# Matching with Multiple Applications: The Limiting Case[*]

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**Abstract**

We give an expression for the expected number of matches between unemployed workers and vacancies when each worker makes a = 2 applications, correcting Albrecht, Gautier, and Vroman (2003). We also show that the limiting matching probability given in our earlier note is correct for any finite a.
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## 1 Introduction

In Albrecht, Gautier, and Vroman (2003), we proposed a generalization of the urn-ball matching function that allowed for multiple applications. Specifically, we considered a situation with $u$ unemployed workers and $v$ vacancies. Each unemployed worker submits $a$ applications, where $a \in \{1, 2, ..., v\}$ is a fixed number. A worker's applications are randomly distributed across the $v$ vacancies with the proviso that any particular worker

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sends at most one application to any particular vacancy. Once the applications are made, each vacancy (of those that received at least one application) chooses one application at random and offers that applicant a job. A worker may get more than one offer. In that case, the worker accepts one of the offers at random.

Let $M(u, v; a)$ be the expected number of matches, i.e., the expected number of accepted offers. We presented an expression for $M(u, v; a)$ for finite $u$ and $v$. We also allowed $u, v \to \infty$ with $v/u = \theta$ fixed and found an expression for the expected number of matches per unemployed worker, i.e., the probability that an unemployed worker finds a job. As pointed out by Tan (2003), our matching function for $a \in \{2, ..., v-1\}$, $u$ and $v$ finite, was incorrect. The matching function for $a = 1$ and $a = v$ was correct. In this note, we give a correct version of the matching function for the case of $a = 2$.[1] The formula for $M(u, v; 2)$ is complicated, but we use it to prove that the expression we gave for the corresponding limiting probability in our earlier note remains correct. We extend our limiting argument from the case of $a = 2$ to any finite, fixed number of applications per worker.

The problem in the finite case can be understood when $a = 2$. Our (incorrect) approach was to reason as follows. Consider any vacancy to which an unemployed worker applies. The number of competitors the worker has at this vacancy is $bin(u-1, \frac{2}{v})$. We can use this fact to compute the probability that the worker fails to receive an offer at this vacancy. Similarly, the number of competitors at the other vacancy to which this worker applies is $bin(u-1, \frac{2}{v})$. Again, we can compute the probability that the worker fails to receive an offer from this vacancy. The probability that a worker receives at least one offer equals 1 minus the probability he or she receives no offers. Our mistake was to assume (implicitly) that the probability a worker receives no offers equals the probability that his first application doesn't generate an offer times the probability that his second application doesn't generate an offer. The problem is that the indicator random variables, "first application leads to an offer" and "second application leads to an offer" are not independent. Equivalently, the numbers of competitors that a worker has at the 2 vacancies are not independent. This is obvious (in retrospect). If, for example, $u = v = 3$ and $a = 2$, then the fact that a worker's first application fails to generate an offer implies that at least one of the other workers also applied to that vacancy, which in turn implies that the chance of the worker's second application being successful increases. This description of where we went wrong suggests why we are correct in the limit. If $u$ and $v$

[1] Tan (2003) gives another expression for the matching function for finite $u$ and $v$.

are large, then the fact that the first application is unsuccessful implies next to nothing about the probability that the second application is successful. Equivalently, if $u$ and $v$ are large, then the numbers of competitors that a worker has at the 2 vacancies to which he or she applies are approximately independent random variables.

In the next section, we derive the correct matching function for the case of $a = 2$. The limiting case for $a = 2$ is given in the following section. The last section provides the general limiting result.

## 2 The Matching Function with a = 2

Let $S$ be the number of competitors a worker faces at the first vacancy to which he or she applies, and let $T$ be the number of competitors at the second vacancy. $S$ and $T$ are each $bin(u-1, \frac{2}{v})$, but these 2 random variables are not independent. We want an expression for $P[S = s, T = t]$ where $0 \leq s, t \leq u - 1$. Once we have this, we can compute the expected number of matches as a function of $u$ and $v$ when $a = 2$ as

$$M(u, v; 2) = u \left( 1 - \sum_{s=0}^{u-1} \sum_{t=0}^{u-1} (\frac{s}{s+1})(\frac{t}{t+1}) P[S = s, T = t] \right),$$

that is, as the number of unemployed times one minus the probability that an individual unemployed worker gets an offer at neither of the vacancies to which he or she applies.

Let $A(s, t)$ be the number of ways in which $u-1$ competitors can send $s$ applications to the first and $t$ applications to the second vacancy given that there are $v$ vacancies ($v \geq 4$). Then

$$P[S = s, T = t] = \frac{A(s, t)}{\binom{v}{2}^{u-1}}.$$

Let $i$ be the number of competitors who applied to both vacancies, where $i \leq \min[s, t]$. This means $(s - i)$ competitors applied only to the first vacancy and $(t - i)$ competitors applied only to the second vacancy. We then have

$$A(s, t) = \sum_{i=\max[0, s+t-(u-1)]}^{\min[s,t]} \binom{u-1}{i} \binom{u-1-i}{s-i} (v-2)^{s-i} \binom{u-1-i-(s-i)}{t-i} (v-2)^{t-i} \binom{v-2}{2}^{u-1-i-(s-i)-(t-i)}.$$

To understand this expression, consider a particular value of $i$. There are $\binom{u-1}{i}$ ways that the $i$ competitors who applied to both vacancies can be chosen from the $u-1$ unemployed. This leaves $u-1-i$ unemployed. There are $\binom{u-1-i}{s-i}$ ways that the $s-i$ competitors who applied only to the first vacancy can be chosen out of the remaining $u-1-i$ unemployed, and there are $(v-2)^{s-i}$ ways that these $s-i$ competitors can spread their other application across the remaining $v-2$ vacancies. Now there remain $u-1-i-(s-i)$ unemployed. There are $\binom{u-1-i-(s-i)}{t-i}$ ways that the $t-i$ competitors who applied only to the second vacancy can be chosen from this group, and these $t-i$ competitors can spread their other application across the remaining vacancies in $(v-2)^{t-i}$ ways. Finally, there are $u-1-i-(s-i)-(t-i)$ workers who applied to neither of the 2 vacancies. There are $\binom{v-2}{2}^{u-1-i-(s-i)-(t-i)}$ ways that their applications can be spread across the other $v-2$ vacancies. To count all the possible ways in which $S=s$ and $T=t$, we now need to sum over all possible $i$. The lower bound for the possible values of $i$ reflects the fact that if $u$ is small relative to $s$ and/or $t$, small values of $i$ may not be possible. For example, if $u=4$, $s=3$ and $t=3$, only $i=3$ is possible. [2]

Since

$$\binom{u-1}{i}\binom{u-1-i}{s-i}\binom{u-1-i-(s-i)}{t-i} = \frac{(u-1)!}{i!(s-i)!(t-i)!(u-1+i-s-t)!}$$

we have

$$A(s,t) = \sum_{i=\max[0,s+t-(u-1)]}^{\min[s,t]} \frac{(v-2)^{u-1-i}\,(v-3)^{u-1-s-t+i}\,2^{-u+1+s+t-i}\,(u-1)!}{i!(s-i)!(t-i)!(u-1+i-s-t)!}$$

Substituting and simplifying yields

$$P[S=s,T=t] = 2^{s+t}\left(1-\tfrac{2}{v-1}\right)^{u-1}\left(1-\tfrac{2}{v}\right)^{u-1} \sum_{i=\max[0,s+t-(u-1)]}^{\min[s,t]} \frac{2^{-i}(v-2)^{-i}(v-3)^{i}(u-1)!}{(v-3)^{s+t}i!(s-i)!(t-i)!(u-1+i-s-t)!}$$

and the corresponding matching function

$M(u,v;2) =$

[2] Our expression for $A(s,t)$ does not apply when $v=3$. The reason is that every unemployed must be a competitor for at least one vacancy. This means that we need to set $\binom{v-2}{2} = 1$ and to account for the fact that for some values of $s$, $t$ and $i$, $P[S=s,T=t]=0$. This latter is taken care of via the indicator function in the following expression for $A(s,t)$ :

$$\sum_{i=\max[0,s+t-(u-1)]}^{\min[s,t]} I[i=s+t-(u-1)]\binom{u-1}{i}\binom{u-1-i}{s-i}(v-2)^{s-i}\binom{u-1-i-(s-i)}{t-i}(v-2)^{t-i}.$$

$$u(1-\sum_{s=0}^{u-1}\sum_{t=0}^{u-1}(\tfrac{s}{s+1})(\tfrac{t}{t+1})2^{s+t}(1-\tfrac{2}{v-1})^{u-1}\left(1-\tfrac{2}{v}\right)^{u-1}\sum_{i=\max[0,s+t-(u-1)]}^{\min[s,t]}\frac{2^{-i}(v-2)^{-i}(v-3)^{i}(u-1)!}{(v-3)^{s+t}i!(s-i)!(t-i)!(u-1+i-s-t)!})$$

## 3 The Matching Function in the Limit ($a = 2$)

The formula we derived for $M(u, v; 2)$, although complicated, reduces to a simple expression in the limit. The key is that in the limit, $S$ and $T$ are independent, so that $P[S = s, T = t] = P[S = s]P[T = t]$.

Since the marginals for $S$ and $T$ are each $bin(u-1, \frac{2}{v})$, we have (using the standard result on the Poisson as the limit of a binomial) that

$$\lim_{u,v\to\infty,\frac{v}{u}\to\theta} P[S = s] = 2^s\theta^{-s}e^{-2/\theta}\frac{1}{s!} \equiv h(s) \text{ for } s = 0, 1, ...$$

$$\lim_{u,v\to\infty,\frac{v}{u}\to\theta} P[T = t] = 2^t\theta^{-t}e^{-2/\theta}\frac{1}{t!} \equiv h(t) \text{ for } t = 0, 1, ...$$

We thus need to show

$$\lim_{u,v\to\infty,v/u=\theta} P\left[S = s, T = t\right] = \frac{2^{s+t}e^{-4/\theta}\theta^{-(s+t)}}{s!t!} \text{ for } s = 0, 1, .. \text{ and } t = 0, 1, ....$$

The first step in doing this is to show that in the limit, $P[i > 0] = 0$; that is, the probability that any competitor applies to both of the vacancies to which an individual has applied is zero. When this is true, only the $i = 0$ term survives in the summation in our expression for $P[S = s, T = t]$. Note that as $u \to \infty$, $\max[0, s + t - (u - 1)] = 0$ for each fixed $s$ and $t$.

To show that in the limit $P[i > 0] = 0$, we first note that for large $v$, the number of applications that a competitor sends to the vacancies to which the individual has applied is approximately $bin(2, \frac{2}{v})$.[3] Then, the probability that a competitor applies to neither or just one of these vacancies, i.e., not to both, is approximately $(1-\frac{2}{v})^2 + \frac{4}{v}(1-\frac{2}{v})$, and the probability that no competitor applies to both vacancies is approximately $\left((1-\frac{2}{v})^2 + \frac{4}{v}(1-\frac{2}{v})\right)^{u-1}$. Thus,

$$P[i > 0] = 1 - \left((1-\frac{2}{v})^2 + \frac{4}{v}(1-\frac{2}{v})\right)^{u-1}.$$

[3] The number of applications that a competitor sends to the vacancies to which an individual has applied is a hypergeometric random variable, so we are using the binomial distribution to approximate the hypergeometric. That is, as $v \to \infty$, we are (legitimately) ignoring the proviso that the worker sends at most one application to any one vacancy.

Setting $u - 1 \approx v\theta$, taking the limit as $v \to \infty$, and applying L'Hôpital's rule gives the result that $\lim\limits_{u,v\to\infty,v/u=\theta} P[i > 0] = 0$.

Given that we need only consider the $i = 0$ term in the summation in the expression for $P[S = s, T = t]$,, we have

$$P[S = s, T = t] = 2^{s+t}\left(1 - \frac{2}{v-1}\right)^{u-1}\left(1 - \frac{2}{v}\right)^{u-1}\frac{(u-1)!}{(v-3)^{s+t}\, s!t!(u-1-s-t)!}.$$

Finally, note that

$$\lim_{u,v\to\infty,\frac{v}{u}\to\theta}\left(1 - \frac{2}{v-1}\right)^{u-1} = \lim_{u,v\to\infty,\frac{v}{u}\to\theta}\left(1 - \frac{2}{v}\right)^{u-1} = e^{-2/\theta}$$

and

$$\lim_{u,v\to\infty,\frac{v}{u}\to\theta}\left(\frac{(u-1)!}{(v-3)^{s+t}\, s!t!(u-1-s-t)!}\right) = \frac{\theta^{-(s+t)}}{s!t!}.$$

We thus have our result that

$$\lim_{u,v\to\infty,\frac{v}{u}\to\theta} P[S = s, T = t] = \frac{2^{s+t}e^{-4/\theta}\theta^{-(s+t)}}{s!t!} = h(s)h(t).$$

The limiting matching function can now be derived as follows:

$$\begin{aligned} m(\theta;2) &\equiv \lim_{u,v\to\infty,\frac{v}{u}\to\theta}\frac{M(u,v;2)}{u} = 1 - \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}(\frac{s}{s+1})(\frac{t}{t+1})h(s)h(t) \\ &= 1 - \left(\sum_{x=0}^{\infty}(1 - \frac{1}{x+1})\frac{(2/\theta)^x e^{-2/\theta}}{x!}\right)^2 = 1 - \left(1 - \frac{\theta}{2}(1 - e^{-2/\theta})\right)^2 \end{aligned}$$

This is precisely the expression that we derived in our 2003 note (page 69) for the case of $a = 2$.

# 4 The Matching Function in the Limit (general $a$)

To extend the limiting argument from the case of $a = 2$ to the general case of $a \in \{2, \ldots\ldots, A\}$, where $A$ is an arbitrary (but fixed) number of applications, we need to show that in the limit the probability that any competitor applies to two or more of the vacancies to which an individual has applied is zero. The argument is the same as the one used for $a = 2$; namely, we need to show

$$\lim_{u,v\to\infty,v/u=\theta} 1 - \left((1 - \frac{a}{v})^a + a\frac{a}{v}(1 - \frac{a}{v})^{a-1}\right)^{u-1} = 0.$$

This is again done using a l'Hôpital's Rule argument.

Now, let $S_1, ..., S_a$ be the numbers of competitors that an individual has for the first vacancy to which he or she applies, the second vacancy to which he or she applies, ..., the last vacancy to which he or she applies. Let $A(s_1, s_2, ..., s_a)$ be the number of ways that $u-1$ potential competitors can make $s_1$ applications to vacancy $1, ..., s_a$ applications to vacancy $a$. Then $P[S_1 = s_1, ..., S_a = s_a] = \frac{A(s_1, s_2, ..., s_a)}{\binom{v}{a}^{u-1}}$.

Given that no worker is competing with the individual for more than one vacancy (legitimate in the limit by the argument given above),

$$
\begin{aligned}
A(s_1, s_2, ..., s_a) &= \binom{u-1}{s_1}\binom{v-a}{a-1}^{s_1}\binom{u-1-s_1}{s_2}\binom{v-a}{a-1}^{s_2}\cdots\binom{u-1-\sum_{j=1}^{a-1}s_j}{s_a}\binom{v-a}{a-1}^{s_a}\binom{v-a}{a}^{u-1-\sum_{j=1}^{a}s_j} \\
&= \frac{(u-1)!}{\prod_{j=1}^{a}s_j!(u-1-\sum_{j=1}^{a}s_j)!}\left(\frac{(v-a)!}{(v-2a+1)!(a-1)!}\right)^{\sum_{j=1}^{a}s_j}\left(\frac{(v-a)!}{(v-2a)!a!}\right)^{u-1-\sum_{j=1}^{a}s_j} \\
&= \frac{(u-1)(u-2)\cdots(u-\sum_{j=1}^{a}s_j)}{\prod_{j=1}^{a}s_j!}\left(\frac{a}{v-2a+1}\right)^{\sum_{j=1}^{a}s_j}\left(\frac{(v-a)!}{(v-2a)!a!}\right)^{u-1} \\
&= \frac{\frac{(u-1)}{v-2a+1}\frac{(u-2)}{v-2a+1}\cdots\frac{(u-\sum_{j=1}^{a}s_j)}{v-2a+1}}{\prod_{j=1}^{a}s_j!}a^{\sum_{j=1}^{a}s_j}\left(\frac{(v-a)(v-a-1)\cdots(v-2a+1)}{a!}\right)^{u-1}
\end{aligned}
$$

Since $\binom{v}{a} = \frac{v(v-1)\cdots(v-a+1)}{a!}$, we have

$$
P[S_1 = s_1, ..., S_a = s_a] = \frac{\frac{(u-1)}{v-2a+1}\frac{(u-2)}{v-2a+1}\cdots\frac{(u-\sum_{j=1}^{a}s_j)}{v-2a+1}}{\prod_{j=1}^{a}s_j!}a^{\sum_{j=1}^{a}s_j}\left(\frac{v-a}{v}\right)^{u-1}\left(\frac{v-a-1}{v-1}\right)^{u-1}\cdots\left(\frac{v-2a-1}{v-a+1}\right)^{u-1}
$$

Finally,

$$
\lim_{u,v\to\infty, v/u=\theta} P[S_1 = s_1, ..., S_a = s_a] = \frac{\left(\frac{a}{\theta}\right)^{\sum_{j=1}^{a}s_j}}{\prod_{j=1}^{a}s_j!}\exp\{-\frac{a}{\theta}\}^a,
$$

which equals the product of $a$ independent Poissons, each with parameter $a/\theta$.

The limiting matching function for $a \in \{2, ..., A\}$ is then

$$
m(\theta; a) \equiv \lim_{u,v\to\infty, v/u=\theta}\frac{M(u,v;a)}{u} = 1 - \left(1 - \frac{\theta}{a}\left(1 - e^{-\frac{a}{\theta}}\right)\right)^a,
$$

as was given in Albrecht, Gautier, and Vroman (2003).

**References:**

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