cases of unitary elasticity and substitutes.

In our preferred calibration, we will assume that money and bonds are complements. We will let $\xi = 0.75$, which is in line with the IV estimates. However, since the parameter is imprecisely estimated, we will consider three cases in what follows: the preferred complements case, $\xi = 0.75$; the unit elasticity case, $\xi = 1$; and a substitutes case, with $\xi = 1.25$. It should be noted that the latter value is much higher than all of our point estimates but one.

### 2.5.2 The Steady State and the Rest of the Model’s Calibration

Canzoneri et al.[5] describes the steady state of the Liquid Bonds Model and its calibration in some detail. Our discussion here can be brief, and once again, the interested reader is referred to that paper for a more complete description.

The model’s calibration can usefully be divided into three sets of parameters. The first set is standard: $\beta$ is set equal to 0.99, $\theta = 0.75$ implies that the average duration of prices is four quarters; $\eta = 7$ implies that the steady state markup is 1.17; $\chi = 1$ implies that the disutility of work is quadratic; and $\alpha = 0.66$ implies that capital’s share is one third. The second set uses sample averages during the Volker-Greenspan era (we do not include data from the financial crisis) to set the steady state values of $\bar{\Pi}$, $\bar{I}$, $\bar{b}/\bar{c}$, $\bar{b}/\bar{m}$, and $\bar{g}/\bar{y}$. Along with our choice of $\beta$, this sets $\bar{I}^* = \bar{\Pi}/\beta$. For each value of $\rho$, we use these sample averages to calculate $\bar{I}^* - \bar{I}$ and then compute $a$, $\nu^*$, and $A$ using,
\[ a = \frac{\bar{m}/\bar{b}}{[\bar{m}/\bar{b} + \beta (\bar{I}^* - \bar{I}) / (\bar{\Pi} - \beta)]^{1/(1-\rho)}} \]  
(17)

\[ \bar{v} = \left[ a^{(1-\rho)} (\bar{m}/\bar{c})^{\rho} + (1 - a)^{(1-\rho)} (\bar{b}/\bar{c})^{\rho} \right]^{-1/\rho} \]  
(18)

\[ \frac{v^*}{\bar{v}} = \left( \frac{1-\beta/\bar{\Pi}}{(a\bar{m}/\bar{m})^{1-\rho}\bar{v} - \bar{\tau}} \right) \left( \frac{1-\beta/\bar{\Pi}}{(a\bar{m}/\bar{m})^{1-\rho}\bar{v} + \bar{\tau}} \right) \]  
(19)

\[ A = \frac{1 - \beta/\bar{\Pi}}{a (\bar{m}/\bar{m})^{1-\rho} [\bar{v}^2 - (v^*)^2]} \]  
(20)

where \( \bar{\tau} \) is set to 0.1 percent of steady state consumption.

When we vary the elasticity of substitution \( (\xi = \frac{1}{1-\rho}) \), some aspects of the steady state must of course be altered. However, our approach to deriving the steady state – fixing the steady state values of \( \bar{\tau}, \bar{\Pi}, \bar{I}, \bar{b}/\bar{c}, \bar{b}/\bar{m}, \) and \( \bar{g}/\bar{y} \) and allowing the parameters of the transactions technology change when we use different values of \( \rho \) – assures that it is just some parameters in the transactions technology that are affected. The steady state values of consumption, hours worked, output, and wages will all be independent of the choice of \( \rho \).

The final set of parameters are determined using estimated values. Parameters in the stochastic processes for government spending and productivity are estimated using detrended quarterly U.S. data. We set \( \rho_g = 0.95, \) SD(\( \varepsilon^g \)) = 0.01; \( \rho_z = 0.93, \) SD(\( \varepsilon^z \)) = 0.01. The response of taxes to debt is set to 0.018. Bohn’s [2], [3], [4] estimates suggest a response in the range of 0.013 to 0.030.
3 Interest Rate Dynamics and the Interest Rate Gap

Observing the natural rate of interest is presumably more important for policy makers when the natural rate makes large and persistent deviations from its steady state. In this section, we show how the natural rate of interest, and the gap between the policy rate and the natural rate, respond to an increase in government spending or a decrease in productivity. These dynamic responses depend upon both the economic environment and the interest rate rule that is in place.

We will focus on two aspects of the economic environment: (1) The elasticity of substitution between money and bonds. In our benchmark calibration (based on our own estimates), money and bonds are complements ($\xi = 0.75$); but we will also consider the possibility that they are substitutes ($\xi = 1.25$). (2) The fiscal response of taxes to a changes in debt, $\phi$. In our benchmark calibration (based on the work of Bohn), $\phi = 0.018$; but we will also consider more aggressive responses.

3.1 Importance of the Economic Environment

First, we show how the interest rates respond to an increase in government purchases or a decrease in productivity, and how the responses depend on the values of $\xi$ and $\phi$. For this exercise we assume that monetary policy is guided by Rule 1, the basic Taylor rule.

Figure 1 shows impulse response functions for an increase in government purchases. The basic Taylor rule has a constant intercept term; there is no explicit attempt to track the unobserved fluctuations in the natural rate. The top panel shows responses of the real natural rate of interest, and the bottom panel shows responses of the gap between the policy rate and its natural rate. Solid lines are for the Standard Model (in which bonds have no liquidity value); dashed lines are for our benchmark case of complements; and dotted lines are for the case of substitutes.
An increase in government purchases crowds out consumer spending, which requires an increase in the natural rate of interest. In the Standard Model, the increase in the natural rate is moderate, and it dies out relatively quickly. The policy rate rises in response to the inflation generated by the spending increase, but by slightly less than the natural rate. This produces a small gap between the two, but it too closes rather quickly. The policy rate tracks its natural rate pretty well in the Standard Model.

There is a very different response in the Liquid Bonds Model, and that response depends upon whether money and bonds are complements or substitutes. In either case, the initial increase in the natural rate is moderate, not unlike the increase in the Standard Model. But here, the natural rate does not return to its steady state value for a very long time. The gap term shows a similar pattern. And these responses are much more pronounced when money and bonds are complements.

Why is this so? In both the Standard Model and the Liquid Bonds Model, the increase in government spending crowds out consumption and the natural rate of interest has to rise. And in both models, the public debt has to rise, since the increase in spending is initially financed by issuing debt. This is illustrated in the bottom panel of Figure 1A. But the rising debt does not increase liquidity in the Standard Model. It does in the Liquid Bonds Model, and this augments the expansionary effect of the spending increase. When money and bonds are complements, bonds have a higher marginal liquidity value, so the extra expansionary effect is enhanced in this case.

Another important parameter in the Liquid Bonds Model is $\phi$, the response of taxes to an increase in debt. In Figure 2, we assume that money and bonds are complements, and we consider two values of $\phi$. The solid lines in Figure 2 are the same as the dashed lines in Figure 1A. The benchmark value of $\phi$ is 0.018; when we increase $\phi$ by 50%, and then 100%, the persistence in the natural rate’s response to a government spending shock is attenuated. The reason for this is straightforward. As noted above, the increase in spending is initially
deficit financed, increasing the amount of liquid debt; a stronger tax response closes the
deficit more rapidly and limits the new debt provision. The persistence of the effect on the
natural rate and the interest rate gap is diminished. We should note that the benchmark
response – $\phi = 0.018$ – is consistent with the estimates of \[2, 3, 4\]three times the steady
state value of $R$.

Figure 3 shows the interest rate responses to a productivity shock. An increase in pro-
ductivity has sizable effects on the natural rate of interest in both models; however, they are
not nearly as long lasting. A productivity shock, unlike a shock to government purchases,
does not have a direct effect on the provision of government debt. A positive productivity
shock increases the supply of output, and this drives the natural rate down. The effect is
more pronounced in the Liquid Bonds Model, but the differences, while long lasting, are not
large.

*Summing up:* In the Standard Model, government spending shocks cause the natural
rate to rise moderately, and productivity shocks cause the natural rate to fall moderately.
But in each case, movements in the natural rate die out relatively quickly. Moreover, the
conventional Taylor rules track movements in the natural rate rather closely. When bonds
provide liquidity, roughly the same conclusions hold with respect to productivity shocks.
However, bond financed government spending shocks induce sustained movements in the
natural rate because of the additional liquidity; the basic Taylor rule cannot make the
policy rate keep up. In this case, observing and tracking the natural rate would seem to
be most important. Finally, this last result is quite pronounced when money and bonds
are complements, and this is our benchmark calibration, based upon our estimates of the
elasticity of substitution.
3.2 Importance of the Interest Rate Rule

Some policy rules make the policy rate track its natural rate better than others. Rule 1 has a constant intercept term, set at the steady state value of the nominal natural rate of interest; this rule makes no direct attempt to track short run fluctuations in the natural rate. Neither does Rule 3, the first difference rule. But there is no intercept term in the first difference rule, and it may track the natural rate better than the basic Taylor rule. Why? A unit root process may be better at picking up prolonged movements in the natural rate. Rule 4 is the benchmark by which we evaluate the other rules, and we want to see that it actually does track the natural rate.

In this section, we examine the performance of these three rules in the Standard Model and in our Liquid Bonds Model. We begin with the Liquid Bonds Model; for this exercise, we set the elasticity of substitution between money and bonds at its benchmark value ($\xi = 0.75$).

The top panel in Figure 4 shows responses of the interest rate gap to an increase in government spending. As might be expected, the basic Taylor rule (Rule 1) does a very poor job of tracking the natural rate: the interest rate gap is still widening after five years. The first difference rule does a better job of tracking the natural rate after the first few quarters. The first difference rule is not pinned down by a fixed intercept term, and the policy rate is free to move widely over time. And finally, the natural rate rule (Rule 4) does indeed track the natural rate almost perfectly.

The bottom panel in Figure 4 shows responses of the interest rate gap to an increase in productivity. Here, the basic Taylor rule does a much better job than it did with the spending increase, but it is still inferior to the first difference rule after the first few quarters. And once again, the natural rate rule tracks the natural rate almost perfectly.

Figure 5 gives the analogous results for the Standard Model. Since bond financed fiscal

---

7 Rule 2 tracks the natural rate better than does Rule 1, and worse than Rule 3. For simplicity of exposition, we do not include it in this section.
expansions do not directly affect liquidity, the basic Taylor rule fares much better in the case of an increase of government spending; it also does somewhat better in the case of a productivity shock. In each case, the first difference rule does somewhat better than the basic Taylor Rule, but only after the first five quarters. And once again, Rule 4 tracks the natural rate almost perfectly.

Summing up: Rules that quickly bring the policy rate in line with its natural rate are presumably better policies, a normative question we pursue in Section 4. The basic Taylor rule makes no explicit attempt to track the natural rate, and in fact, it does a bad job of it. This is especially true for bond financed spending increases in the Liquid Bonds Model. The first difference rule does a much better job in both models, but only after the first four or five quarters. This may be because a unit root process is better at picking up prolonged movements in the natural rate. Rule 4 assumes that the natural rate of interest is directly observable, and as a consequence it tracks the natural rate almost perfectly; however, it is only a benchmark.

4 Tracking the Natural Rate and Household Utility

As noted in the introduction, the natural rate rule (Rule 4) is a very good rule; it virtually eliminates the costs of nominal rigidities in both models. And, as shown in the last section, the natural rate rule tracks the natural rate almost perfectly. In this section, we use it as a benchmark to assess the importance of being to be able to track the natural rate using rules that are actually implementable.

In the last section, we saw that the basic Taylor rule (Rule 1) did a poor job of tracking the natural rate, especially in the Liquid Bonds Model. The first difference rule (Rule 3) did a better job, but only after four or five quarters. The smoothed Taylor rule (Rule 2) is more inertial than the basic Taylor rule, but the policy rate is still pinned down by a fixed
intercept; this rule falls somewhere in between the other two. Here, we ask how well these rules do when compared to the natural rate rule. How important is it for a central bank to be able to track the natural rate?

The metric we use is the expected discounted value of household utility, conditional on the economy starting in its steady state. We follow the custom of comparing the policy rules in terms of consumption units: what percent of consumption would households be willing to give up each period – assuming that the work effort is held constant – to move from one rule to another.

The top panel of Table 3 presents the basic welfare results. In it, the comparisons are all with respect to the basic Taylor rule, which turns out to be the worst. Households would give up a little over a tenth of a percent of steady state consumption to have interest rate smoothing (Rule 2) in either the Standard Model or the Liquid Bonds Model with complements, our benchmark case; in the substitutes case, the number falls by half.

The first difference rule does much better when compared with the basic Taylor rule: a quarter of a percent of consumption in the Standard Model, and a half a percent in the Liquid Bonds Model with complements. These are large numbers by the standards of the New-Keynesian literature. The fact that the first difference rule does particularly well in the Liquid Bonds Model with complements is perhaps not surprising. In that environment, government spending shocks cause sustained movements in the natural rate that are hard for either of the conventional Taylor rules to track; the first difference rule is not pinned down by a fixed intercept, and the policy rate is free to move widely over time. The gain is not as large in the Liquid Bonds Model with substitutes: about a third of a percent of consumption. Finally, it is interesting to note that the first difference rule performs virtually as well as the natural rate rule, and it requires absolutely no information about the natural rate, not even its steady state value.

The second panel in Table 3 shows that the relative performances of the various rules
depend on the parameter $\phi$, which measures the response of taxes to changes in the level of debt. Here again, we assume that money and bonds are complements, our benchmark case. When we double $\phi$, the welfare gains are smaller. This is not surprising in light of Figure 2. A quicker response of taxes to debt closes the deficit faster and limits the provision of liquid debt. However, the higher values of $\phi$ are at or beyond the upper end of the range of Bohn’s estimates.

*Summing up:* In the Standard Model, households would be willing to give up a quarter of a percent of steady state consumption each period to replace the basic Taylor rule with the natural rate rule. In our benchmark calibration of the Liquid Bonds Model, the household would be willing to give up a half of a percent of consumption to get the natural rate rule. Finally, the first difference rule performs virtually as well as the natural rate rule in all of these cases.

## 5 Accounting for Model Uncertainty

Using our benchmark calibration of the Liquid Bonds Model, we have shown that it is very important for the central bank to be able to track the natural rate of interest. But is the Liquid Bonds Model an accurate representation of the “true” model of the economy? Most central bankers would admit to considerable uncertainty about their models, and indeed, model uncertainty is one of the reasons why it is so difficult for them to track the natural rate in practice.

In this section, we take model uncertainty into account in two separate ways. In the first exercise, we consider a particular example of model misspecification. We assume that the central bank can actually compute the natural rate of interest, but that it uses the wrong model in doing so. Can the first difference rule do better than the (misspecified) natural rate?

---

8This exercise is in the spirit of a literature that looks for policies that do well in a variety of model. See for example McCallum[22] and Levin et.al.[19].
rate rule in this environment? In the second exercise, we consider a much more general characterization of model uncertainty in the spirit of Hansen and Sargent.\(^9\) Can the first difference rule perform well in this environment?

### 5.1 Central Bank Uses the Wrong Model

Suppose the Liquid Bonds Model is the true model, but the central bank does not recognize that government bonds provide liquidity; it believes the Standard Model to be the true model of the economy. Suppose further (and counter to our previous presumptions) that the central bank is able to compute the natural rate of interest using the Standard Model.

The third panel of Table 3 gives the welfare comparisons for this particular misspecification. As might be expected, the natural rate rule does worse than the first difference rule in this environment, but the difference in welfare is very small. Households would be willing to give up two one-hundredths of a percent of steady state consumption (or even less in the substitutes case) to have the central bank use the first difference rule rather than the natural rate rule.

While the welfare differences here are small, they do tell a cautionary tale. If the natural rate is imprecisely estimated, the far simpler first difference rule may be the better part of valor.

### 5.2 Robustness

There can be little doubt that government bonds provide liquidity. But, does that liquidity come precisely in the manner prescribed by our Liquid Bonds Model? We have already admitted to uncertainty about the elasticity of substitution between money and bonds, and we have seen that a reduction in that elasticity significantly increases the natural

\(^9\)This exercise is in the spirit of Hansen and Sargent[11] and Giordani and Soderlind [9].
rate’s persistence in response to government spending shocks. Variations in other model parameters may also affect the natural rate’s dynamics – the tax response to debt, φ, comes readily to mind. Figure 2 showed that increasing φ significantly lowered the natural rate’s persistence in response to a spending shock. More generally, bond liquidity may also matter to firms, affecting marginal costs in addition to marginal utilities; the “true” transactions technology may look very different than the one we have modeled.

The range of possible misspecification is broad and rather ill defined. We will therefore not even attempt to specify a set of competing models with corresponding probabilities. Instead, we will consider model uncertainty in the spirit of Hansen and Sargent[11].

More specifically, agents within the economy believe that the true model lies in a well specified neighborhood around the reference model, which we assume to be the benchmark Liquid Bond Model. Following Giordani and Soderlind[9], we linearize the reference model, and represent it by the law of motion:

\[
\begin{bmatrix}
    x_{1,t+1} \\
    E_t(x_{2,t+1})
\end{bmatrix} = A 
\begin{bmatrix}
    x_{1,t} \\
    x_{2,t}
\end{bmatrix} + BR_t + \begin{bmatrix}
    C \\
    O
\end{bmatrix} (\varepsilon_{t+1} + \zeta_{t+1})
\]

(21)

where \(O\) is a matrix of zeros, \(x_{1,t}\) is a vector of state variables, \(x_{2,t}\) is a vector of forward looking (or jump) variables, \(\varepsilon_t\) is a vector of primitive shocks and \(\zeta_t\) is a vector that generates model misspecifications in a manner that will be described shortly. The linearization of the Liquid Bonds Model is tedious; it is relegated to an appendix.

In the Hansen-Sargent approach to robustness, the potential for model misspecification is represented by additional additive errors, \(\zeta_t\). It is standard in this literature to use the metaphor of an evil agent who chooses these \(\zeta_t\) to maximize the resulting cost to households. Here we want to evaluate our four generic policy rules; so, we follow Giordani and Soderlind’s

\[\text{Later, we will use the software provided by these authors to calculate the economy’s dynamics in the “worst case model” – that is under the worst possible misspecification in the neighborhood of the reference model.}\]
approach.

More precisely, the evil agent chooses $\zeta_{t+1}$ (as a linear function of $x_{1,t}$) to maximize $E_0 \sum_{t=0}^{\infty} \beta^t (\Pi_t - \bar{\Pi})^2$, subject to the law of motion, one of our interest rate rules, and what might be called a neighborhood constraint:

$$E_0 \sum_{t=0}^{\infty} \beta^t \zeta'_{t+1} \zeta_{t+1} \leq \Theta$$

A single parameter, $\Theta$, defines the neighborhood around the reference model that is thought to contain the set of plausible misspecifications, and this is the set of possible misspecifications that the evil agent is allowed to choose from. Since $\zeta_{t+1}$ can be a linear function of the state variables, $x_{1,t}$, this approach allows for parameter uncertainty. $\zeta_{t+1}$ cannot be a function of expectations of the forward looking variables, $x_{2,t}$. In this formulation, only the state variables have shocks, and the evil agent’s actions have to be conflated with these primitive shocks.

Critics of the Hansen-Sargent approach have argued that if $\Theta$ is very big, the agents’ behavior may be determined by rather improbable misspecifications, and this might make them look overly cautious. The trick is to choose a value of $\Theta$ that is big enough to reflect a reasonable degree of caution, but not so big as to make agents look foolish.  

Since there is no set of specified alternative models to provide the context for evaluating the policy rules, we compare the welfare of the representative agent under the worst case model for the alternative policy rules. Welfare results are presented in the lower panel of Table 3, and they show that the first difference rule outperforms the Taylor rules. Letting the benchmark Liquid Bonds Model be the reference model, households would be willing to

---

11 Hansen and Sargent suggest a way to do this. The bigger is $\Theta$, the greater is the probability of statistically distinguishing between the reference model and the worst case model. The $\Theta$ we use here sets the probability of detection at one third, which is somewhat higher than the probabilities recommended by Hansen and Sargent. We are being conservative in our specification of uncertainty, therefore the welfare gains we report below may understate our basic result.
give up three quarters of a percent of steady state consumption to have the central bank use the first difference rule instead of the basic Taylor rule, Rule 1. If instead the substitutes case is chosen as the reference model, the welfare gain falls to two fifths of a percent, which is still a large number.

*Summing up:* When model uncertainty is taken into account, the case for the first difference rule is much bolstered. If the central bank uses the wrong model to estimate the natural rate, then the result can easily be worse than the simple first difference rule that requires no estimation. If the central bank cannot estimate the natural rate precisely, then discretion may be the better part of valor. And when model uncertainty is profound, Hansen and Sargent’s approach to robustness suggests that the first difference rule’s dominance is clear.

### 6 Conclusion

In Section 2, we showed that – in some economic environments and with some standard monetary policy rules – shocks to the economy can make the natural rate of interest deviate substantially from its steady state value for a very long time. In Section 3, we showed that in this kind of environment, household utility improves significantly when the central bank makes the policy rate track its natural rate precisely. This may be very difficult to do in practice. Fortunately, a first difference rule – a rule that does not require any information about the natural rate – performs virtually as well as the natural rate rule. In Section 4, we took model uncertainty into account in several ways, and in each case the dominance of the first difference rule was only enhanced. More detailed results have already been summarized at the ends of the preceding sections.

There is a timely lesson to be learned from our analysis. As of this writing, many OECD countries are undertaking, or contemplating, large cuts in government spending to stabilize
their sovereign debts. If bonds do provide liquidity services, then our results suggest that the natural rate of interest will be on the move and hard to track. The first difference rule seems to be made for just this situation.
Table 1: Elasticity of Substitution between Money and Bonds in Liquidity

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - \rho$</td>
<td>1.305 (0.678)</td>
<td>1.600 (0.542)</td>
<td>1.4589 (0.594)</td>
<td>0.895 (1.234)</td>
<td>1.427 (0.959)</td>
<td>1.103 (1.285)</td>
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<tr>
<td>crisis dummy</td>
<td>0.360 (0.345)</td>
<td></td>
<td>0.527 (0.722)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.503 (0.085)</td>
<td>0.514 (0.081)</td>
<td>0.502 (0.094)</td>
<td>0.641 (0.145)</td>
<td>0.661 (0.137)</td>
<td>0.638 (0.146)</td>
</tr>
<tr>
<td>Method</td>
<td>IV</td>
<td>IV</td>
<td>IV</td>
<td>OLS</td>
<td>OLS</td>
<td>OLS</td>
</tr>
<tr>
<td>$\xi = \frac{1}{1-\rho}$</td>
<td>0.77</td>
<td>0.63</td>
<td>0.69</td>
<td>1.12</td>
<td>0.70</td>
<td>0.91</td>
</tr>
</tbody>
</table>

Data Sources

- $b_t$: Debt held by the public, excluding the Federal Reserve System, at the end of the fiscal year. Source: Budget of the United States Government FY2014, Historical Tables, Table 7.1.

- $m_t$: Adjusted monetary base, end of the fiscal year, Source: Federal Reserve Bank of St. Louis (FRED).

- $I_t^{AAA}$: Moody’s yield on seasoned corporate bonds rated AAA, last month of the fiscal year. Source: Federal Reserve Board Release H.15.

- $I_t^{10}$: Yield on United States Treasury securities at 10 years constant maturity, last month of the fiscal year. Source: Federal Reserve Board Release H.15.

- $\Pi_t$: Gross inflation rate for personal consumption expenditures, fiscal Q4/Q4. Source: Bureau of Economic Analysis, National Income and Product Accounts, Table 1.1.4.
Table 2: Model Calibration

A. Fixed Parameters

<table>
<thead>
<tr>
<th></th>
<th>( \beta )</th>
<th>( \theta )</th>
<th>( \eta )</th>
<th>( \chi )</th>
<th>( \alpha )</th>
<th>( \tau )</th>
<th>( \phi )</th>
<th>( \gamma/y )</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>0.99</td>
<td>0.75</td>
<td>7</td>
<td>1</td>
<td>0.67</td>
<td>0.01</td>
<td>0.018</td>
<td>0.25</td>
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</tbody>
</table>

B. Estimated Parameters

<table>
<thead>
<tr>
<th></th>
<th>( \Pi )</th>
<th>( I )</th>
<th>( b/c )</th>
<th>( m/c )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.007</td>
<td>1.014</td>
<td>2.0</td>
<td>0.3</td>
</tr>
</tbody>
</table>

C. Implied Parameters

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( a )</th>
<th>( v^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi = 0.75 )</td>
<td>0.096</td>
<td>0.307</td>
<td>0.432</td>
</tr>
<tr>
<td>( \xi = 1.0 )</td>
<td>0.090</td>
<td>0.391</td>
<td>0.460</td>
</tr>
<tr>
<td>( \xi = 1.25 )</td>
<td>0.084</td>
<td>0.479</td>
<td>0.490</td>
</tr>
</tbody>
</table>

Data Sources

<table>
<thead>
<tr>
<th></th>
<th>( \Pi )</th>
<th>( I )</th>
<th>( b/c )</th>
<th>( m/c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of debt held by the public, excluding Federal Reserve System to quarterly personal consumption expenditures, average 1980-2007. Source: Budget of the United States Government FY2014, Historical Tables, Table 7.1 and National Income and Product Accounts, Table 1.1.5.</td>
<td></td>
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</tr>
<tr>
<td>Ratio of adjusted monetary base to quarterly personal consumption expenditures, average 1980-2007. Source: Federal Reserve Bank of St. Louis (FRED) and National Income and Product Accounts, Table 1.1.5.</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Table 3: Welfare Comparisons

Liquid Bonds and Standard Models ($\phi = 0.018$), gain over Rule 1

<table>
<thead>
<tr>
<th>Rule</th>
<th>complements</th>
<th>unit elasticity</th>
<th>substitutes</th>
<th>Standard Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule 2: smoothing</td>
<td>0.13</td>
<td>0.08</td>
<td>0.06</td>
<td>0.11</td>
</tr>
<tr>
<td>Rule 3: first difference</td>
<td>0.51</td>
<td>0.37</td>
<td>0.29</td>
<td>0.25</td>
</tr>
<tr>
<td>Rule 4: natural rate</td>
<td>0.51</td>
<td>0.37</td>
<td>0.29</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Liquid Bonds Model (complements), gain over Rule 1

<table>
<thead>
<tr>
<th>Rule</th>
<th>$\phi = 0.018$</th>
<th>$\phi = 0.027$</th>
<th>$\phi = 0.036$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule 2: smoothing</td>
<td>0.13</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>Rule 3: first difference</td>
<td>0.51</td>
<td>0.39</td>
<td>0.33</td>
</tr>
<tr>
<td>Rule 4: natural rate</td>
<td>0.50</td>
<td>0.39</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Wrong Model, gain of first difference rule over natural rate rule

<table>
<thead>
<tr>
<th>Rule</th>
<th>complements</th>
<th>unit elasticity</th>
<th>substitutes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule 3: first difference</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
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</tbody>
</table>

Worst Case Misspecification, gain over Rule 1

<table>
<thead>
<tr>
<th>Rule</th>
<th>complements</th>
<th>unit elasticity</th>
<th>substitutes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule 2: smoothing</td>
<td>0.44</td>
<td>0.28</td>
<td>0.20</td>
</tr>
<tr>
<td>Rule 3: first difference</td>
<td>0.77</td>
<td>0.54</td>
<td>0.41</td>
</tr>
</tbody>
</table>
Figure 1: Government Spending Shock, Rule 1, Importance of Elasticity of Substitution

Standard Model: solid line; Liquid Bonds model, complements: dashed line; substitutes: dotted line
Figure 1A: Government Spending Shock, Rule 1, Importance of Debt

gap = real interest rate − real natural rate

debt to (steady state) gdp ratio

Standard Model: solid line; Liquid Bonds Model, complements: dashed line; substitutes: dotted line
Figure 2: Government Spending Shock, Rule 1, Importance of Fiscal Response

Liquid Bonds Model, money and bonds are compliments.

\[ \text{phi} = 0.018 \text{ (benchmark case): solid line; phi} = 0.027 \text{: dashed line; phi} = 0.036 \text{: dotted line} \]
Figure 3: Productivity Shock, Rule 1, importance of Elasticity of Substitution

Standard Model: solid line; Liquid Bonds model, complements: dashed line; substitutes: dotted line
Figure 4: Liquid Bonds Model, Complements, Importance of Policy Rules

gap = real interest rate − real natural interest rate, government spending shock

gap = real interest rate − real natural interest rate, productivity shock

Rule 1: solid line; Rule 3: dashed line; Rule 4: dotted line
Figure 5: Standard Model, Importance of Policy Rules

gap = real interest rate − real natural interest rate, Government Spending Shock

Rule 1: solid line; Rule 3: dashed line; Rule 4: dotted line

Figure 5: Standard Model, Importance of Policy Rules

gap = real interest rate − real natural interest rate, Productivity Shock

Rule 1: solid line; Rule 3: dashed line; Rule 4: dotted line
References


7 Appendix: linearization & model dynamics

In this section we linearize the model and write it in the form

\[
\begin{bmatrix}
  x_{1,t+1} \\
  E_t(x_{2,t+1})
\end{bmatrix} = A \begin{bmatrix}
  x_{1,t} \\
  x_{2,t}
\end{bmatrix} + BI_t + \begin{bmatrix}
  C \\
  O
\end{bmatrix} \varepsilon_t
\]

where \( x_{1,t} \) is a vector of predetermined variables and \( x_{2,t} \) is a vector of forward looking (jump) variables, and \( \varepsilon_t \) is a vector of exogenous shocks.

7.1 Government

The government’s flow budget constraint can be written as follows

\[
b_t + m_t = \Pi_t^{-1}(I_{t-1}b_{t-1} + m_{t-1}) + g_t - t_t \tag{22}
\]

We let \( d_t = b_t + m_t \) and write (22) as

\[
d_{t+1} = \Pi_{t+1}^{-1} [I_t d_t - (I_t - 1)m_t] + g_{t+1} - t_{t+1}
\]

to get the log-linear version

\[
\hat{d}_{t+1} = -\left(\frac{I d - (I - 1)m}{\Pi d}\right) \hat{\Pi}_{t+1} + \left(\frac{I}{\Pi}\right) \hat{d}_t - \left[\frac{(I - 1) m}{\Pi d}\right] \hat{m}_t + I \left(\frac{d - m}{\Pi d}\right) \hat{I}_t + \left(\frac{g}{\hat{\alpha}}\right) \hat{g}_{t+1} - \left(\frac{t}{\hat{\alpha}}\right) \hat{t}_{t+1}
\]

We then define the predetermined state variable

\[
q_t = \hat{d}_t + \left(\frac{I d - (I - 1)m}{\Pi d}\right) \hat{\Pi}_t = \hat{d}_t + \delta_0 \hat{\Pi}_t \tag{23}
\]
and write the government budget constraint as

\[
q_{t+1} = \left( \frac{I}{\Pi} \right) \left[ q_t - \delta_0 \hat{\Pi}_t \right] - \left[ \frac{(I - 1) m}{\Pi d} \right] \hat{m}_t \\
+ I \left( \frac{d - m}{\Pi d} \right) \hat{I}_t + \left( \frac{g}{d} \right) \hat{g}_{t+1} - \left( \frac{t}{d} \right) \hat{t}_{t+1}
\]

Later, we will replace \( \hat{g}_{t+1} \) and \( \hat{t}_{t+1} \) by their transition functions, and we will eliminate \( \hat{m}_t \). Although \( \hat{d}_t \) will not appear in the final representation we will use, we keep it as a variable for now, to be eliminated using (23) later.

### 7.2 Consumers

The log-linearized versions of expressions (??)-(6) are

\[
\tilde{\tau}_t = \gamma_\tau \hat{\tau}_t \\
\tilde{\vartheta}_t = \gamma_\vartheta \hat{\vartheta}_t
\]

(24)

(25)

where

\[
\gamma_\tau \equiv \frac{v + v^*}{v - v^*}
\]

and

\[
\gamma_\vartheta \equiv \frac{v}{v - v^*}
\]

The log-linear version of the CES aggregator

\[
\tilde{m}_t^\rho = a^{1-\rho} m_t^\rho + (1 - a)^{1-\rho} (d_t - m_t)^\rho
\]
\[ \hat{m}_t = \left( \hat{m} \right)^{-\rho} \left\{ a^{1-\rho} m^\rho - (1-a)^{1-\rho} m (d-m)^{\rho-1} \right\} \hat{m}_t + \left[ (1-a)^{1-\rho} d (d-m)^{\rho-1} \right] \hat{d}_t \]

which gives

\[ \hat{v}_t = \hat{c}_t - \gamma_m \hat{m}_t - \gamma_d \hat{d}_t \]  

(26)

with

\[ \gamma_m \equiv \left( \frac{v}{c} \right)^\rho \left[ a^{1-\rho} m^\rho - (1-a)^{1-\rho} m (d-m)^{\rho-1} \right] \]

and

\[ \gamma_d \equiv \left( \frac{v}{c} \right)^\rho \left[ (1-a)^{1-\rho} d (d-m)^{\rho-1} \right] \]

Log-linearizing the households’ optimality condition (10)

\[ \left\{ 1 - A [\hat{v}_t^2 - (v^*)^2] \left( \frac{(1-a) \hat{m}_t}{b_t} \right)^{1-\rho} \right\} = \frac{I_t}{I^*_t} \]

we get

\[ \frac{I}{I^*} (\hat{I}_t - \hat{I}_t) = 2A v^2 \left( \frac{(1-a) \hat{m}}{d-m} \right)^{1-\rho} \hat{v}_t 
\]

\[ (1-\rho) \left( \frac{(1-a) \hat{m}}{d-m} \right)^{1-\rho} A [\hat{v}^2 - (v^*)^2] \left\{ \frac{1}{\hat{m}} \left( \gamma_m \hat{m}_t + \gamma_d \hat{d}_t \right) \right\} 
\]

\[ \hat{I}_t - \hat{I}_t = \delta_1 \hat{v}_t + \delta_2 \left[ \gamma_m + \left( \frac{m}{d-m} \right) \right] \hat{m}_t + \delta_2 \left[ \gamma_d - \left( \frac{d}{d-m} \right) \right] \hat{d}_t \]  

(27)

where

\[ \delta_1 = 2A \left( \frac{I^*}{I} \right) (v)^2 \left[ \frac{(1-a) \hat{m}}{d-m} \right]^{1-\rho} \]
and
\[ \delta_2 = A \left( \frac{I^*}{I} \right) (1 - \rho) [v^2 - (v^*)^2] \left[ \frac{(1 - a) \tilde{m}}{d - m} \right]^{1 - \rho} \]

Log-linearizing (9),
\[ 1 - (I^*_t)^{-1} = A [v^*_t - (v^*)^2] \left( \frac{a \tilde{m}_t}{m_t} \right)^{1 - \rho} \]
we get
\[ \hat{I}^*_t = I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1 - \rho} \left\{ \delta_1 \hat{v}_t + \delta_2 \left[ (\gamma_m - 1) \hat{m}_t + \gamma_d \hat{d}_t \right] \right\} \] (28)

The rest of the consumer block is:
\[ \hat{I}^*_t - E_t \hat{\Pi}_{t+1} = \sigma (E_t \hat{c}_{t+1} - \hat{c}_t) + \frac{2Av}{1 + \vartheta} E_t (\hat{v}_{t+1} - \hat{v}_t) \]
\[ \hat{w}_t = \chi \hat{n}_t + \sigma \hat{c}_t + \frac{2Av}{1 + \vartheta} \hat{v}_t \] (30)

7.3 Firms and the supply side

The supply side equations associated with Calvo pricing need not be reproduced here. Their log-linearization yields
\[ \hat{\Pi}_t = \beta E_t \left( \hat{\Pi}_{t+1} \right) + \lambda_p \hat{m}_t c_t \] (31)
\[ \hat{m}_t c_t = \hat{w}_t - \hat{y}_t + \hat{n}_t \] (32)
\[ \hat{y}_t = \alpha \hat{n}_t + \hat{z}_t \] (33)
\[ \hat{y}_t = \tau \gamma_c \hat{c}_t + (1 + \tau) \gamma_c \hat{c}_t + \gamma_g \hat{g}_t \] (34)

where \( \gamma_c = \frac{c}{y}, \gamma_g = \frac{g}{y} \), and the slope coefficient, \( \lambda_p \equiv \frac{(1 - \theta) (1 - \beta \theta)}{\theta [1 + \eta (\frac{1 - \alpha}{\alpha})]} \)
7.4 System Reduction

We define
\[ x_t = \sigma \hat{c}_t + \left( \frac{2A v}{1 + \vartheta} \right) \hat{v}_t \]
and write (29) as
\[ \hat{I}_t = E_t \hat{\Pi}_{t+1} + E_t x_{t+1} - x_t \] (35)
Substituting this in (28), we get
\[ E_t \hat{\Pi}_{t+1} + E_t x_{t+1} = x_t + I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1-\rho} \left\{ \delta_1 \left( \frac{1 + \vartheta}{2A v} \right) \left( x_t - \sigma \hat{c}_t \right) + \delta_2 \left[ (\gamma_m - 1) \hat{m}_t + \gamma_d \hat{d}_t \right] \right\} \]
We then use (26), to get
\[ \hat{m}_t = \frac{1}{\gamma_m} \left\{ 1 + \sigma \left( \frac{1 + \vartheta}{2A v} \right) \hat{c}_t - \left( \frac{1 + \vartheta}{2A v} \right) x_t - \gamma_d \hat{d}_t \right\} \] (36)
and use this to eliminate \( \hat{m}_t \) and get
\[ E_t \hat{\Pi}_{t+1} + E_t x_{t+1} = \delta_3 x_t + \delta_4 \hat{d}_t + \delta_5 \hat{c}_t \] (37)
with
\[ \delta_3 = 1 + I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1-\rho} \left( \frac{1 + \vartheta}{2A v} \right) \left\{ \delta_1 - \left[ \frac{\delta_2 (\gamma_m - 1)}{\gamma_m} \right] \right\} \]
\[ \delta_4 = I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1-\rho} \left( \frac{\gamma_d \delta_2}{\gamma_m} \right) \]
\[ \delta_5 = I \left[ \frac{(d - m) a}{(1 - a) m} \right]^{1-\rho} \left\{ \frac{\delta_2 (\gamma_m - 1)}{\gamma_m} \left[ \frac{\sigma (1 + \vartheta)}{2A v} + 1 \right] - \sigma \delta_1 \left( \frac{1 + \vartheta}{2A v} \right) \right\} \]
Next, we make the same substitutions in (27), to get

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = x_t + \tilde{I}_t + \delta_1 \left( \frac{1 + \vartheta}{2Av} \right) (x_t - \sigma \tilde{c}_t) + \delta_2 \left[ \gamma_d - \left( \frac{d}{d - m} \right) \right] \tilde{d}_t + \delta_2 \frac{\gamma_m + \left( \frac{m}{d - m} \right) }{\gamma_m} \left\{ \left[ 1 + \sigma \left( \frac{1 + \vartheta}{2Av} \right) \right] \tilde{c}_t - \left( \frac{1 + \vartheta}{2Av} \right) x_t - \gamma_d \tilde{d}_t \right\}
\]

which simplifies to

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = \tilde{I}_t + \delta_6 x_t + \delta_7 \tilde{d}_t + \delta_8 \tilde{c}_t
\]

(38)

with

\[
\begin{align*}
\delta_6 &= 1 + \left( \frac{1 + \vartheta}{2Av} \right) \left\{ \delta_1 - \frac{\delta_2}{2} \left[ \gamma_m + \left( \frac{m}{d - m} \right) \right] \right\} \\
\delta_7 &= \delta_2 \left[ \gamma_d - \left( \frac{d}{d - m} \right) \right] - \delta_2 \gamma_d \left[ \gamma_m + \left( \frac{m}{d - m} \right) \right] \\
\delta_8 &= \frac{\delta_2}{2} \left[ \gamma_m + \left( \frac{m}{d - m} \right) \right] \left[ \sigma \left( \frac{1 + \vartheta}{2Av} \right) \right] + 1 - \sigma \delta_1 \left( \frac{1 + \vartheta}{2Av} \right)
\end{align*}
\]

Then, we use (37) to eliminate \( \tilde{c}_t \) from (38) and get

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = \tilde{I}_t + \delta_6 x_t + \delta_7 \tilde{d}_t + \frac{\delta_8}{\delta_5} \left[ E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} - \delta_3 x_t - \delta_4 \tilde{d}_t \right]
\]

which, using (23), gives

\[
E_t \tilde{\Pi}_{t+1} + E_t x_{t+1} = \left( \frac{\delta_5}{\delta_5 - \delta_8} \right) \left[ \tilde{I}_t + \delta_9 x_t + \delta_{10} \tilde{q}_t - \delta_9 \delta_0 \tilde{\Pi}_t \right]
\]

(39)

with

\[
\begin{align*}
\delta_9 &= \delta_6 - \delta_3 \left( \frac{\delta_8}{\delta_5} \right) \\
\delta_{10} &= \delta_7 - \delta_4 \left( \frac{\delta_8}{\delta_5} \right)
\end{align*}
\]
Next, we write the Phillips curve as

\[ \hat{\Pi}_t = \beta E_t \hat{\Pi}_{t+1} + \lambda_p \hat{mc}_t \]

We use (37) and (38) to calculate

\[ \hat{c}_t = \left( \frac{1}{\delta_5 - \delta_8} \right) \left\{ \hat{I}_t + (\delta_6 - \delta_3) x_t + (\delta_7 - \delta_4) \left[ q_t - \delta_0 \hat{\Pi}_t \right] \right\} \]

(40)

and we put this and

\[ \hat{v}_t = \left( \frac{1 + \vartheta}{2A_v} \right) (x_t - \sigma \hat{c}_t) \]

in the market clearing condition,

\[ \hat{y}_t = \tau \gamma_c \gamma_r \hat{v}_t + (1 + \tau) \gamma_c \hat{c}_t + \gamma_g \hat{g}_t , \]

to get

\[ \hat{y}_t = \tau \gamma_c \gamma_r \left( \frac{1 + \vartheta}{2A_v} \right) x_t + \left[ (1 + \tau) \gamma_c - \tau \gamma_c \gamma_r \sigma \left( \frac{1 + \vartheta}{2A_v} \right) \right] \left( \frac{1}{\delta_5 - \delta_8} \right) \left\{ \hat{I}_t + (\delta_6 - \delta_3) x_t + (\delta_7 - \delta_4) \left[ q_t - \delta_0 \hat{\Pi}_t \right] \right\} + \gamma_g \hat{g}_t \]

which gives

\[ \hat{y}_t = \gamma_g \hat{g}_t + \delta_{11} \hat{I}_t + \left[ \tau \gamma_c \gamma_r \left( \frac{1 + \vartheta}{2A_v} \right) + \delta_{11} (\delta_6 - \delta_3) \right] x_t + \delta_{11} (\delta_7 - \delta_4) \left( q_t - \delta_0 \hat{\Pi}_t \right) \]

with

\[ \delta_{11} = \left[ (1 + \tau) \gamma_c - \tau \gamma_c \gamma_r \sigma \left( \frac{1 + \vartheta}{2A_v} \right) \right] \left( \frac{1}{\delta_5 - \delta_8} \right) \]
We can write the marginal-cost deviation in terms of the output gap as usual

\[
\hat{mc}_t = \hat{w}_t - \hat{y}_t + \frac{1}{\alpha} (\hat{y}_t - \hat{a}_t) = \chi \hat{\eta}_t + \sigma \hat{c}_t + \frac{2A\nu}{1 + \vartheta} \hat{v}_t - \hat{y}_t + \frac{1}{\alpha} (\hat{y}_t - \hat{z}_t)
\]

\[
= \left( \frac{1 + \chi - \alpha}{\alpha} \right) \hat{y}_t - \left( \frac{1 + \chi}{\alpha} \right) \hat{z}_t + x_t
\]

and use the solution for the output gap to get

\[
\hat{mc}_t = \left( \frac{1 + \chi - \alpha}{\alpha} \right) \left( \gamma g \hat{g}_t + \delta_{11} \hat{I}_t \right) + \left( \frac{1 + \chi - \alpha}{\alpha} \right) \delta_{11} (\delta_7 - \delta_4) \left( q_t - \delta_0 \hat{\Pi}_t \right) + \left\{ 1 + \left( \frac{1 + \chi - \alpha}{\alpha} \right) \left[ \tau \gamma c \gamma_t \left( \frac{1 + \vartheta}{2A\nu} \right) + \delta_{11} (\delta_6 - \delta_3) \right] \right\} \left\{ x_t - \left( \frac{1 + \chi}{\alpha} \right) \hat{z}_t \right\}
\]

This leads to the Phillips curve

\[
\beta E_t \hat{\Pi}_{t+1} = \delta_{12} \hat{\Pi}_t - \lambda_p \left( \frac{1 + \chi - \alpha}{\alpha} \right) \left( \gamma g \hat{g}_t + \delta_{11} \hat{I}_t \right) - \lambda_p \left[ \delta_{13} x_t + \delta_{14} q_t - \left( \frac{1 + \chi}{\alpha} \right) \hat{z}_t \right]
\]

with

\[
\delta_{12} = 1 + \lambda_p \delta_0 \delta_{11} \left( \frac{1 + \chi - \alpha}{\alpha} \right) (\delta_7 - \delta_4)
\]

\[
\delta_{13} = 1 + \left( \frac{1 + \chi - \alpha}{\alpha} \right) \left[ \tau \gamma c \gamma_t \left( \frac{1 + \vartheta}{2A\nu} \right) + \delta_{11} (\delta_6 - \delta_3) \right]
\]

and

\[
\delta_{14} = \left( \frac{1 + \chi - \alpha}{\alpha} \right) \delta_{11} (\delta_7 - \delta_4)
\]

Next, we eliminate \( \hat{m}_t \) from the government budget constraint,

\[
q_{t+1} - \left( \frac{g}{d} \right) \hat{g}_{t+1} + \left( \frac{t}{d} \right) \hat{I}_{t+1} = \left( \frac{I}{\Pi} \right) \left[ q_t - \delta_0 \hat{\Pi}_t \right] + \left[ \frac{(I - 1) \, m}{\Pi d} \right] \hat{m}_t + I \left( \frac{d - m}{\Pi d} \right) \hat{I}_t
\]
Substituting (40) in (36), we have

\[
\hat{m}_t = \frac{1}{\gamma_m} \left\{ \delta_{15} \left[ \hat{I}_t + \left( \delta_6 - \delta_3 \right) x_t + (\delta_7 - \delta_4) \left( q_t - \delta_0 \hat{\Pi}_t \right) \right] - \left( \frac{1 + \vartheta}{2Av} \right) x_t - \gamma_d \left( q_t - \delta_0 \hat{\Pi}_t \right) \right\}
\]

where

\[
\delta_{15} = \left[ 1 + \sigma \left( \frac{1 + \vartheta}{2Av} \right) \right] \left( \frac{1}{\delta_5 - \delta_8} \right)
\]

which simplifies to

\[
\hat{m}_t = \left( \frac{\delta_{15}}{\gamma_m} \right) \hat{I}_t + \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) \left( \delta_6 - \delta_3 \right) - \left( \frac{1 + \vartheta}{2Av\gamma_m} \right) \right] x_t
\]

\[
+ \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) (\delta_7 - \delta_4) - \left( \frac{\gamma_d}{\gamma_m} \right) \right] q_t - \left( \frac{\delta_0}{\gamma_m} \right) \left[ \delta_{15} (\delta_7 - \delta_4) - \gamma_d \right] \hat{\Pi}_t
\]

We specify the tax rule

\[
\begin{pmatrix} t \\ d \end{pmatrix} \hat{t}_{t+1} = \phi_d \left( \tilde{\alpha}_t \right) + u_{t+1} = \phi_d \left( q_t - \delta_0 \hat{\Pi}_t \right) + u_{t+1}
\]

where \( u \) is generated by an AR(1) process, and we write the budget constraint as

\[
q_{t+1} - \left( \frac{g}{d} \right) \hat{g}_{t+1} + u_{t+1} = \left[ \frac{I}{\Pi} - \frac{Im}{\Pi d} \left( 1 + \frac{\delta_{15}}{\gamma_m} \right) + \left( \frac{m\delta_{15}}{\Pi d\gamma_m} \right) \right] \tilde{I}_t + \delta_{16} \hat{\Pi}_t - \delta_{17} x_t + \delta_{18} q_t
\]

with

\[
\delta_{16} = \left[ \frac{(I - 1) m}{\Pi d} \right] \left( \frac{\delta_0}{\gamma_m} \right) \left[ \delta_{15} \left( \delta_7 - \delta_4 \right) - \gamma_d \right] - \delta_0 \left( \frac{I}{\Pi} - \phi_d \right)
\]

\[
\delta_{17} = \left[ \frac{(I - 1) m}{\Pi d} \right] \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) \left( \delta_6 - \delta_3 \right) - \left( \frac{1 + \vartheta}{2Av\gamma_m} \right) \right]
\]

and

\[
\delta_{18} = \left( \frac{I}{\Pi} - \phi_d \right) - \left[ \frac{(I - 1) m}{\Pi d} \right] \left[ \left( \frac{\delta_{15}}{\gamma_m} \right) \left( \delta_7 - \delta_4 \right) - \left( \frac{\gamma_d}{\gamma_m} \right) \right]
\]

The first two equations of the dynamic system are then the exogenous processes for
productivity and government purchases, \( \hat{z}_t = \rho \hat{z}_{t-1} + \varepsilon^z_t \) and \( \hat{g}_t = \rho \hat{g}_{t-1} + \varepsilon^g_t \). When we include a tax shock, the third equation is \( u_t = \rho u_{t-1} + \varepsilon^u_t \). The fourth equation is,

\[
q_{t+1} - \left( \frac{q}{d} \right) \hat{g}_{t+1} + u_{t+1} = \left[ \frac{I}{\Pi} - \frac{Im}{\Pi d} \left( 1 + \frac{\delta_{15}}{\gamma_m} \right) + \left( \frac{m\delta_{15}}{\Pi d \gamma_m} \right) \right] \hat{I}_t + \delta_{16} \hat{\Pi}_t - \delta_{17} x_t + \delta_{18} q_t
\]

and the remaining two equations are,

\[
E_t \hat{\Pi}_{t+1} + E_t x_{t+1} = \left( \frac{\delta_5}{\delta_5 - \delta_8} \right) \left[ \hat{I}_t + \delta_9 x_t + \delta_{10} q_t - \delta_{10} \delta_0 \hat{\Pi}_t \right]
\]

and

\[
\beta E_t \hat{\Pi}_{t+1} = \delta_{12} \hat{\Pi}_t - \lambda_p \left( \frac{1 + \chi - \alpha}{\alpha} \right) (\gamma g \hat{g}_t + \delta_{11} \hat{I}_t)
- \lambda_p \left[ \delta_{13} x_t + \delta_{14} q_t - \left( \frac{1 + \chi}{\alpha} \right) \hat{z}_t \right]
\]