An Equilibrium Search Model of Synchronized Sales*

James Albrecht
Georgetown University and IZA

Fabien Postel-Vinay
University of Bristol,
Sciences Po Paris,
CEPR, and IZA

Susan Vroman
Georgetown University and IZA

June 2012

Abstract

We demonstrate the existence of periodic nonstationary equilibria with self-generating cycles in a simple model of random search. Our results provide a theory of synchronized sales based on product market search by heterogeneous consumers. That is, our model explains how it can be optimal for all sellers to follow a repeated pattern of posting a high price for several periods and then posting a low price for one period.

*We thank two anonymous referees and the editor for their constructive reports on this paper. We also thank seminar participants at the Tinbergen Institute, the 2008 Economics and Music conference, the 2008 and 2010 NBER Summer Institutes, the 2009 SED meetings, the 2009 Search Theory Conference at Osaka University, and the 2011 SaM workshop in Le Mans for useful comments. The usual disclaimer applies.

1 Introduction

In this paper, we demonstrate the existence of periodic nonstationary equilibria with self-generating cycles in a simple model of random sequential search. While existence of equilibrium cross-sectional price dispersion is a well-understood feature of this class of model, the possibility of dispersion across dates — in the sense of recurring price cycles — has hardly been explored. We show that the periodic nonstationary equilibrium is dynamically stable whereas the corresponding stationary dispersion equilibrium is dynamically unstable. Our results offer a simple theory of synchronized sales based on product market search by heterogeneous consumers.[1] That is, our model explains how it can be optimal for sellers to follow a repeated pattern of posting a high price for several periods and then posting a low price for one period.

We consider a market in which consumers search for an acceptable price for a single unit of a semi-durable good. We use shoes as an example although our theory would apply equally well (perhaps up to some minor formal changes) to cars, houses, domestic appliances, etc. Our model could also apply to the labor market, although this would require a more substantial reinterpretation. As is standard in random search models, consumers know the distribution of prices across sellers but not the price charged by any particular seller, so it may take time for a consumer to find a new pair of shoes at an acceptable price. The framework we use is based on the Diamond (1987) model of consumer search, which in turn is closely related to the Albrecht and Axell (1984) job search model.[2] In Albrecht and Axell (1984), there are two worker types with different values of leisure, which in turn leads to

[1]Of course, not all sales are synchronized. As discussed in Nakamura and Steinsson (2008), the importance and nature of sales differs for different kinds of goods. They note that sales are particularly important in apparel (which includes shoes) and document that these sales have strong seasonal patterns; that is, most sellers temporarily lower and then raise their prices back to the usual level at more or less the same time. Sales are less important and don't necessarily exhibit seasonal patterns for other categories of goods. In particular, sales on products sold in supermarkets typically aren't synchronized across sellers. See, for example, Pesendorfer (2002) on sales of ketchup products in Missouri supermarkets and Ariga et al. (2010) on sales of House Vermont Curry, a curry paste widely sold in Japanese supermarkets.

[2]An important distinction between Diamond (1987) and Albrecht and Axell (1984) is that firms are homogeneous in the former but heterogeneous in the latter. Gaumont et al. (2006) is the closest labor-market analogue to Diamond (1987).

reservation wage heterogeneity. Similarly, in Diamond (1987), there are two consumer types, who get different utility from consuming the good. With the shoe market in mind, we call the two consumer types *fashionistas* and *sensible shoppers*. Fashionistas are assumed to receive higher utility from consuming the good (wearing their shoes) than sensible shoppers do. In addition — and in a departure from Diamond (1987) — we assume that the fashionistas' shoes depreciate or go out of style faster than do the shoes of the sensible shoppers. Once their shoes become unfashionable or wear out, consumers are assumed to receive no utility from wearing their old shoes. There is nothing essential about assuming that the two consumer types differ along two dimensions, as opposed to only one — this simply allows us to make our point with less algebra. When a consumer's shoes wear out (or go out of style), the consumer returns to the shopper pool. In the basic model we assume for simplicity that every shopper finds one and only one pair of shoes that she likes in each period, an assumption that we relax in our robustness analysis.

Our model has both stationary and nonstationary equilibria. There is a unique stationary equilibrium for each parameter configuration. For some parameter values — essentially, when the sensible shoppers aren't worth bothering with — only a high price, which is equal to the fashionistas' reservation price, is posted. Alternatively, when the sensible shoppers are relatively many and/or they are not so different from the fashionistas, only a low price, which is equal to the sensible shoppers' reservation price, is posted. Finally, there is an intermediate set of parameters that are consistent with stationary equilibrium price dispersion; that is, some sellers post a high price that is acceptable only to the fashionistas while the others post a low price that is acceptable to both consumer types. These stationary equilibrium results are well known in the equilibrium search literature. However, the idea that there can be nonstationary equilibria in this type of model is new.

We consider *periodic* nonstationary equilibria in which all sellers post high prices (acceptable only to fashionistas) for T_h periods followed by T_ℓ periods in which all sellers post low prices (acceptable to both consumer types). Our results are as follows. First, a unique periodic equilibria exists for every parameter combination that leads to stationary equilibrium price dispersion. Second, depending on parameter values, these equilibria are such that $T_h = 1, 2, 3, \ldots$, but always have $T_\ell = 1$. That is, the equilibrium pattern is one in which sellers post a high price for T_h periods followed by a single period in which all sellers post the low price. Third, the value of T_h in a periodic equilibrium is uniquely determined by the parameters of the model. In particular, for those parameter combinations that are consistent with stationary equilibrium price dispersion in which relatively few sellers post the fashionistas' reservation price, the periodic equilibrium is one in which all sellers post a high price for one period, followed by the low price, followed by the high price, etc. If the stationary dispersion equilibrium is one in which sufficiently more sellers post the high price, then the periodic equilibrium is one with $T_h = 2$ (and $T_\ell = 1$). Indeed, there are parameter combinations consistent with a periodic equilibrium in which sellers post high prices for all possible T_h followed by a single period in which the low price is posted. The higher is T_h, the higher is the fraction of sellers who offer the high price in the corresponding stationary dispersion equilibrium. Fourth, as noted above, we show that the unique periodic nonstationary equilibrium is dynamically stable whereas the corresponding unique dispersion stationary equilibrium is dynamically unstable. Finally, we check robustness by relaxing the assumption that shoppers sample a pair of shoes with probability one in each period. In this version of the model, we continue to find unique periodic nonstationary equilibria with $T_h = 1, 2, 3, \ldots$ and $T_\ell = 1$, but we also find different cyclical equilibria for some other parameter combinations.

The mechanism that lies behind our nonstationary equilibria is straightforward. Demand from sensible shoppers accumulates during high-price periods. At some point, which depends on the strength of the sensible shoppers' demand and on how quickly they accumulate in the pool of shoppers, it makes sense to "hold a sale" to exploit the pent-up demand. Once that demand is satisfied, sellers revert to the high-price regime. The periodic equilibria in our model are thus self generating. The high- and low-price phases lead to compositional changes in the pool of shoppers that naturally drive the cycles. In this sense, the nonstationary equilibria in our model differ from the cycles featured in equilibrium search models such as Diamond and Fudenberg (1989), Fershtman and Fishman (1992), and Burdett and Coles (1998). These are models with multiple stationary equilibria in which agents' beliefs determine which equilibrium obtains. Nonstationary equilibria can be supported if agents believe that the economy will first move towards one stationary equilibrium and then towards another in alternating phases. Our nonstationary equilibria are different. At any parameter configuration that is consistent with a periodic nonstationary equilibrium in our model, there is one and only one corresponding stationary equilibrium.

As we mentioned above, our model offers a simple and compelling theory of synchronized sales. Sellers post high prices for several periods and then "hold a sale" by posting a low price for a single period. Depending on parameters, these sales can occur at different intervals – weekly, monthly, quarterly, etc. Sales are, of course, worth studying in their own right, but they also play an important role in empirical macroeconomics. First, sales are important for price indices. Just as price indices need to account for the fact that consumers can substitute across different goods, so too do price indices need to account for the fact that consumers can substitute across the same good at different points in time (Feenstra and Shapiro 2003). Second, there is an extensive empirical literature on price flexibility (e.g., Klenow and Malin 2011) – how often are prices adjusted and by how much? To address these questions, researchers need to understand transitory price changes due to sales.

There is a literature that interprets synchronized sales as intertemporal price discrimination. Conlisk et al. (1984) considers a monopolist who faces a new cohort of consumers entering the market in each period, some with a low valuation and some with a high valuation. Consumers stay in the market until they make a purchase; after buying, they exit. In this setting it is optimal for the monopolist to charge prices that only the high-valuation consumers will accept for several periods before setting a low price for one period to take advantage of the buildup of the low-valuation consumers. Sobel (1984) extends this result to an oligopolistic framework and, using punishment strategies, shows that there can be synchronized sales, i.e., an equilibrium in which all sellers periodically cut their prices to the same level for one period. Dutta et al. (2007) derives a related result in an overlapping generations framework. Relative to this literature, our contribution is to show how search frictions can generate synchronized sales as an equilibrium outcome in a competitive environment with a continuum of firms. Papers such as Conlisk et al. (1984) show that repeated entry of new consumers allows a monopolist (or oligopolists in the case of Sobel (1984) and Dutta et al. (2007)) to sustain above-marginal-cost pricing in a market for a durable good by engaging in intertemporal price discrimination, thus getting around the Coase (1972) conjecture. As in these papers, our baseline model has an influx of new shoppers every period since fashionistas always want to buy a new pair of shoes. More fundamentally, the Coase conjecture is based on competition between the price that a seller posts today and the prices that the seller will post in the future. With random search and a continuum of sellers, such competition is not possible. Indeed, if the buyer rejects the seller's offer, then the buyer and seller do not expect to meet again. In the oligopoly framework, there are generally multiple equilibria, and punishment strategies are required to sustain synchronized sales. In our model, there is a unique stable equilibrium – stationary with a single price for some parameters, periodic for other parameter configurations. By avoiding multiplicity, our structure thus gives a more tractable and compelling explanation of synchronized sales.[3]

The outline of the rest of the paper is as follows. In the next section, we give the basics of our model. In Section 3, we present the stationary equilibria, and in Section 4 we analyze

[3] There is, of course, also a substantial literature that attempts to explain non-synchronized sales. Some early papers in the equilibrium search literature, e.g., Varian (1980) interpret equilibrium price dispersion as a theory of sales. Other authors, e.g., Slade (1998) and Aguirregabiria (1999) model sales as a response to inventory accumulation. A related strand of the literature (e.g., Lazear 1986) focuses on clearance sales, i.e., price reductions to clear out inventories before a new product is introduced. Finally, there are papers that model sales as a way to build a customer base. For example, in Nakamura and Steinsson (2011), when a firm has a sale it is offering a low price to attract new customers, who then "get in the habit" of buying from that firm. Similarly, in Shi (2011), firms hold sales to attract buyers and build customer relationships. His paper is related to ours in the sense that he also uses an equilibrium search model, albeit with directed rather than random search. The underlying mechanisms in our model (intertemporal price discrimination) and his (building a customer base) are, however, quite different.

the nonstationary, periodic equilibria. In Section 5 we generalize our model to check the robustness of our results. Section 6 concludes.

2 The Basic Model

Environment. Consider the market for a consumption good that is semi-durable (shoes) in the sense that it does not necessarily fully depreciate each period. We assume that time is discrete, indexed by t, and that all agents are infinitely lived and discount the future at rate $\beta < 1$ per period.[4] Agents either buy and wear shoes or run a shoe store. By wearing a pair of shoes, a consumer enjoys a constant utility of $v_k > 0$ each period, where $k = 0, 1$ is the consumer's type (to be defined momentarily), until the shoes go out of fashion (or wear out), which occurs with constant probability δ_k per period. Once her shoes go out of fashion (or wear out), the consumer no longer enjoys wearing them (her utility per period falls to 0) and goes shopping for a new pair.

Shopping is modeled as a search process – the market for shoes is affected by search frictions in that every shopper finds one and only one pair of shoes that she likes in each shopping period. However, the consumer may or may not buy the shoes – that depends on the price. Shoe prices are posted by the shoe stores and cannot be bargained over. The consumer decides whether to buy the shoes or to continue shopping next period.

There are two consumer types. Type-1 consumers (fashionistas) enjoy a high utility, v_1, which we normalize to one, in each period that they wear shoes, whereas type-0 consumers (sensible shoppers) only enjoy $v_0 \equiv v$, where $0 < v < 1$. The two types also differ in terms of how quickly their shoes depreciate or go out of style. In particular, fashionistas' shoes go out of style each period with probability $\delta_1 = 1$ while sensible shoppers' shoes depreciate

[4]Discrete time seems like the natural modeling choice for high-frequency phenomena such as sales and indeed is the assumption used in the sales literature discussed above. The assumption that time is discrete is not, however, without loss of generality. The technical benefit of discrete time is that it allows us to formalize our theory of synchronized sales as a simple scalar (one-dimensional), autonomous dynamic system. It is well known that, in continuous time, one-dimensional autonomous dynamic systems cannot exhibit periodic behavior, whereas one-dimensional autonomous discrete-time systems can. Periodic behavior in continuous time can be achieved in our model by introducing a second state variable but does not, in our opinion, add anything in terms of insight. A sketch of our model in continuous time can be found on our web pages.

each period with probability $\delta_0 = \delta < 1$. The normalization $\delta_1 = 1$ is not without loss of generality, but one can think of a period as the time it takes for the fashionistas' shoes to go out of style.[5] The total population of consumers is normalized to one, and the share of fashionistas in the population is λ.

Values and prices. Stores post prices and shoppers arbitrage between buying shoes and holding a numéraire good of which they receive a fixed endowment each period and from which they derive a utility that is normalized to zero. Because there are only two consumer types, each with a different reservation price, shoe stores either choose to post a high price, which is equal to the reservation price of fashionistas and which sensible shoppers would turn down, or a low price equal to the reservation price of sensible shoppers, which both consumer types would accept. Let γ_t denote the fraction of stores posting the high price in period t.

For a type-k consumer we denote the lifetime value of shopping in period t by S_t^k and that of wearing shoes by W_t^k. The net value of buying shoes at price p_t is $W_t^k - p_t$, so the reservation price r_t^k of a type-k consumer is defined by $W_t^k - r_t^k = \beta S_{t+1}^k$.

We now turn to formal definitions of the value functions. Starting with the value of wearing shoes for sensible shoppers,

$$W_t^0 = v + \beta \left\{ (1-\delta) W_{t+1}^0 + \delta S_{t+1}^0 \right\}. \tag{1}$$

The first term is the current-period utility. At the end of the period, the shoes remain wearable with probability $(1-\delta)$, in which case the consumer continues without shopping, while with probability δ the shoes wear out and the individual becomes a shopper. The

[5] What this normalization does for us is reduce the dimensionality of the dynamic system that characterizes our model's equilibrium by ensuring that the fashionistas' reservation price is always equal to one — see equation (5) below. With $\delta_1 < 1$, the fashionistas' reservation price would have non-trivial, forward-looking dynamics and add a (non-predetermined) state variable to the model. The resulting two-dimensional model would be algebraically considerably more cumbersome without adding much economic insight.

corresponding value of being a (sensible) shopper is

$$\begin{aligned} S_t^0 &= \gamma_t \beta S_{t+1}^0 + (1-\gamma_t)\left(W_t^0 - r_t^0\right) \\ &= \beta S_{t+1}^0. \end{aligned} \tag{2}$$

In the current period, the shopper receives no utility. With probability γ_t, the shopper encounters a store with a high price and continues as a shopper into the next period. With probability $1-\gamma_t$, she finds shoes at an acceptable price and purchases shoes, which generate value $W_t^0 - r_t^0$, the lifetime value of wearing the shoes less the price. Value function (2) implies for all t that

$$S_t^0 = 0. \tag{3}$$

Then, value function (1) implies for all t that

$$r_t^0 = W_t^0 = \frac{v}{1-\beta(1-\delta)} \equiv r^0. \tag{4}$$

Next consider the fashionistas. Their lifetime value of wearing shoes is

$$W_t^1 = 1 + \beta S_{t+1}^1, \tag{5}$$

reflecting our assumption that a fashionista's shoes go out of style with probability 1 at the end of the current period. This implies that the fashionistas' reservation price is the same for all t, namely,

$$r_t^1 = 1 \equiv r^1. \tag{6}$$

The lifetime value of shopping for these consumers is

$$\begin{aligned} S_t^1 &= \gamma_t\left(W_t^1 - r^1\right) + (1-\gamma_t)\left(W_t^1 - r^0\right) \\ &= \beta S_{t+1}^1 + (1-\gamma_t)\left(r^1 - r^0\right). \end{aligned} \tag{7}$$

As was the case for the sensible shoppers, fashionistas receive no utility while searching for a new pair of shoes. With probability γ_t, they encounter a store posting the high price, purchase the shoes, and receive the value of wearing new shoes less the high price. With probability $1-\gamma_t$, they find the shoes at a store posting the low price and receive the value of wearing new shoes less the low price. The condition that ensures that fashionistas are willing to spend more than sensible shoppers are to buy shoes is simply $r^1 = 1 > r^0$; that is,

$$v < 1 - \beta(1-\delta). \tag{8}$$

If this restriction does not hold, then the roles of the fashionistas and sensible shoppers are reversed.

Consumer stocks. Let s_t denote the share of shoppers in the total consumer population and φ_t the share of fashionistas among the shoppers at the beginning of period t. These shares depend on the history of stores' pricing strategies. However, fashionistas never resist buying shoes and therefore always leave the population of shoppers with probability 1 at the end of each period no matter what fraction of stores post the high price. The stock of fashionistas who are shopping is thus constant at their population share, so

$$s_t\varphi_t \equiv \lambda. \tag{9}$$

The total number of shoppers then evolves according to

$$\begin{aligned} s_{t+1} &= s_t - s_t\varphi_t - (1-\gamma_t)\, s_t (1-\varphi_t) + \lambda + \delta\left(1 - \lambda - s_t(1-\varphi_t)\gamma_t\right) \\ &= \lambda + (1-\lambda)\,\delta + \gamma_t s_t (1-\delta) - \lambda\gamma_t(1-\delta), \end{aligned} \tag{10}$$

where the second equality uses (9). The number of shoppers at the beginning of period $t+1$ equals the number at the start of period t minus the $s_t\varphi_t$ fashionistas who all buy shoes minus the $(1-\gamma_t)\, s_t(1-\varphi_t)$ sensible shoppers who encountered stores posting the low price plus all the λ fashionistas whose shoes went out of fashion at the end of the period plus the $\delta\left(1-\lambda-s_t(1-\varphi_t)\gamma_t\right)$ sensible shoppers who enjoyed their shoes during the period but whose shoes wore out at the end of the period.

Stores' decisions. Stores seek to maximize expected sales revenue each period.[6] This is proportional to the probability that the posted price will be accepted by a randomly met shopper times the sale price – the wholesale price at which shoe stores buy their stock is assumed constant and is normalized to zero. Hence a store chooses to post the high price r^1, at which only fashionistas are prepared to buy, rather than the low price r^0, which is acceptable to all consumers, if and only if $r^1\varphi_t > r^0$. Of course, they make the converse choice (are indifferent between the two options) if and only if this latter inequality holds in reverse (becomes an equality). By using (6) and (9), one can rewrite the conditions under which stores post the high price, both prices or the low price as

$$\begin{aligned} \gamma_t &= 1 \text{ if } s_t < \frac{\lambda}{r^0} \\ \gamma_t &\in (0,1) \text{ if } s_t = \frac{\lambda}{r^0} \\ \gamma_t &= 0 \text{ if } s_t > \frac{\lambda}{r^0}. \end{aligned} \tag{11}$$

Note that any individual store's decision rule only depends on the beginning-of-period stock of shoppers and is independent of the current period choices made by competing firms. In other words, there is no within-period strategic interaction among stores in this market. Moreover, it is clear from (11), which only depends on the aggregate state variable s_t, that whatever choice is optimal for one store in a given period is optimal for all stores in that period. Therefore, unless stores are indifferent between posting the high or the low price (i.e., unless $s_t = \frac{\lambda}{r^0}$), they either all post the high price ($\gamma_t = 1$) or they all post the low price ($\gamma_t = 0$), and there is no within-period price dispersion in equilibrium. Substituting the decision rule (11) into the law of motion (10) thus yields a slightly more compact expression

[6] This assumes that finding a good deal at a particular store does not lead a consumer to return to the same store again after her shoes have depreciated. In equilibrium, this assumption is sensible – there is no reason for a consumer to patronize the same store more than once. It is also worth noting that the assumption that sellers maximize expected current period revenue distinguishes our model, which is set in a product market, from labor market models like Albrecht and Axell (1984) and Gaumont et al. (2006). In a labor market, the relationship between the buyer (the firm) and the seller (the worker) extends beyond the current period.

for the law of motion of s_t in the form of a piecewise linear difference equation:

$$s_{t+1} = F(s_t) := \begin{cases} s_t(1-\delta)+\delta & \text{if } s_t < \dfrac{\lambda}{r^0} \\ \\ \delta + \lambda(1-\delta) & \text{if } s_t > \dfrac{\lambda}{r^0} \end{cases} \tag{12}$$

3 Stationary Equilibria

In this section we investigate the existence and properties of stationary long-run equilibria in which all endogenous variables remain constant over time.

Stationary equilibrium with $\gamma = 1$ (all stores post the high price). Equation (12) implies that if such a stationary equilibrium exists, it features a constant number of shoppers s_h^* given by

$$s_h^* = 1. \tag{13}$$

Condition (11) further implies that it is indeed optimal for stores to post the high price in this situation if and only if

$$s_h^* = 1 < \frac{\lambda}{r^0}. \tag{14}$$

Stationary equilibrium with $\gamma = 0$ (all stores post the low price). Following similar steps, if such a stationary equilibrium exists, the number of shoppers s_ℓ^* is given by

$$s_\ell^* = \lambda + (1-\lambda)\delta, \tag{15}$$

and the consistency condition (11) for this to be an equilibrium is

$$s_\ell^* = \lambda + (1-\lambda)\delta > \frac{\lambda}{r^0}. \tag{16}$$

Stationary equilibrium with $\gamma \in (0,1)$ (price dispersion). If a stationary equilibrium with price dispersion exists, the number of shoppers s_γ^* is found using (10) and is given by

$$s_\gamma^* = \frac{\lambda + (1-\lambda)\delta - \lambda\gamma(1-\delta)}{1-\gamma(1-\delta)}. \tag{17}$$

For this to be an equilibrium, (11) must hold as an equality. Substituting the above expression for s_γ^*, this condition becomes

$$s_\gamma^* = \frac{\lambda + (1-\lambda)\,\delta - \lambda\gamma\,(1-\delta)}{1-\gamma\,(1-\delta)} = \frac{\lambda}{r^0}. \tag{18}$$

This defines a unique share of firms $\gamma^* \in (0,1)$ if and only if

$$\lambda + \delta\,(1-\lambda) < \frac{\lambda}{r^0} < 1. \tag{19}$$

To see that γ^* is indeed unique, note that the function $\gamma \mapsto \dfrac{\lambda + (1-\lambda)\,\delta - \lambda\gamma\,(1-\delta)}{1-\gamma\,(1-\delta)}$ increases monotonically from $\lambda + \delta\,(1-\lambda)$ to 1 as γ increases from 0 to 1. Condition (19) defines a nonempty set of parameter values for which stationary equilibria with price dispersion exist. Inspection of (18) further reveals that γ^*, the equilibrium share of stores posting the high price, is a decreasing function of r^0 and an increasing function of λ. Posting the high price is more likely to be profitable if there are more fashionistas around or if sensible shoppers are relatively less eager to buy shoes.

Example. Let $\beta = \dfrac{1}{1+0.012} = 0.988$, so we think of a period as one quarter, and let $\lambda = \delta = 0.5$. Then, from (19) , $0.253 < v < 0.337$ is consistent with equilibrium price dispersion. If, for example, $v = 0.3$, then (from equation (18)) a fraction $\gamma^* = 0.544$ of the sellers post the fashionista reservation price ($r^1 = 1$). The other sellers post the sensible shopper reservation price, $r^0 = 0.593$ (from equation (4)). If $v < 0.253$, then only the high price is posted, while if $v > 0.337$, only the reservation price of the sensible shoppers, which is increasing in v up to the point that inequality (8) binds (when $v = 0.506$ for this example), is posted.[7]

Stationary equilibria: Summary and stability properties. Conditions (14), (16), and (19) define the pattern of stationary equilibria. These conditions partition the parameter

[7]Note that there is nothing "special" about this example in the sense that for any $(\beta, \delta, \lambda) \in (0,1)^3$, there is a interval of v's, i.e., sensible shopper period utilities, in $(0,1)$ that are consistent with stationary equilibrium price dispersion.

space, so a unique stationary equilibrium always exists. This equilibrium only features price dispersion (in the form of a two-point distribution) when condition (19) holds. Other stationary equilibria are single-price. This is essentially the result presented in inequality (5) of Diamond (1987) and in Proposition 1 of Gaumont, et al. (2006).

Before we move on, we remark on the stability properties of the various stationary equilibria listed in this section. First, both the high-price stationary equilibrium s_h^* and the low price stationary equilibrium s_ℓ^*, whenever they exist, are clearly globally stable, as the top and bottom equations in (12), taken separately, each have slope in $[0, 1)$ and therefore generate stable dynamics. However, whenever parameters are such that (19) holds, then the only stationary equilibrium is the one with price dispersion, $s_\gamma^* = \frac{\lambda}{r^0}$. Anticipating results in the next two sections, we show below that the stationary equilibrium with price dispersion is dynamically unstable. Aside from our interest in offering a theory of sales, instability of the unique stationary equilibrium in itself constitutes a motive to look for nonstationary stable long-run equilibria.

4 Nonstationary Equilibria

The existence of stationary equilibria with price dispersion in this type of model is well known. What has not been shown before is that these equilibria coexist with nonstationary equilibria. We now investigate the existence of such equilibria.

4.1 A Simple Example

The simplest example of a nonstationary equilibrium is a periodic one in which all stores post the high price in even periods and the low price in odd periods (or vice versa), i.e. $\gamma_{2k} = 1$ and $\gamma_{2k+1} = 0$. We now investigate the conditions under which such equilibria with periodicity 2 exist.

Let s_h (s_ℓ) denote the number of shoppers at the beginning of a period in which $\gamma = 1$

($\gamma = 0$). Equation (12) implies

$$\begin{cases} s_h = \lambda + (1-\lambda)\delta \\ s_\ell = (1-\delta)\, s_h + \delta = (1-\delta)\,(\lambda + (1-\lambda)\delta) + \delta. \end{cases} \tag{20}$$

Our candidate nonstationary equilibrium has the economy alternating between one period in which the number of shoppers takes on the value s_h and one period featuring s_ℓ. For it to be a valid equilibrium, the stores' optimality condition (11) must be satisfied in both types of period; i.e.,

$$s_\ell > \frac{\lambda}{r^0} \qquad \text{and} \qquad s_h < \frac{\lambda}{r^0}$$

must hold simultaneously. Substituting for s_h and s_ℓ, these conditions imply the following restriction on the parameters:

$$\lambda + (1-\lambda)\,\delta < \frac{\lambda}{r^0} < (1-\delta)\,(\lambda + (1-\lambda)\,\delta) + \delta. \tag{21}$$

This restriction defines a nonempty set of parameter values. This type of nonstationary equilibrium only coexists with stationary equilibria of the $\gamma \in (0,1)$ type, i.e. stationary equilibria featuring price dispersion. More precisely, the minimum value of $\frac{\lambda}{r^0}$ that is consistent with stationary price dispersion is the same as the minimum $\frac{\lambda}{r^0}$ that is consistent with an equilibrium with periodicity 2, while the maximum $\frac{\lambda}{r^0}$ that is consistent with stationary price dispersion is greater than the maximum $\frac{\lambda}{r^0}$ that is consistent with an equilibrium with periodicity 2. That is, the lower bound of (21) equals the lower bound of (19), while the upper bound of (21) is below the upper bound of (19).

The intuition underlying the periodic nonstationary equilibrium is straightforward. If only the high price is posted in period t, then the sensible shoppers will delay replacing their shoes. This changes the composition of the pool of shoppers moving into period $t+1$; in particular, the fraction of fashionistas in the shopper pool falls between t and $t+1$ ($\varphi_{t+1} < \varphi_t$). In period $t+1$, firms react to the increased presence of sensible shoppers by posting the lower price, etc.

4.2 A Theory of Synchronized Sales

Our simple example shows that an equilibrium with self-generating cycles exists, but this example is limited as a theory of synchronized sales. A theory of synchronized sales should allow for richer nonstationary periodic equilibria in which the market spends T_h periods in a high-price regime ($\gamma = 1$) and then spends one period in a low-price regime ($\gamma = 0$). Equilibria of this type explain synchronized sales more generally, e.g., weekly, monthly or quarterly sales. We start with a more general setup and consider the possibility of equilibria with T_ℓ periods in a low-price regime ($\gamma = 0$) followed by T_h periods ($\gamma = 1$) in a high-price regime, with T_h and T_ℓ left unrestricted for now. We show that equilibria exist in which T_h takes on values greater than one but that no equilibria exist with $T_\ell > 1$. Thus, we indeed have a theory of periodic synchronized price reductions rather than one of cycles with several low price periods followed by high-price periods. We also show that $(T_h, 1)$ equilibria exist for all positive integer $T_h = 1, 2, 3, \ldots$. Furthermore, there is a $(T_h, 1)$ equilibrium for each $\frac{\lambda}{r^0}$ satisfying (19). These equilibria are ordered in the sense that the higher is the fraction of firms, γ, that would post the high price in the stationary dispersion equilibrium (equivalently, the higher is $\frac{\lambda}{r^0}$), the higher is T_h. The length of the interval of $\frac{\lambda}{r^0}$'s consistent with a $(T_h, 1)$ equilibrium decreases with T_h, and as $T_h \to \infty$, the value of $\frac{\lambda}{r^0}$ consistent with periodic equilibria collapses to 1, the upper limit of the interval given by (19).

Now we turn to the analysis. Because we are interested in periodic equilibria, it is convenient to reset the time subscript t to zero every time the economy switches regimes. Moreover, in what follows we add an $\{h, \ell\}$ superscript to the endogenous variable s to indicate the economy's current regime (high- or low-price, $\gamma = 1$ or 0) With these notational conventions, characterization of the economy's dynamic behavior follows from equation (12).

The high- and low-price regimes. The law of motion (12) implies that, in the high-price regime, for $t \in \{1, \cdots, T_h - 1\}$, the stock of shoppers evolves following

$$s_{t+1}^h = (1 - \delta)\, s_t^h + \delta. \tag{D_h}$$

Equation (D_h) implies that the number of shoppers keeps increasing so long as the economy stays in the high-price regime. In the limit, if the high price prevailed forever, the number of shoppers would approach one – all sensible shoppers would eventually go barefoot, unable (or unwilling) to afford shoes, while fashionistas would continue shopping every period.

In the low-price regime, for $t \in \{1, \cdots, T_\ell - 1\}$, the dynamic behavior of the economy is simply characterized by

$$s^\ell_{t+1} = \lambda + (1 - \lambda)\,\delta. \tag{D_ℓ}$$

The number of shoppers stays constant (after one period) in the low-price regime. As always, all fashionistas shop in every period, while sensible shoppers are now willing to buy the first pair of shoes they sample – and they sample one with certainty in their first period of shopping. Therefore, in this regime, the number of shoppers each period is made up of the λ fashionistas, who shop no matter what, and the fraction δ of the $1 - \lambda$ sensible shoppers whose shoes just wore out.

The steady-state values of s^h and s^ℓ were already given in equations (13) and (15).

Switching points. The dynamic systems (D_h) and (D_ℓ) only apply in the "interior" of each regime, i.e., when the economy is not about to switch from one regime to the other. At switching points – i.e. at $t = T_h$ in the high-price regime and $t = T_\ell$ in the low-price regime – the law of motion (12) implies the following. At a switch from the high-price into the low-price regime, one has

$$s^\ell_1 = (1 - \delta)\,s^h_{T_h} + \delta, \tag{$S_{h\to\ell}$}$$

and at a switch from the low- into the high-price regime

$$s^h_1 = \lambda + (1 - \lambda)\,\delta. \tag{$S_{\ell\to h}$}$$

Candidate nonstationary equilibrium for given T_h and T_ℓ. In order to construct a candidate nonstationary equilibrium, we solve systems (D_h), (D_ℓ), ($S_{h\to\ell}$), and ($S_{\ell\to h}$)

recursively for the sequence of populations of shoppers s_t. This gives

$$
\begin{aligned}
s_1^\ell &= 1 - (1-\delta)^{T_h+1}(1-\lambda) && \\
s_t^\ell &\equiv \lambda + (1-\lambda)\,\delta && \text{for } t \in \{2, \cdots, T_\ell\} \\
s_1^h &= \lambda + (1-\lambda)\,\delta && \\
s_t^h &= 1 - (1-\delta)^t (1-\lambda) && \text{for } t \in \{2, \cdots, T_h\}.
\end{aligned}
$$

Note that any candidate nonstationary equilibrium characterized by the set of equations above is independent of the number of periods spent in the low-price regime, T_ℓ. For given values of T_h, we have therefore constructed a family of candidate nonstationary equilibria, each member of which is characterized by a duration of the high-price regime, T_h. We now turn to the consistency requirements for these candidate equilibria to constitute valid equilibria of our model.

Consistency conditions. For the candidate equilibrium described above to be valid, it is necessary that the following inequalities hold:

$$
s_{T_\ell}^\ell > \frac{\lambda}{r^0} \quad \text{and} \quad s_{T_h}^h < \frac{\lambda}{r^0} \tag{22}
$$

Note that we only need to check these conditions at dates T_h and T_ℓ, respectively. Thanks to the monotonicity properties of s_t^h and s_t^ℓ in the candidate periodic equilibrium, inequality (22) ensures that $s_t^h < \frac{\lambda}{r^0}$ holds for all $t < T_h$ when the market is in the high-price regime and that $s_t^\ell > \frac{\lambda}{r^0}$ holds for all $t < T_\ell$ when the market is in the low-price regime.

Substitution of the various expressions that characterize our candidate equilibrium into (22) leads to conditions on the parameters that are slightly different depending on whether one considers $T_\ell = 1$ or $T_\ell \geq 2$. The difference is due to the fact that it takes two periods in the low-price regime for the number of shoppers to reach its constant value of $\lambda + (1-\lambda)\,\delta$. For $T_\ell = 1$, the consistency conditions can be rewritten as:

$$
1 - (1-\delta)^{T_h}(1-\lambda) < \frac{\lambda}{r^0} < 1 - (1-\delta)^{T_h+1}(1-\lambda). \tag{23}
$$

This latter condition coincides with (21) when $T_h = 1$. In addition, condition (23) defines a nonempty set of parameter values for $T_h = 2, 3, \ldots$. Note that the minimum value of $\frac{\lambda}{r^0}$ consistent with the periodic equilibrium that spends two periods in the high-price regime followed by one period in the low-price regime, i.e., the $(2,1)$ equilibrium, equals the maximum value of $\frac{\lambda}{r^0}$ consistent with the $(1,1)$ equilibrium; the minimum value of $\frac{\lambda}{r^0}$ consistent with the $(3,1)$ equilibrium equals the maximum value of $\frac{\lambda}{r^0}$ consistent with the $(2,1)$ equilibrium, etc. Further, as $T_h \to \infty$, the minimum and maximum values of $\frac{\lambda}{r^0}$ consistent with a $(T_h, 1)$ equilibrium both converge to 1, i.e. the highest $\frac{\lambda}{r^0}$ consistent with stationary price dispersion. That is, the parameter ranges for the various $(T_h, 1)$ cycles are non-overlapping, ordered, and cover the entire interval of $\frac{\lambda}{r^0}$'s for which stationary price dispersion equilibria exist. Figure 1 provides a diagrammatic rendition of that partition of the parameter space, also indicating the nature of equilibrium in each region.

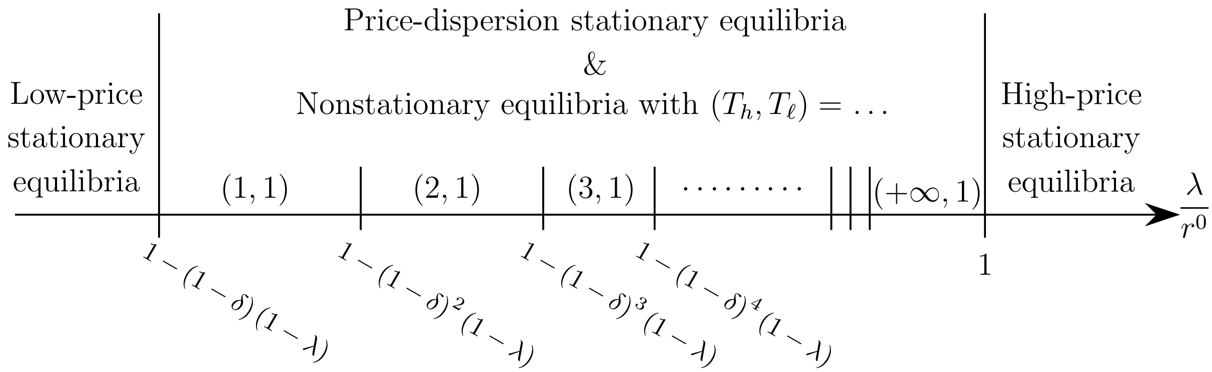

Figure 1: Equilibrium characterization

Finally, for a cyclical equilibrium with $T_\ell > 1$ to exist, the following inequalities must hold:

$$1 - (1-\delta)^{T_h}(1-\lambda) < \frac{\lambda}{r^0} < 1 - (1-\delta)(1-\lambda), \tag{24}$$

which is clearly not possible with $T_h \geq 1$. There is a clear intuition for the non-existence of periodic equilibria with $T_\ell > 1$ in this simple model. As discussed before (see the discussion after equation (D_ℓ)), the size and composition of the shopper pool stay constant after one period in the low-price regime. Therefore, if stores do not find it in their interest to switch back to the high price after just one period of sales, they never will – and in that case, the economy stays in the low-price stationary equilibrium described in Section 3. In other words, if stores are ever to switch back to the high price (as they are by definition in a periodic equilibrium), then they want to do so after just one period in the low-price regime.

To summarize, we find that nonstationary equilibria with periodic cycles in which stores charge the high price for some number of periods followed by the low price for one period exist. These equilibria characterize periodic synhronized sales. We thus find that periodic synchronized sales result from search behavior in a market in which consumers differ in terms of their willingness to pay for a single unit of the good and in terms of their desire to switch to the latest style. After several periods of high prices, the pool of searching consumers accumulates a large enough fraction of the sensible shoppers that a sale is optimal for all the stores. This pattern prevails in a market with search frictions despite the absence of market power.

Example. Consider again the case of $\beta = 0.988$ with $\lambda = \delta = 0.5$, and recall that the stationary equilibrium exhibits price dispersion for $0.253 < v < 0.337$. The equilibrium with $T_h = 1$ and $T_\ell = 1$ exists for $0.289 < v < 0.337$, the $(2, 1)$ equilibrium exists for $0.270 < v < 0.289$, the $(3, 1)$ equilibrium exists for $0.261 < v < 0.270$, etc. Skipping ahead, the $(10, 1)$ equilibrium exists for $0.25306 < v < 0.25313$, and the range of v for which the $(100, 1)$ equilibrium exists essentially collapses to the point $v = 0.253$. Periodic equilibria exist, however, for arbitrarily large T_h.[8]

[8]Again, there is nothing special about this example in the sense that given (β, δ, λ), there is always a range of v's that are consistent with periodic equilibrium. The v's that are consistent with periodic equilibria are, of course, exactly the same as those that are consistent with stationary equilibrium price dispersion. Thus, if the reader has a particular view about the market for shoes, that is, about β (how frequently are sellers able to changes their prices?), δ (how long on average do shoes last before a sensible shopper needs to buy another pair?), and λ (what fraction of the market do fashionistas represent?), it is straightforward to compute the range of $v's$ that are consistent with synchronized sales.

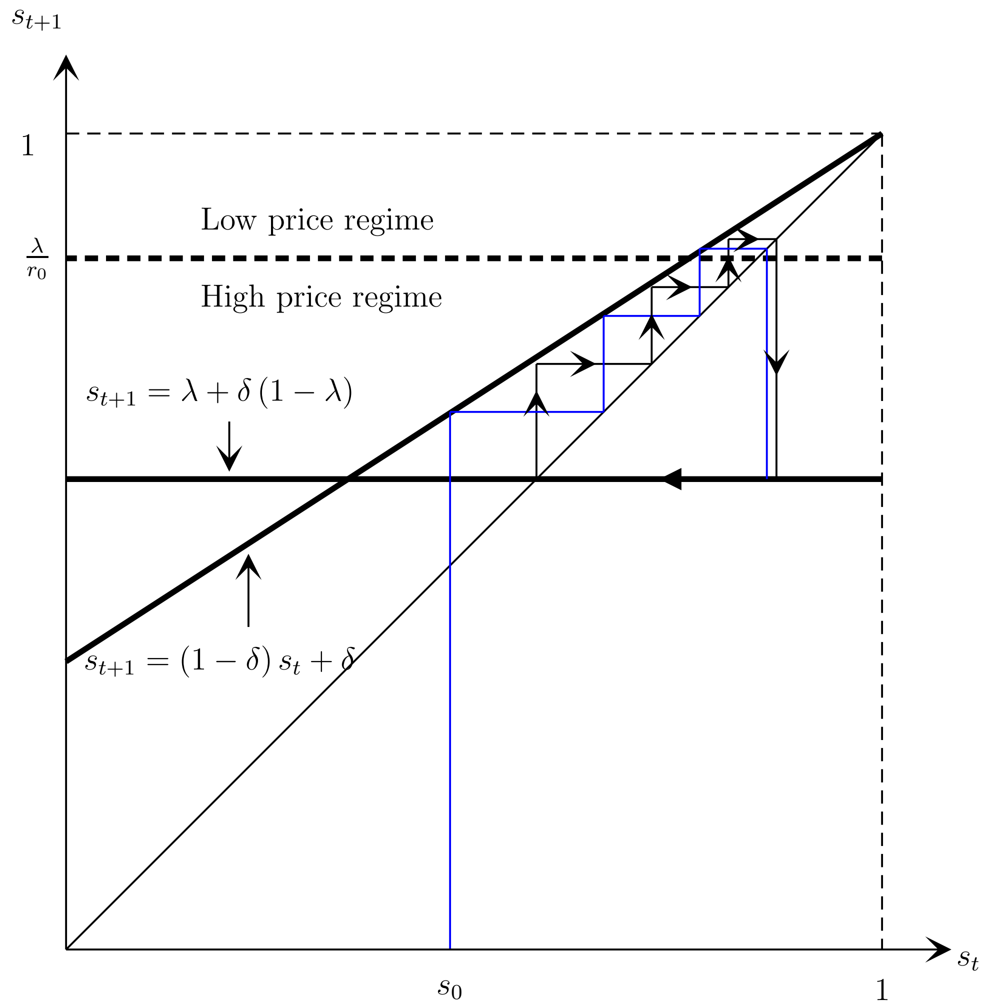

Figure 2: Global stability of periodic equilibria

4.3 Stability

We have established the coexistence of two long-run equilibria whenever condition (19) holds: one stationary equilibrium with price dispersion, described in Section 3, and one non-stationary equilibrium with synchronized sales but no within-period price dispersion, described in this section 4. We now prove that the latter is globally dynamically stable, whereas the former is dynamically unstable.

Establishing global stability of the periodic equilibrium with sales is straightforward in this simple model. As is clear from equation (D_ℓ), the number of shoppers, which is the state variable, is reset to the value $s_t = \lambda + (1-\lambda)\,\delta$ as soon as the economy hits the low-price regime. For that reason the value $s_t = \lambda + (1-\lambda)\,\delta$ is always a point of support for any periodic equilibrium with synchronized sales, i.e., it is the point reached by the economy immediately after the sales period. This establishes that the economy has reached the periodic long-run equilibrium with synchronized sales as soon as it enters the low-price regime, which it is bound to do after a finite number of periods if condition (19) holds. Formally, suppose the economy is initially in the high-price regime at $t = 0$, i.e. $s_0 < \frac{\lambda}{r^0}$. Then s_t starts increasing following the law of motion (D_h), until it surpasses the regime-switching threshold $\frac{\lambda}{r^0}$, which it does in finite time because, from (19), $\frac{\lambda}{r^0} < 1$, whereas the stable steady-state associated with (D_h) is $s_h^* = 1$.

Stability of the periodic equilibrium is illustrated in Figure 2. Starting from some initial value $s_0 < \frac{\lambda}{r^0}$, the stock of shoppers s_t grows along the high-price regime dynamics, as depicted by the plain solid "stairs" in Figure 2. After a few periods (three, in the case depicted in the figure), s_t crosses the $\frac{\lambda}{r^0}$ threshold and enters the low-price regime, implying that the stock of shoppers gets reset to $s_{t+1} = \lambda + \delta\,(1-\lambda) < \frac{\lambda}{r^0}$, at which point the economy is locked into the periodic equilibrium with synchronized sales, shown in the figure by the solid lines with arrows.

Finally, note that the same reasoning that served to establish global stability of the periodic equilibrium also implies dynamic *in*stability of the stationary equilibrium with price dispersion. That is, if the stationary price dispersion equilibrium is disturbed even slightly, the market will move into the periodic nonstationary equilibrium.

5 Robustness

In order to present our results as cleanly as possible, we have made several simplifying assumptions. In particular, we assumed that shoppers always find a pair of shoes ($\alpha = 1$). In this section, we relax this assumption and consider the case of $\alpha < 1$.[9] This complicates

[9]In order to preserve the one-dimensional nature of our model (whose only state variable is the stock of shoppers s_t), we maintain the normalization that the rate at which the fashionistas' shoes go out of style is $\delta_1 = 1$. Footnote 5 explains the technical content of that normalization. As explained before, that

the model somewhat, but we still find equilibria that can be interpreted as synchronized sales, i.e., several periods in the high-price regime followed by one period in the low-price regime. We also find cyclical equilibria with richer dynamics, for example, two periods in the high-price regime followed by one period in the low-price regime followed by three periods in the high-price regime then one period in the low-price regime and then repeating this pattern. Finally, we find some cyclical equilibria with one period in the high-price regime and several periods in the low-price regime.

5.1 The Model with $\alpha < 1$

Values and prices. Compared to the $\alpha = 1$ case analyzed above, the lower probability $\alpha < 1$ of finding an acceptable pair of shoes only affects the values of shoppers. The fashionistas' and sensible shoppers' values of *wearing* shoes, W_t^1 and W_t^0, are still defined by (5) and (1), respectively. The values of shopping, on the other hand, must be amended to account for the possibility of staying in the pool of shoppers for more than one period. Specifically, for sensible shoppers:

$$\begin{aligned} S_t^0 &= (1-\alpha)\,\beta S_{t+1}^0 + \alpha\left\{\gamma_t \beta S_{t+1}^0 + (1-\gamma_t)\left(W_t^0 - r_t^0\right)\right\} \\ &= \beta S_{t+1}^0, \end{aligned} \tag{25}$$

where the second equality uses the definition of the sensible shoppers' reservation price, $W_t^0 - r_t^0 = \beta S_{t+1}^0$. Equation (25) is essentially the same as equation (2) with the added possibility (with probability $1-\alpha$) of not finding a suitable pair of shoes and thus continuing shopping for at least one more period. The solution to (25) is $S_t^0 \equiv 0$ and $r_t^0 \equiv r^0 = \dfrac{v}{1-\beta\,(1-\delta)}$, as in the $\alpha = 1$ case.

normalization can be justified by thinking of a period as the time it takes for the fashionistas' shoes to become unfashionable. As such, the generalization we propose in this section is one that says that it takes more time for a fashionista to find a pair of shoes that she likes than it does for her to lose interest in that pair of shoes.

Turning to the fashionistas, their value of shopping is given by:

$$
\begin{aligned}
S_t^1 &= (1-\alpha)\,\beta S_{t+1}^1 + \alpha\left\{\gamma_t\left(W_t^1 - r_t^1\right) + (1-\gamma_t)\left(W_t^1 - r^0\right)\right\} \\
&= \beta S_{t+1}^1 + \alpha\,(1-\gamma_t)\left(r_t^1 - r^0\right). \qquad (26)
\end{aligned}
$$

Their reservation price is directly obtained from (5) and the definition of r_t^1: $W_t^1 - r_t^1 = \beta S_{t+1}^1$. Combining those two equations immediately yields $r_t^1 \equiv r^1 = 1$, again the same as in the $\alpha = 1$ case.

Consumer stocks. Following the steps we took in Section 2, we now derive the law of motion of s_t, the total stock of shoppers, and $s_t\varphi_t$, the stock of fashionistas among shoppers. As in the $\alpha = 1$ case, the latter is very simple: because all fashionistas' shoes go out of style with certainty after one period ($\delta_1 = 1$), all of the fashionistas shop in every period, so that $s_t\varphi_t \equiv \lambda$. The total stock of shoppers then evolves following:

$$
\begin{aligned}
s_{t+1} &= s_t\left\{1 - \alpha\varphi_t - \alpha\,(1-\varphi_t)\,(1-\gamma_t)\right\} + \left\{\lambda - s_t\,(1-\alpha)\,\varphi_t\right\} \\
&\qquad + \delta\left\{1 - \lambda - s_t\,(1-\varphi_t)\,(1-\alpha+\alpha\gamma_t)\right\} \\
&= s_t\,(1-\alpha+\alpha\gamma_t)\,(1-\delta) - \alpha\lambda\gamma_t\,(1-\delta) + \alpha\lambda + (1-\alpha\lambda)\,\delta, \qquad (27)
\end{aligned}
$$

where the second equality uses $s_t\varphi_t \equiv \lambda$. This law of motion is interpreted as follows. The stock of shoppers at the beginning of period $t+1$ is the sum of three terms (in curly brackets). The first term is the stock of shoppers who were present last period, less the $\alpha s_t\varphi_t$ fashionistas among them who found a suitable pair of shoes, less the $\alpha s_t\,(1-\varphi_t)\,(1-\gamma_t)$ sensible shoppers who found a pair of shoes at a low price (the high price being unacceptable to sensible shoppers). The second term reflects the inflow of fashionistas who had shoes in period t and all of whom returned to shopping in period $t+1$: those are all the λ fashionistas in the population, minus the $(1-\alpha)\,s_t\varphi_t$ who were already shopping in period t but failed to find shoes. Similarly, the last term in curly brackets reflects the inflow of sensible shoppers who had shoes in period t but whose shoes just wore out (which occurs with probability δ).

Stationary equilibria. The revenue obtained from posting any given price is still proportional to the price times the probability that a customer met at random will accept that price: it is simply scaled down to account for the lower probability of any given customer visiting the store. This scaling down does not affect the comparison between the revenue from posting the high price and that from posting the low price, implying that the stores' decisions about what price to post are still governed by the same rule as in the $\alpha = 1$ case, i.e. decision rule (11).

Given this, it is straightforward to show that, just as in the $\alpha = 1$ case, the model has a unique stationary equilibrium, which can take one of three forms depending on parameter values. The first type of stationary equilibrium is the one in which all firms post the high price ($\gamma = 1$) and the stock of shoppers is $s_h^* = 1$. This stationary equilibrium is consistent with the stores' decision rule (11) if and only if $s_h^* = 1 < \frac{\lambda}{r^0}$. In the second type of stationary equilibrium, all firms post the low price ($\gamma = 0$), which implies $s_\ell^* = \frac{\delta + \alpha\lambda(1-\delta)}{\alpha + \delta(1-\alpha)}$. Application of (11) shows that this type of equilibrium obtains if $s_\ell^* = \frac{\alpha\lambda + \delta(1-\alpha\lambda)}{\alpha + \delta(1-\alpha)} > \frac{\lambda}{r^0}$. Finally, the third type of equilibrium features price dispersion. The stock of shoppers and the fraction of stores posting the high price ($0 < \gamma^* < 1$) are jointly determined by the steady-state version of the law of motion of s_t (27) and the stores' indifference condition from (11):

$$s_\gamma^* = \frac{\alpha\lambda + \delta(1-\alpha\lambda) - \alpha\gamma\lambda(1-\delta)}{\alpha + \delta(1-\alpha) - \alpha\gamma(1-\delta)}, \qquad \text{with} \qquad \frac{\lambda(1-\beta(1-\delta))}{v} = s_\gamma^*.$$

This pair of equations defines a unique value $\gamma^\star \in (0,1)$ if and only if:

$$\frac{\delta + \alpha\lambda(1-\delta)}{\alpha + \delta(1-\alpha)} < \frac{\lambda}{r^0} < 1. \tag{28}$$

The conditions for any one type of equilibrium to prevail generalize conditions (14), (16), and (19) found in the $\alpha = 1$ case and, as before, these conditions partition the parameter space, so that the model still has a unique stationary equilibrium for any set of parameter values.

5.2 Nonstationary Equilibria

Because the stores' choices of which price to post are still governed by (11), the indifference condition $s_t = \frac{\lambda}{r^0}$ must hold for there to be price dispersion within any given period. If the indifference condition does not hold, either all stores post the high price ($\gamma_t = 1$), or they all post the low price ($\gamma_t = 0$). Then, (27) yields the following law of motion for the stock of shoppers:

$$s_{t+1} = F(s_t) := \begin{cases} s_t (1-\delta) + \delta & \text{if } s_t < \dfrac{\lambda}{r^0} \\ \\ s_t (1-\alpha)(1-\delta) + \delta + \alpha\lambda (1-\delta) & \text{if } s_t > \dfrac{\lambda}{r^0} \end{cases} \tag{29}$$

Existence, uniqueness and stability. If parameters are such that a high-price stationary equilibrium exists — that is, if $\frac{\lambda}{r^0} > 1$ — then, as can be readily seen from the stable dynamics governed by the top equation in (29), and as was already the case with $\alpha = 1$, the model's unique stationary equilibrium is globally stable and no stable nonstationary equilibrium can coexist with it. The same applies, *mutatis mutandis*, to the case in which a low-price stationary equilibrium exists, i.e. when $\frac{\lambda}{r^0} < \frac{\delta + \alpha\lambda(1-\delta)}{\alpha + \delta(1-\alpha)}$. Since we are interested in nonstationary equilibria, we exclude these cases from the subsequent analysis and assume that (28) holds. We now have the following result:

Proposition 1 *Suppose inequalities* (28) *hold. Then the model has a unique, globally stable equilibrium, which is periodic.*

The proof is in Appendix A and follows directly from the analysis of piecewise continuous difference equations in Keener (1980).[10]

Proposition 1 guarantees several things. First, it guarantees existence and uniqueness of a nonstationary equilibrium. Second, it ensures that the equilibrium is periodic (as opposed to chaotic), arguably a desirable property for a theory of periodic sales. Beyond that, however, there is nothing in Proposition 1 to help us characterize the actual dynamic behavior of our

[10]We thank Leo Kaas for this reference.

economy. Even though we now know that equilibrium is periodic, its periodicity may be very large and its dynamic behavior may look much more complex than that of a market with sales held at regular intervals. To gauge whether our model still offers a descriptively appealing theory of synchronized sales, we need to further characterize these periodic equilibria. We do so, at least partially, in the next paragraph.

Further equilibrium characterization. In the $\alpha = 1$ case, we established that the economy never stays in the low-price regime for two consecutive periods.[11] In other words, low-price spells are at most one period long, and therefore look like sales. We now establish a parallel, if slightly more complex equilibrium property that holds in the more general case $\alpha \leq 1$.

Proposition 2 *Suppose* (28) *holds, so that there exists a unique periodic equilibrium.*

1. *If* $\dfrac{\delta + \alpha\lambda(1-\delta) + \delta(1-\alpha\lambda)(1-\delta)}{\alpha + \delta(1-\alpha) + \delta(1-\alpha)(1-\delta)} \leq \dfrac{\lambda}{r^0} \leq 1$*, then the economy never spends two consecutive periods in the low-price regime and periodically spends two or more consecutive periods in the high-price regime.*

2. *If* $\dfrac{\delta + \alpha\lambda(1-\delta) + \delta(1-\alpha)(1-\delta)}{\alpha + \delta(1-\alpha) + \delta(1-\alpha)(1-\delta)} < \dfrac{\lambda}{r^0} < \dfrac{\delta + \alpha\lambda(1-\delta) + \delta(1-\alpha\lambda)(1-\delta)}{\alpha + \delta(1-\alpha) + \delta(1-\alpha)(1-\delta)}$*, then the economy oscillates between one period in the high-price regime and one period in the low-price regime.*

3. *If* $\dfrac{\delta + \alpha\lambda(1-\delta)}{\alpha + \delta(1-\alpha)} \leq \dfrac{\lambda}{r^0} \leq \dfrac{\delta + \alpha\lambda(1-\delta) + \delta(1-\alpha)(1-\delta)}{\alpha + \delta(1-\alpha) + \delta(1-\alpha)(1-\delta)}$*, then the economy never spends two consecutive periods in the high-price regime and periodically spends two or more consecutive periods in the low-price regime.*

The proof is in Appendix B. The three cases covered in the statement of Proposition 2 partition the set of parameter values for which (28) holds, i.e. the set of parameter values for which there exists a unique periodic equilibrium.

[11] Except of course if it stays there forever, i.e. if the only long-run equilibrium is the stationary low-price equilibrium. See Sections 3 and 4.

Case 1 in Proposition 2 can be described as the 'synchronized sales' case. Although the periodic equilibrium pattern can be fairly rich (as will be illustrated in simulations below), spells of low prices never last more than one period in that parameter configuration, while spells of high prices tend to last longer. Case 2 is the limiting '$T_h = 1, T_\ell = 1$' case, analogous to the first example analyzed at the beginning of Section 4. Case 3 is in a sense the opposite of case 1 and is more difficult to interpret. In this case, high prices are the exception rather than the rule: equilibrium prices are low most of the time, with occasional one-period spells of high prices. Further note that, while cases 1 and 2 both persist as possible equilibrium configurations as $\alpha \to 1$,[12] the set of parameter values under which case 3 arises collapses as $\alpha \to 1$.

Simulations. To illustrate some of the patterns that can occur in the market with $\alpha < 1$, we now present a set of simulations. We think of a period as one month. Fashionistas go shopping every month. We assume that sensible shoppers need a new pair of shoes every year on average, so that $\delta = 1/12$. We set $\alpha = 0.5$ (it takes a shopper two months on average to find a suitable pair of shoes), and $\lambda = 0.5$ (the population is evenly divided into fashionistas and sensible shoppers). We let r^0 take different values within the bounds that are consistent with periodic equilibria, which are given by condition (28). In this case the condition imposes $r^0 \in [0.5, 0.8667]$. Figures 3 to 7 show examples of simulated time paths of the number of shoppers, s_t. In all simulations the economy starts with a number of shoppers $s_0 = 0.46$, except for Figure 7, where for aesthetic reasons the starting value was set to $s_0 = 0.68$. All simulations run for 48 periods.

Figure 3 is an example that resembles the $\alpha = 1$ case. The market converges to a $(T_h, 1)$ price cycle, with $T_h = 5$ (there is a sale every sixth month). The duration of a typical high-price spell, T_h, increases in similar examples as r^0 gets closer to the lower bound of the set of values consistent with periodic equilibria defined by (28).

[12]Although, as established in Section 4, the set of possible patterns under case 1 becomes less rich as $\alpha \to 1$ — see below for simulations.

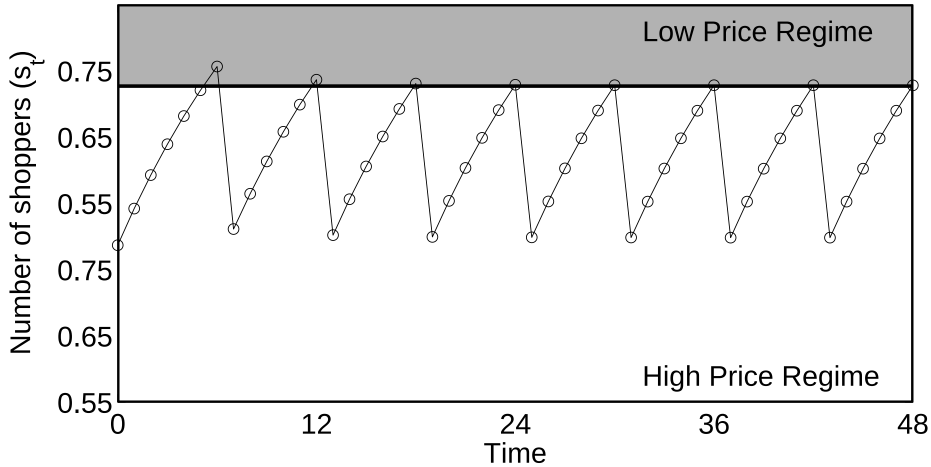

Figure 3: Simulation: $r^0 = 0.6341$

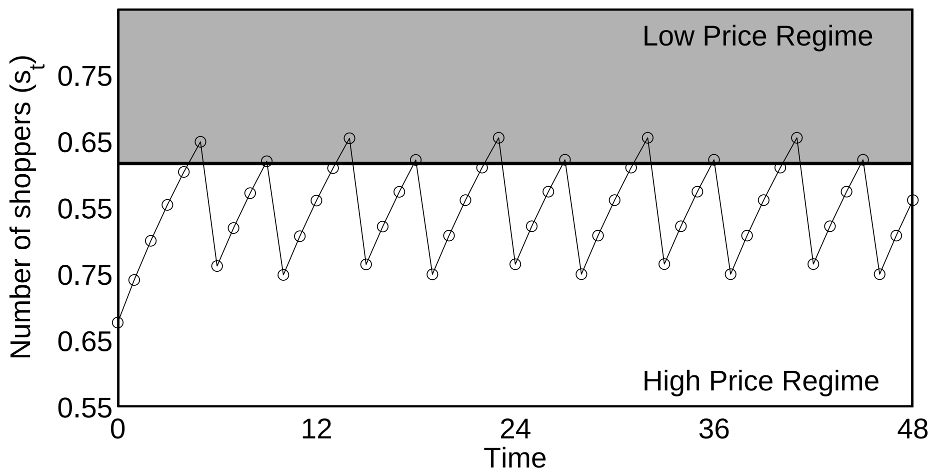

Figure 4: Simulation: $r^0 = 0.6817$

Figure 4 shows a more complex pattern, where the economy converges to a cycle of three periods of high prices, followed by one period of sales, followed by four periods of high prices, followed by one period of sales, etc. Other examples of this type of cyclical pattern, which can be concisely described as a $(T_h, 1, T_h + 1, 1)$ cycle, arise for different values of r^0, with the value of T_h again increasing as r^0 moves closer to the lower bound of (28)

To further illustrate the richness of the set of possible cyclical patterns, Figures 5 and 6 show two intermediate examples of a $(T_h, 1, T_h + 1, 1, T_h + 1, 1)$ cycle (Figure 5), and a $(T_h, 1, T_h, 1, T_h + 1, 1)$ cycle (Figure 6), with $T_h = 3$ in both instances.

Finally, Figure 7 shows an example of case 3 in Proposition 2 in which the economy is

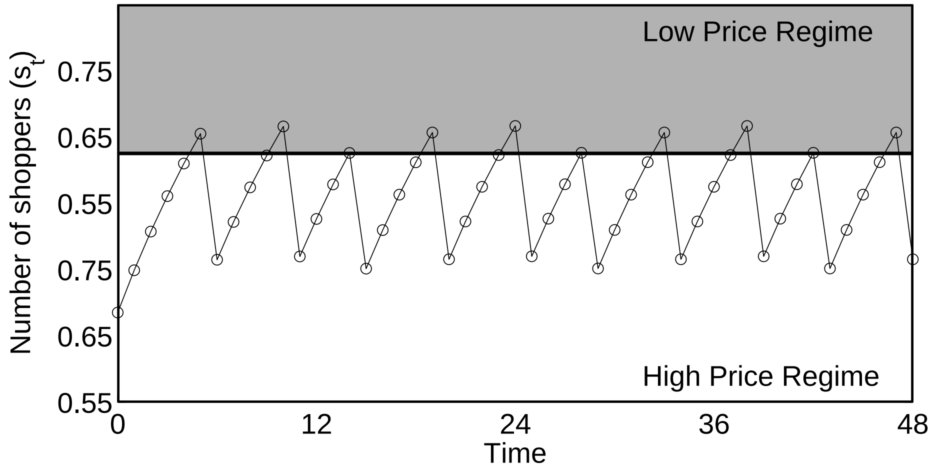

Figure 5: Simulation: $r^0 = 0.6778$

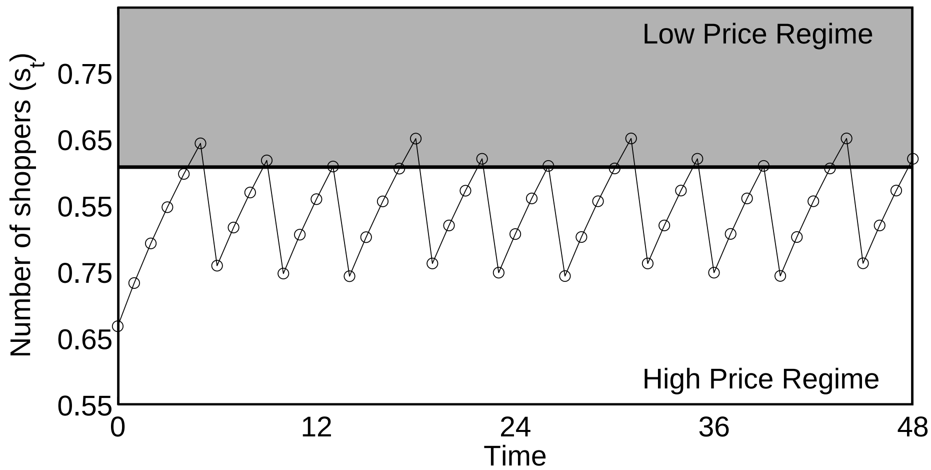

Figure 6: Simulation: $r^0 = 0.6857$

"normally" (i.e. in three periods out of four) in the low-price regime but experiences a period of high prices every third period.[13]

As these simulations show, the model with $\alpha < 1$ delivers the synchronized salesthat we found in our basic model with $\alpha = 1$ as well as a variety of other, more complex cycles.

[13]While Figure 7 shows an example of a simple $(1, T_\ell)$ cycle, it is possible to obtain more complex patterns (e.g. a $(1, T_\ell, 1, T_\ell + 1)$ cycle) in simulations. Such examples are available on request.

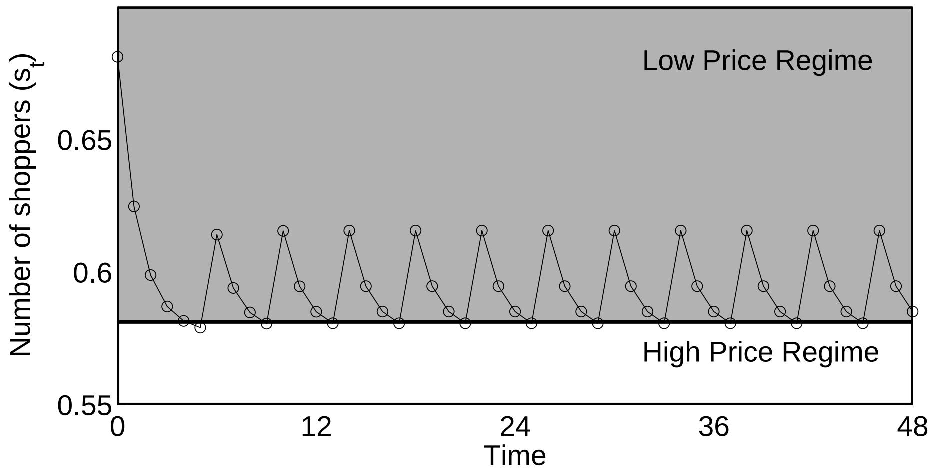

Figure 7: Simulation: $r^0 = 0.8604$

6 Concluding Remarks

In this paper, we propose a new theory of synchronized sales. We do this by proving the existence of nonstationary periodic equilibria in a model of product market search. We show that these periodic equilibria coexist with the stationary equilibria of the model that exhibit price dispersion. More precisely, consider any parameter configuration that leads to a stationary equilibrium in which a fraction $\gamma \in (0,1)$ of the sellers post a price that only the high-demand consumers accept while the other sellers post a lower price that both consumer types accept. Then there is a corresponding nonstationary equilibrium in which the following pattern repeats – all sellers post a high price for T_h periods and then post a low price for one period. Equilibria of this $(T_h, 1)$ type exist for $T_h = 1, 2, 3, \ldots$ and these equilibria are ordered in the sense that the higher is T_h, the higher is the value of γ in the corresponding stationary equilibrium. Finally, these periodic equilibria, which are new to the search literature, are globally dynamically stable, whereas the corresponding stationary dispersion equilibria are dynamically unstable.

An equilibrium in which all sellers post a high price for several periods and then post a low price for one period before returning to the high-price regime is exactly what one means by "synchronized sales," and, depending on parameters, our theory can generate sales at arbitrary frequencies. These sales are driven by self-generating changes in the composition of consumer types in the shopper pool and, in contrast to earlier models of sales with periodic cycles, these sales arise in a model in which firms do not have market power.

References

[1] Aguirregabiria, Victor (1999) "The Dynamics of Markups and Inventories in Retailing Firms," *Review of Economic Studies*, 66(2), 275-308.

[2] Albrecht, James and Bo Axell (1984) "An Equilibrium Model of Search Unemployment," *Journal of Political Economy*, 92(5) 824-840.

[3] Ariga, Kenn, Kenji Matsui, and Makoto Watanabe (2010), "Hot and Spicy: Ups and Downs on the Price Floor and Ceiling at Japanese Supermarkets," mimeo.

[4] Burdett, Kenneth and Melvyn Coles (1998) "Separation Cycles," *Journal of Economic Dynamics and Control*, 1998, 22(7) 1069–1090.

[5] Coase, Ronald (1972) "Durability and Monopoly," *Journal of Law and Economics*, 15(1), 143-49.

[6] Conlisk, John, Eitan Gerstner and Joel Sobel (1984) "Cyclic Pricing by a Durable Goods Monopolist," *Quarterly Journal of Economics*, 99(3) 489-505.

[7] Diamond, Peter (1987) "Consumer Differences and Prices in a Search Model," *Quarterly Journal of Economics*, 102(2) 429-436.

[8] Diamond, Peter and Drew Fudenberg (1989) "Rational Expectations Business Cycles in Search Equilibrium," *Journal of Political Economy*, 97(3) 606-619.

[9] Dutta, Prajit, Alexander Matros, and Jörgen Weibull (2007) "Long-Run Price Competition," *RAND Journal of Economics*, 38(2), 291-313.

[10] Feenstra, Robert and Matthew Shapiro (2003) "High Frequency Substitution and the Measurement of Price Indexes," in Robert Feenstra and Matthew Shapiro, eds., Scanner Data and Price Indexes, Univ. of Chicago and NBER, 123-146.

[11] Fershtman, Chaim and Arthur Fishman (1992) "Price Cycles and Booms: Dynamic Search Equilibrium," *American Economic Review*, 82(5) 122-1233.

[12] Gaumont, Damien, Martin Schindler, and Randall Wright (2006) "Alternative Theories of Wage Dispersion," *European Economic Review*, 51(4) 831-848.

[13] Keener, James P. (1980) "Chaotic Behavior in Piecewise Continuous Difference Equations," *Transactions of the American Mathematical Society*, 261(2), 589-604.

[14] Klenow, Peter and Benjamin Malin (2011) "Microeconomic Evidence on Price-Setting," in Benjamin Friedman and Michael Woodford, eds., Handbook of Monetary Economics, Amsterdam: Elsevier.

[15] Lazear, Edward (1986) "Retail Pricing and Clearance Sales," *American Economic Review*, 76(1), 14-32.

[16] Nakamura, Emi and Jón Steinsson (2008) "Five Facts About Prices: A Reevaluation of Menu Cost Models," *Quarterly Journal of Economics*, 123(4) 1415-1464.

[17] Nakamura, Emi and Jón Steinsson (2011) "Price Setting in Forward-Looking Customer Markets," *Journal of Monetary Economics*, 58(3), 220-233.

[18] Pesendorfer, Martin (2002) "Retail Sales. A Study of Pricing Behavior in Supermarkets," *Journal of Business,* 75(1), 33-66.

[19] Shi, Shouyong (2011) "Customer Relationship and Sales," mimeo.

[20] Slade, Margaret (1998) "Optimal Pricing with Costly Adjustment and Persistent Effects: Empirical Evidence from Retail-Grocery Prices," *Review of Economic Studies*, 65(1), 87-105.

[21] Sobel, Joel (1984) "The Timing of Sales," *Review of Economic Studies*, 51(3) 353-368.

[22] Varian, Hal (1980) "A Model of Sales," *American Economic Review*, 70(4), 651-659.

A Proof of Proposition 1

The model's dynamic behavior is governed by the piecewise linear difference equation (29), which belongs to the class of difference equations analyzed by Keener (1980). Following Keener's notation, let $a = \lim_{s\downarrow\lambda/r^0} F(s) = (1-\delta)\lambda\left[\alpha + (1-\alpha)/r^0\right] + \delta$ and $b = \lim_{s\uparrow\lambda/r^0} F(s) = (1-\delta)\lambda/r^0 + \delta$. The assumption that (28) holds immediately implies that $a < \dfrac{\lambda}{r^0}$ and $b > \dfrac{\lambda}{r^0}$, so that $F(a) = (1-\delta)a + \delta > a$ and $F(b) = (1-\alpha)(1-\delta)b + \delta + \alpha\lambda(1-\delta) < b$. Simple algebra further shows that $F(a) = F(b) + \alpha\delta(1-\delta)(1-\lambda) > F(b)$, which establishes that the difference equation (29) satisfies properties F-I through F-IV in Keener (1980). Moreover, wherever differentiable, F has a slope that is strictly less than 1.

The rest of this proof mainly consists of showing that the set of series of preimages of $\dfrac{\lambda}{r^0}$ that lie in the interval $(a, F(a)]$ (Ω in Keener's notation, see Definition 3.3 (b) in Keener, 1980), is finite, so that Lemma 3.6 and Corollary 3.16 (b) apply and prove the proposition. To that end, let us consider the following three cases, which cover the whole parameter space under condition (28):

Case 1: $F(a) > \dfrac{\lambda}{r^0} > F(b)$**.** In this case, Lemma 3.2 in Keener (1980) applies and immediately proves the proposition.

Case 2: $\dfrac{\lambda}{r^0} > F(a) > F(b)$**.** In this case the interval $(a, F(a)]$ lies entirely below $\dfrac{\lambda}{r^0}$, so that preimages of $\dfrac{\lambda}{r^0}$ greater than $\dfrac{\lambda}{r^0}$ cannot be in the set Ω and we only need to focus on such preimages that are below $\dfrac{\lambda}{r^0}$. Now the relevant dynamic equation for values of s below $\dfrac{\lambda}{r^0}$ is the top equation in (29). Working that equation backward, preimages of $\dfrac{\lambda}{r^0}$ all have the form $1 - \dfrac{1-\lambda/r^0}{(1-\delta)^n}$, with n a positive integer. Clearly there can only be a finite number of such terms within $(a, F(a)]$, which proves that Ω is finite in this case.

Case 3: $F(a) > F(b) > \frac{\lambda}{r^0}$. In this case the interval $(a, F(a)]$ straddles $\frac{\lambda}{r^0}$, the point of discontinuity of F. However, consider a potential preimage of $\frac{\lambda}{r^0}$ (say σ) in $\left(a, \frac{\lambda}{r^0}\right)$. Then, because F is strictly increasing over $\left(a, \frac{\lambda}{r^0}\right)$, it follows that $F(\sigma) > F(a)$, so that $F(\sigma) \notin (a, F(a)]$, implying that σ is not in Ω. We can thus focus on preimages that lie in $\left(\frac{\lambda}{r^0}, F(a)\right]$, an interval which is entirely above $\frac{\lambda}{r^0}$. The same reasoning as in Case 2 then applies, only now involving the bottom equation in (29) rather than the top one.

B Proof of Proposition 2

The model's dynamic behavior is governed by the piecewise linear difference equation (29). Consider a date t_c at which the economy crosses from the high into the low-price regime, i.e. a date such that $s_{t_c} < \frac{\lambda}{r^0}$ and $s_{t_c+1} = F(s_{t_c}) > \frac{\lambda}{r^0}$. We now ask whether $s_{t_c+2} = F^2(s_{t_c})$ is greater or smaller than $\frac{\lambda}{r^0}$.

Because $s_{t_c} < \frac{\lambda}{r^0}$, $s_{t_c+1} = F(s_{t_c}) > \frac{\lambda}{r^0}$, and $F(\cdot)$ is increasing within each regime, $s_{t_c+2} = F^2(s_{t_c}) < F^2\left(\frac{\lambda}{r^0}^{-}\right)$.[14] Here, $F^2\left(\frac{\lambda}{r^0}^{-}\right) = (1-\delta)^2(1-\alpha)\frac{\lambda}{r^0} + \delta(1-\delta)(1-\alpha) + \delta + \alpha\lambda(1-\delta)$. It is then straightforward to show that:

$$F^2\left(\frac{\lambda}{r^0}^{-}\right) < \frac{\lambda}{r^0} \iff \frac{\lambda}{r^0} > \frac{\delta + \alpha\lambda(1-\delta) + \delta(1-\alpha)(1-\delta)}{\alpha + \delta(1-\alpha) + \delta(1-\alpha)(1-\delta)}. \tag{30}$$

This establishes that, if inequality (30) holds, then the economy cannot spend two consecutive periods in the low-price regime: even if it crosses into the low-price regime "to the furthest possible extent," inequality (31) implies that it will revert to the high-price regime after one period.

Conversely, suppose the economy crosses from the low- into the high-price regime at date t_c, i.e. $s_{t_c} > \frac{\lambda}{r^0}$ and $s_{t_c+1} = F(s_{t_c}) < \frac{\lambda}{r^0}$. We examine whether $s_{t_c+2} = F^2(s_{t_c})$ is greater or smaller than $\frac{\lambda}{r^0}$. In this case, $s_{t_c} > \frac{\lambda}{r^0}$ implies $s_{t_c+2} = F^2(s_{t_c}) > F^2\left(\frac{\lambda}{r^0}^{+}\right)$.[15] Here,

[14] Where $F\left(\frac{\lambda}{r^0}^{-}\right) := \lim_{s\uparrow\lambda/r^0} F(s)$.

[15] Where $F\left(\frac{\lambda}{r^0}^{+}\right) := \lim_{s\downarrow\lambda/r^0} F(s)$.

$F^2\left(\frac{\lambda}{r^0}^+\right) = (1-\delta)^2(1-\alpha)\frac{\lambda}{r^0} + \delta(1-\delta) + \delta + \alpha\lambda(1-\delta)^2$. Then:

$$F^2\left(\frac{\lambda}{r^0}^+\right) > \frac{\lambda}{r^0} \iff \frac{\lambda}{r^0} < \frac{\delta + \alpha\lambda(1-\delta) + \delta(1-\alpha\lambda)(1-\delta)}{\alpha + \delta(1-\alpha) + \delta(1-\alpha)(1-\delta)}, \tag{31}$$

which establishes that, if inequality (31) holds, the economy cannot spend two consecutive periods in the high-price regime.

The proposition is then proven by noticing that case 2 in the statement corresponds to parameter combinations such that both inequalities (30) and (31) hold simultaneously, so that the economy never spends more than one period in any regime, leaving the two-period bang-bang equilibrium as the sole possibility.