Supplement to
An Equilibrium Search Model of Synchronized Sales:
A Sketch of the Model in Continuous Time

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This note sketches a continuous-time version of the discrete time model in our paper, "An Equilibrium Search Model of Synchronized Sales."

The notation is taken from the discrete-time model, except that what were probabilities are now Poisson rates. Moreover, we now assume that shoes depreciate at the same rate $0 < \delta < +\infty$ for fashionistas and sensible shoppers. (Equality of depreciation rates is merely assumed to streamline the notation. What matters is the assumption that no consumer type has 'instant depreciation', i.e. an infinite depreciation rate. This assumption is what leads to non-trivial dynamics in the fashionistas' reservation price.)

VALUES.

- Value of wearing shoes for sensible shoppers:

$$\rho W^0 = v^0 + \delta \left(S^0 - W^0\right) + \dot{W}^0.$$

  (A dot over a variable denotes its time-derivative.)

- Value of shopping for a sensible shopper:

$$\rho S^0 = \alpha \left(1 - \gamma\right) \left(W^0 - r^0 - S^0\right) + \dot{S}^0 = \dot{S}^0,$$

where the second equality comes from the definition of the reservation price $W^0 - r^0 = S^0$. The only solution that satisfies transversality is:

$$S^0 \equiv 0 \qquad \text{and} \qquad r^0 = W^0 \equiv \frac{v^0}{\rho+\delta} \equiv \overline{r}^0.$$

- Value of wearing shoes for a fashionista:

$$\rho W^1 = v^1 + \delta\left(S^1 - W^1\right) + \dot{W}^1.$$

- Value of shopping for a fashionista:

$$\begin{aligned} \rho S^1 &= \alpha\left\{\gamma\left(W^1 - r - S^1\right) + (1-\gamma)\left(W^1 - \overline{r}^0 - S^1\right)\right\} + \dot{S}^1 \\ &= \alpha(1-\gamma)\left(r - \overline{r}^0\right) + \dot{S}^1, \end{aligned}$$

where r denotes the reservation price of a fashionista.

- Substituting $r = W^1 - S^1$ gives:

$$\dot{r} = \left[\rho + \delta + \alpha(1-\gamma)\right] r - \left[v^1 + \alpha(1-\gamma)\,\overline{r}^0\right].$$

STOCKS.

- Equation of motion for the stock of shoppers (recalling $s\varphi \equiv \frac{\delta\lambda}{\alpha+\delta}$):

$$\begin{aligned} \dot{s} &= -\alpha s\varphi - \alpha(1-\gamma)\,s\,(1-\varphi) + \delta(1-s) \\ &= -\left[\delta + \alpha(1-\gamma)\right] s + \delta - \alpha\gamma\frac{\delta\lambda}{\alpha+\delta}. \end{aligned}$$

- Condition for either regime to prevail:

$$r\varphi \gtrless r^0 \iff r \gtrless \overline{r}^0 \frac{\delta+\alpha}{\delta\lambda}\cdot s.$$

NON-STATIONARY EQUILIBRIA.

In this section we only consider nonstationary equilibria that alternate between periods in which $\gamma = 0$ and periods in which $\gamma = 1$.

Introducing the following notation:

$$\overline{r}^h = \frac{v^1}{\rho+\delta}, \quad \overline{r}^\ell = \frac{(\rho+\delta)\,\overline{r}^h + \alpha\overline{r}^0}{\rho+\delta+\alpha}, \quad \overline{s}^h = \frac{\delta+\alpha(1-\lambda)}{\alpha+\delta}, \quad \text{and} \quad \overline{s}^\ell = \frac{\delta}{\alpha+\delta}$$

the dynamics of the economy are now governed by the following systems.

- Dynamics in the high-price ($\gamma = 1$) regime:

$$\dot{r} = (\rho+\delta)\left(r - \overline{r}^h\right)$$
$$\dot{s} = -\delta\left(s - \overline{s}^h\right)$$

- Dynamics in the low-price ($\gamma = 0$) regime:

$$\dot{r} = (\rho+\delta+\alpha)\left(r - \overline{r}^\ell\right)$$
$$\dot{s} = -(\delta+\alpha)\left(s - \overline{s}^\ell\right)$$

- The boundary between the two regimes is defined by $r = \overline{r}^0 \frac{\delta+\alpha}{\delta\lambda} \cdot s$.

- The phase diagram in Figure 1 is constructed under the assumption that:

$$\frac{\delta+\alpha(1-\lambda)}{\delta} > \frac{\lambda\overline{r}^h}{\overline{r}^0} > \frac{\rho+\delta+\alpha(1-\lambda)}{\rho+\delta} \tag{1}$$

  which guarantees that a stationary equilibrium with a single price does not exist (see next paragraph).

- In the phase diagram, red is used to represent the dynamics under the high-price regime, and blue under the low-price regime. E^h denotes the steady-state of the high-price regime, and E^ℓ that of the low-price regime. Both are saddle-path stable, but with the above condition on parameters, E^h lies in the region of the (s, r) plane where the low-price regime prevails, and E^ℓ lies in the region of the (s, r) plane where the high-price regime prevails, so that there is no stationary equilibrium with a single price in this case.

- The path highlighted in green on the phase diagram shows an example of cyclical equilibrium. Starting from point A, the economy is in the high-price regime and starts moving to the right

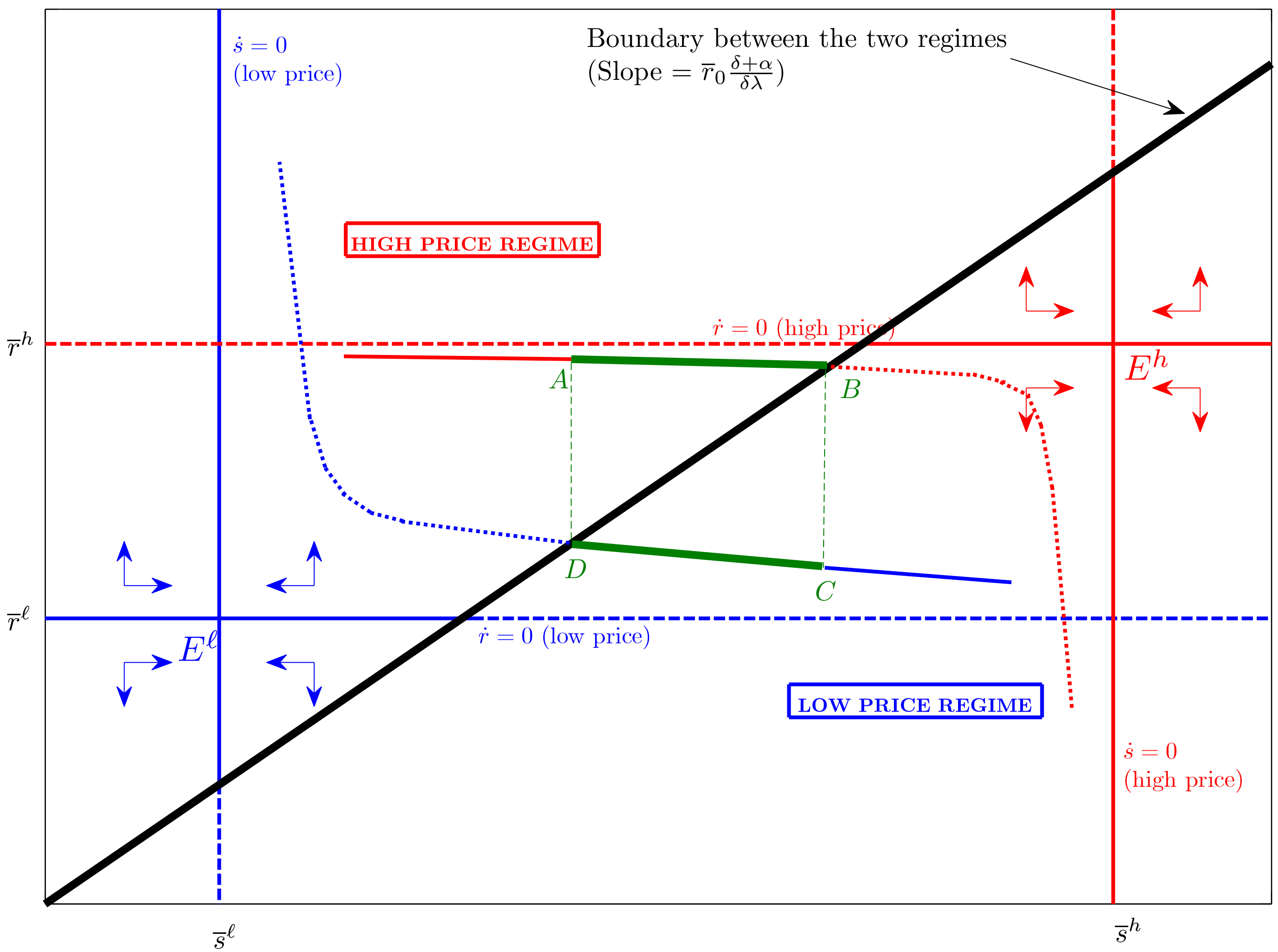

Figure 1: Phase Diagram

along the solid green line. During the high-price phase, the number of shoppers s increases because sensible shoppers keep coming back into the shoppers' pool, while none exit because the price is too high. The composition of the shoppers' pool therefore gradually shifts toward more and more sensible shoppers, as in the basic model. Meanwhile, the reservation price r of fashionistas declines because fashionistas anticipate that they are getting closer to a period of low prices. At point B, the economy hits the boundary between the two regimes. Stores start posting the low price $\overline{r}^0$. The reservation price of fashionistas jumps down to point C (more on this below), and the economy moves to a period of low prices, where sensible shoppers exit the pool of shoppers as they find shoes, so that s gradually declines, while r rises as fashionistas see a new period of high prices looming. The economy eventually hits the boundary again at point D, at which stage it switches back into the high-price regime: the reservation price r jumps back to point A.

- It is straightforward to solve the above dynamic system in closed form and show that at most one (T_ℓ, T_h)-cycle exists for any values of (T_ℓ, T_h). (See the appendix below. Depending on the values of (T_ℓ, T_h), of course conditions on the parameters have to be satisfied for existence.)

- Why the jumps? The periodic equilibrium involves the reservation price of fashionistas jumping discretely each time the economy hits the boundary between the two regimes. What happens there? The reservation price in fact depends on three things: the predetermined state variable s, the regime the economy is in (h or ℓ), and how long agents expect to stay in that regime (assuming expectations are coordinated).[1] At points B and D of the cycle, both the regime and the expected forward time in that regime jump, hence the jump in r.

- **Issues: Stability and multiplicity of equilibria.** Suppose the economy approaches, say, point B in the phase diagram. The reason the economy jumps down to point C, and not to a different low-price trajectory when it hits the boundary at point B, is that agents expect

[1] Formally, the relevant state variable for consumers in this economy can be described as including (i) the current number of shoppers s, (ii) an indicator of the current price regime, and (iii) the entire sequence of future dates at which the economy is expected to switch regimes. The first element (s) of the state variable is predetermined, the other two are not.

that the economy will stay in the low-price regime for a length of time T^ℓ, then go back to the high-price regime for a length of time T^h, and so on. In principle, though, they could coordinate their expectations on any other sequence of future times spent in the two regimes, which would set the economy on a different equilibrium path. In other words, while it is fairly straightforward to exhibit equilibria that take the form of simple (T_ℓ, T_h)-cycles, it seems that infinitely many alternative equilibria exist, with potentially many other cyclical patterns.

APPENDIX: CONSTRUCTING A (T_ℓ, T_h)-CYCLE.

- In what follows, time is re-set to 0 at each regime switch, and superscripts h and ℓ are used as a reminder of which regime the economy is in.

- Solution in the high-price regime:

$$r_t^h = \left(r_0^h - \overline{r}^h\right) e^{(\rho+\delta)t} + \overline{r}^h \tag{2}$$

$$s_t^h = \left(s_0^h - \overline{s}^h\right) e^{-\delta t} + \overline{s}^h, \tag{3}$$

where s_0^h is the predetermined initial stock of shoppers and r_0^h is the (free) initial value of r in the high-price regime.

- Solution in the low-price regime:

$$r_t^\ell = \left(r_0^\ell - \overline{r}^\ell\right) e^{(\rho+\delta+\alpha)t} + \overline{r}^\ell \tag{4}$$

$$s_t^\ell = \left(s_0^\ell - \overline{s}^\ell\right) e^{-(\delta+\alpha)t} + \overline{s}^\ell, \tag{5}$$

where s_0^ℓ is the predetermined initial stock of shoppers and r_0^ℓ is the (free) initial value of r in the low-price regime.

- Now fix (T_ℓ, T_h). A (T_ℓ, T_h)-cycle starting with s_0^h shoppers at point A is such that:

$$\text{Point } B \text{ is on the boundary:} \qquad r_{T_h}^h = \overline{r}^0 \frac{\delta+\alpha}{\delta\lambda} \cdot s_{T_h}^h \tag{6}$$

$$\text{Point } D \text{ is on the boundary:} \qquad r_{T_\ell}^\ell = \overline{r}^0 \frac{\delta+\alpha}{\delta\lambda} \cdot s_{T_\ell}^\ell \tag{7}$$

$$s \text{ doesn't jump at } B: \qquad s_0^\ell = s_{T_h}^h \tag{8}$$

$$s \text{ doesn't jump at } D: \qquad s_0^h = s_{T_\ell}^\ell. \tag{9}$$

- From (2) and (4):

$$s_{T_h}^h = \left(s_0^h - \overline{s}^h\right) e^{-\delta T_h} + \overline{s}^h \qquad\qquad s_{T_\ell}^\ell = \left(s_0^\ell - \overline{s}^\ell\right) e^{-(\delta+\alpha)T_\ell} + \overline{s}^\ell.$$

Substituting (8) and (9), and combining, we obtain:

$$s_0^h = \overline{s}^h \left(1 - e^{-\delta T_h}\right) e^{-(\delta+\alpha)T_\ell} + \overline{s}^\ell \left(1 - e^{-(\delta+\alpha)T_\ell}\right),$$

which uniquely defines s_0^h, and hence $s_{T_h}^h = s_0^\ell$ and $s_{T_\ell}^\ell = s_0^h$.

- Finally, (6) and (7) uniquely define r_0^h and r_0^ℓ.

- One then has to check consistency of the solution, i.e. that all the initial and terminal values thus derived are in the correct regions of the (s, r) plane.

- **Numerical Example.** Figure 2 shows the simulated path of r_t, the fashionistas' reservation price, using the following parameter values: $\alpha = \delta = 1/3$, $\lambda = 1/2$, $\rho = 0.005$, $\overline{r}^0 = 1$, $\overline{r}^h = 3.01$ (so that (1) holds), and $(T_h, T_\ell) = (6, 1)$, so that there is one month of sale every six months.

- The price actually posted by stores is r_t during high price-periods and $\overline{r}^0 = 1$ during low-price periods. Note that, as was already visible on Figure 1, the price gradually declines over the course of a typical high-price period, but those fluctuations are of very small magnitude compared to the jump in price that occurs when the economy enters or exits a period of sale.

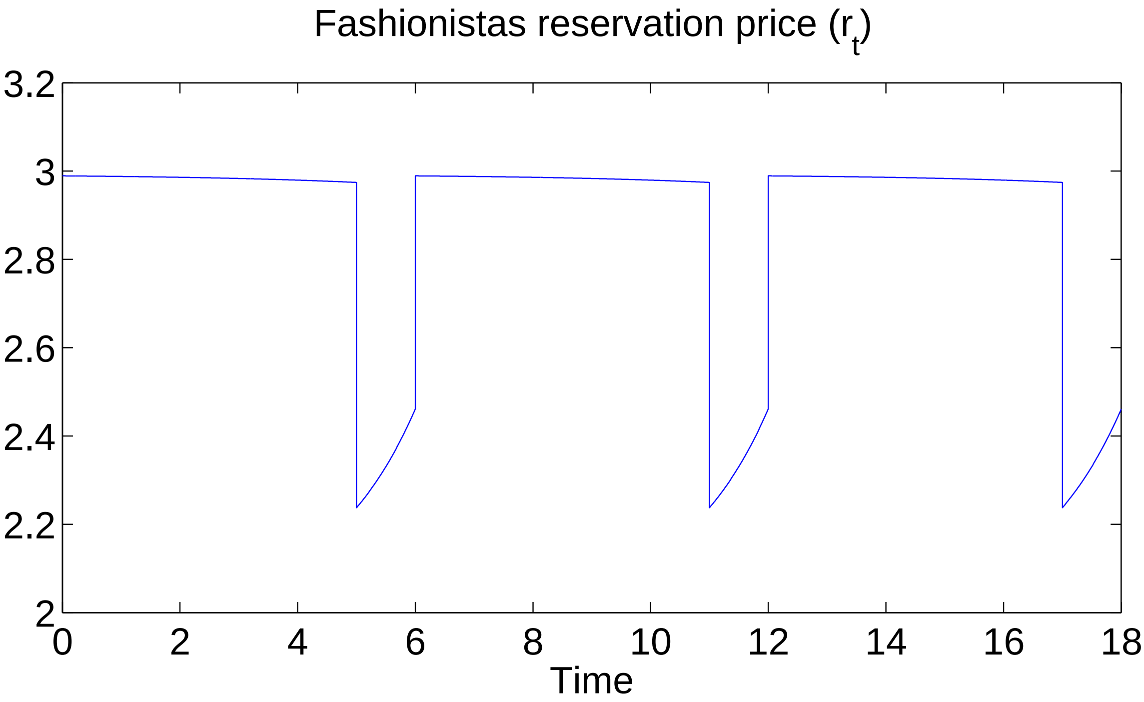

Figure 2: A Simulation