A CHARACTERIZATION FOR THE SPHERICAL SCORING RULE

ABSTRACT. Strictly proper scoring rules have been studied widely in statistical decision theory and recently in experimental economics because of their ability to encourage assessors to honestly provide their true subjective probabilities. In this article, we study the spherical scoring rule by analytically examining some of its properties and providing some new geometric interpretations for this rule. Moreover, we state a theorem which provides an axiomatic characterization for the spherical scoring rule. The objective of this analysis is to provide a better understanding of one of the most commonly available scoring rules, which could aid decision makers in the selection of an appropriate tool for evaluating and assessing probabilistic forecasts.

KEY WORDS: decision theory, probability assessment, probability elicitation, proportionality, strictly proper scoring rule

AMS SUBJECT CLASSIFICATION: Primary 62C99, Secondary 62A01, 90B50, 91B06
JEL CLASSIFICATION NUMBER: C44, C89

1. INTRODUCTION

One of the most important processes in the analysis of many decision models is the method by which decision makers are able to quantify uncertainty. In most cases, decision makers do this either by expressing their own beliefs through the use of probabilities or by eliciting this information from experts whom they feel are more adequate to make such judgments.

With such a personalistic view of probability, methods by which decision makers could induce experts to provide ‘good’ assessments had to be established. In this regard, several
means of eliciting probabilities have been suggested in the literature. One of the most popular means of achieving this is through the use of scoring rules.

Scoring rules are functions that assign a ‘reward’ for a probability assessment given an outcome of an uncertain event. From an \textit{ex post} point of view, scoring rules are very useful because they can provide a means by which assessors can be evaluated with respect to their predictive abilities. In addition, scores generated from this mechanism can also provide feedback for assessors to help them improve themselves (De Groot and Fienberg, 1983; Schervish, 1989; Johnstone, 2007).

On the other hand, scoring rules can serve another purpose in an \textit{ex ante} sense. Through the use of incentive schemes based on scoring rules, decision makers are able to elicit honest probabilities from an assessor, who would reasonably want to maximize his utility derived from such a scheme. In this context, this scoring rule becomes a tool for probability elicitation (Hendrickson and Buehler, 1972; Kadane and Winkler, 1988).

Several scoring rules have emerged in the literature. Three of the most popular and well-studied rules are the quadratic, logarithmic, and spherical scoring rules.

The first characterization for any scoring rule was made in the late 1960s by Shuford et al. (1966) for the logarithmic scoring rule. More recently, Selten (1998) provided a formal axiomatic characterization for the quadratic scoring rule using a result which was earlier discovered by Savage (1971). The spherical scoring rule first introduced by Roby (1965) in the context of psychological testing has several interesting geometric properties and has a strong connection to the statistical notion of surprise (Good, 1971). Statistical texts commonly cite the existence and use of the spherical rule but do not provide a comparable characterization for this rule.

The main purpose of this article is to provide an axiomatic characterization for the spherical scoring rule. It will be shown that this rule is the only strictly proper, order invariant
scoring rule which satisfies a certain proportionality requirement.

The remainder of the article proceeds as follows. The next section lays out the basic framework, definitions and notations. In Section 3, the spherical rule is presented in some detail and several geometric insights are developed. Section 4 contains the main result of this article, which is the presentation of the relevant axioms and the axiomatic characterization theorem for the spherical scoring rule. It is then followed by some discussion on the different axioms and their relationship with other rules on Section 5. Section 6 concludes the article.

2. SOME FOUNDATIONS

In this section, we describe briefly relevant concepts from the literature on scoring rules and state some of the notations that will be used throughout the article. For a more detailed discussion about scoring rules, the reader is referred to Lad (1996), Winkler (1996), or Bernardo and Smith (2000).

Consider the set \( \Omega = \{E_1, E_2, \ldots \} \) representing a countable collection of mutually exclusive and exhaustive outcomes of the experiment of interest with at least 2 elements. Moreover, let \( \mathcal{F} \) be a \( \sigma \)-field of subsets of \( \Omega \) and let \( \mathcal{P} \) be the class of probability measures on the space \( (\Omega, \mathcal{F}) \). Note that it makes sense to view the elements of \( \mathcal{P} \) as probability measures since we assume that our assessors are coherent, that is, their probability assessments on the events on \( \Omega \) are non-negative and sum up to 1 so as to avoid Dutch book scenarios.

From a geometric point of view, if we assume that the set \( \Omega \) contains \( n \) elements then we can represent the measure \( R \in \mathcal{P} \) as some point \( r \) with \( i \)th element \( r_i \). The collection of all these points \( r \) representing some measure \( R \) in \( \mathcal{P} \) can be visualized as the probability simplex

\[
\Delta_n = \left\{ r \in \mathbb{R}^n : \sum_{j=1}^{n} r_j = 1 \right\}.
\]
Given this setup, we can now formally define a scoring rule. A scoring rule \( S \) is a function that assigns every element of \((\mathcal{P}, \Omega)\) a value in the extended real number line. This function assigns a score to every particular forecast that an assessor provides and the actual outcome of the experiment. In particular, if an assessor provides a forecast \( R \in \mathcal{P} \) and \( E \in \Omega \) occurs then the forecaster receives a score of \( S_i(r) \).

In an *ex post* sense, the score received by the assessor can be used to generate a reward scheme such that it is beneficial for him to provide an honest assessment in order for him to maximize his score. In most cases, it is assumed that the utility of the assessor is linear with respect to this score. When this does not hold, modifications to the scoring rules can be made to take into account the repercussions of nonlinearity (e.g., see Winkler and Murphy, 1970).

From an *ex ante* point of view, scoring rules are also useful. Suppose that the probability measure \( P \) represents an assessor’s true beliefs and suppose that he chooses to report \( R \in \mathcal{P} \). Then his expected score is given by:

\[
\mathbb{E}_P(S(R)) = \int \sum_i S_i(R) dP(\omega) = \sum_i p_i S_i(r).
\]

Alternatively, we can also analyze his choice to report \( R \) in terms of his expected loss:

\[
L(r, p) = \mathbb{E}_P(S(p)) - \mathbb{E}_P(S(r)) = \sum_i p_i S_i(p) - \sum_i p_i S_i(r).
\]

Formally, we say that a scoring rule \( S \) is strictly proper if and only if

\[
\mathbb{E}_P(S(p)) > \mathbb{E}_P(S(r)) \tag{1}
\]

for all \( p \neq r \). Using the expected loss framework, a strictly proper scoring rule implies that \( L(p, r) \) is minimized (i.e., attains a value of 0) only when \( p = r \). Thus, an assessor who wants to maximize expected score has an incentive to report his true beliefs, i.e., to set \( r = p \).
In the literature, the three most commonly cited strictly proper scoring rules are:

- **Quadratic**: \( S_i(p) = 2p_i - ||p||^2 \),
- **Logarithmic**: \( S_i(p) = \log p_i \),
- **Spherical**: \( S_i(p) = \frac{p_i}{||p||} \),

where \( ||p|| \) refers to the 2-norm of the vector associated with the point \( p \) given by

\[
||p|| = \left( \sum p_j^2 \right)^{1/2}.
\]

From these strictly proper scoring rules, we can create infinitely many more strictly proper scoring rules by taking positive affine transformation of the said rules. That is, if \( S \) is a strictly proper scoring rules, then so is \( \alpha S + \beta \), for \( \alpha > 0 \). Such transformations are useful in converting the range of possible values for these scoring rules into other ranges which may be more appropriate to the decision context.

At times, these transformations can be used to attain some sense of consistency among scoring rules by creating some notion of ‘standardization’ (e.g., Winkler, 1969). A scoring rule is said to be standardized with respect to an event \( i \) if:

\[
S_i(e_{jn}) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}
\]

where \( e_{jn} \) is the \( j \)th natural basis vector of \( \mathbb{R}^n \). An example of which is the spherical rule.²

3. THE SPHERICAL SCORING RULE

In this section, we begin our investigation of the spherical scoring rule by examining some of its properties from an analytic and geometric point of view.

Traditionally, the name for this rule has been attributed to the fact that every assessment \( p \) can be mapped into an \((n - 1)\)-sphere (with radius \( ||p|| \)) through the mapping of the point
$(S_1(p), \ldots, S_n(p))$ in $\mathbb{R}^n$ space. In addition, if we consider the contours of this rule on the simplex $\Delta_n$, these form concentric $(n-1)$-spheres around the center of the simplex.

Geometrically, these contours represent a locus of points for which the spherical score attains the same expected score. In particular,

$$E_p(S(p)) = \sum_i p_i \frac{r_i}{||p||} = ||p||. \tag{3}$$

This shows that the expected score for reporting honestly is simply the (Euclidean) length of the vector associated with the point $p$ representing the assessment $P \in \mathcal{P}$. A consequence of this is that not all honest assessments get the same expected score. Points in the center get smaller expected scores than points near the corners of the simplex.

Given $p, r \in \Delta_n$, we can re-express $E_p(S(r))$ as follows:

$$E_p(S(r)) = \sum_i p_i \frac{r_i}{||r||}$$

$$= ||p|| \sum_i \frac{p_i r_i}{||r||}$$

$$= ||p|| \frac{||r||}{||p||} \cos \theta,$$

where $\theta$ represents the interior ‘angle’ between the vectors $r$ and $p$. This implies two things. First, the expected score is related to the angle of the deviation. Second, it reinforces the fact that the spherical rule is strictly proper since the score is maximized only if the angle between the two vectors is equal to zero.

Geometrically, we can see from Figure 1 that the expected score of $r$ given that the true belief is $p$ can be interpreted as the norm of an orthogonal projection of a vector of the same size (norm) as that of $p$ but in the direction of $r$. This supports the notion that the angular deviation is a sufficient statistic (when taken together with $p$) for the expected score $E_p(S(r))$. Moreover, this angular deviation can be used to represent a measure of ‘distance’ between the reported forecast and the true belief since the expected loss can be measured from the angle that is made by this vector from the vector $p$. 
Another geometric property that the spherical rule has is symmetry. We say that a scoring rule $S$ is symmetric if $\exists$ a function $f$ over $\Delta_n$ such that

$$S_k(r) = f(r_k, r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n)$$

for $k = 1, 2, \ldots, n$. Since the contours are concentric about the center, we can see that if we drew a plane through the simplex passing through one of the corners and the orthocenter, it would produce mirror images on both sides. Lad (1996, pp. 348–349), provides an illustration of this property for the spherical rule. Note however that scoring rules need not be symmetric in general. Winkler (1994) gives a detailed discussion on how general families of asymmetric rules can be created and used in practice.

4. AN AXIOMATIZATION

This section presents a set of properties which will distinguish the spherical scoring rule from any other scoring rule. For this purpose, we state and discuss three axioms, which will be used in the characterization theorem for the spherical scoring rule:
AXIOM 1: (Invariance) Let \( r, \vartheta(r) \in \Delta_n \) be two admissible assessments such that the elements of \( \vartheta(r) \) are mapped from the elements of \( r \) under the permutation operator \( \vartheta \). In this case, we say that \( \forall i \)

\[ S_i(r) = S_{im_{\vartheta}(i)}(\vartheta(r)), \]

where \( im_{\vartheta}(i) \) is the corresponding element of the \( i \)th element of \( r \) under the permutation operator \( \vartheta \).

AXIOM 2: (Strict Propriety) \( S \) is strictly proper.

AXIOM 3: (Proportionality) For any \( p, r \in \Delta_n \), the expected loss of \( p \) given \( r \) is related proportionally to the expected loss of \( r \) given \( p \) in the following manner:

\[ L(p, r) = \frac{|r|}{|p|}L(r, p). \]

Axiom 1 requires the scoring rule to maintain the same functional form regardless of the labeling of the events. As for Axiom 2, when decision makers elicit probabilities, strictly proper scoring rules are generally preferred although a non-proper scoring rule such as the truncated logarithmic rule is sometimes used. Though the first two axioms reduce the set of possible scoring rules to be considered, all the scoring rules mentioned in Section 2 satisfy the first two axioms. However, the inclusion of Axiom 3 characterizes the spherical scoring rule up to a certain transformation.

The idea behind Axiom 3 is the following: Consider an assessor who has two distinct assessments \( p \) and \( r \). If one of the two assessments truly represents his true belief, then he expects to incur a loss if he reports the other. The main question that proportionality answers is what the relationship of \( L(p, r) \) and \( L(r, p) \) should be? In this case, Axiom 3 states that they should be related to quantities representing the norms of the vectors associated with \( p \) and \( r \).

Recalling that the norm of a vector represents the expected score for truthful reporting, Figure 2 illustrates how a sense of ‘proportionality’ is achieved in the decomposition of this
expected score. The similar triangles in this figure clearly shows that the losses incurred by reversing two assessments are proportional to what they expect to achieve by reporting truthfully. Thus, the two losses are equal in relative terms. That is, for the spherical rule it is a fixed proportion of what they expect to achieve by foregoing an assessment in favor of the other.

Another interesting consequence of this axiom is that under the spherical scoring rule, the comparison of losses takes into account the informativeness of the assessments. This is because the norm of a vector $p$ measures to some extent the ‘sharpness’ of this assessment. As $p$ moves away from the center of the simplex towards its edges, the value of the norm increases and a prediction is viewed to be more informative. From a practical viewpoint, it seems reasonable that when faced with $n$ events which one has complete ignorance over, most people by the principle of insufficient reason would tend to defer to a uniform distribution rather than making a categorical forecast.

Using these axioms, we have the following lemma:

**Lemma 1** The spherical scoring rule is the only standardized scoring rule that satisfies Axioms 1–3.
As an extension, we can also say that the spherical rule is the only standardized scoring rule that is invariant, strictly proper, and satisfies the following relationship for any two distinct points \( p, r \):

\[
\frac{L(p, r)}{L(r, p)} = \frac{\mathbb{E}_p(S(r))}{\mathbb{E}_p(S(p))}.
\]

(4)

This states that among standardized rules satisfying Axioms 1 and 2, it is the only one in which the loss ratio is equal to the ratio of expected gains.

However, this characterization is limited because it deals only with a certain subset of scoring rules. What is interesting is that using these same axioms, we have the following result:

**THEOREM 1** A scoring rule satisfies Axioms 1 – 3 if and only if it is a positive affine transformation of the spherical scoring rule.

The beauty of this result is that it applies to all scoring rules. The reason why we are able to attain this result is that Axioms 1–3 imply a boundedness condition on \( S \). This means that the range of values for \( S \) is some interval \([a, b]\), which can be transformed to \([0, 1]\) through some positive affine transformation. But by Lemma 1, this would force \( S \) to be related to the spherical rule because it is the only standardized rule that satisfies all three requirements. Therefore, this provides a characterization for the spherical scoring rule (up to positive affine transformations). And with this result, we have established a set of properties which distinguishes it from other scoring rules.

5. DISCUSSION

We now discuss the relevance and implications of the three axioms.

The first axiom helps us ignore the manner by which we label events and implies a symmetry condition: the functional form of each \( S_i \) is equivalent to \( S_j \) for \( j \neq i \). In theory, scoring
rules could have different forms for each \( i \), but invariance prevents this from happening.

Axiom 2, the most well-studied property of scoring rules, can be treated in ways other than condition (1). For example, instead of expected scores, we talk about expected losses and treat strict propriety as the notion of incentive compatibility in a loss minimization framework. Hendrickson and Buehler (1971) provides a necessary and sufficient condition for a scoring rule to be strictly proper.

Axiom 3 is the most distinctive one in the sense that it pins down the spherical scoring rule among all strictly proper, invariant rules. By itself, it is clearly not sufficient. Take the simple case of a constant scoring rule that assigns the same value to whatever outcome may occur. This satisfies Axiom 3 but not Axiom 2. For standardized rules, note that condition (4) can be used in lieu of Axiom 3, but unfortunately, this does not hold in the general setting since no scoring rule family (i.e., up to positive affine transformations) satisfies this condition. The reason being is that loss ratio is invariant under such transformations while the ratio of expected scores is not.

Now if we were to change this third requirement, we can actually generate the two other scoring rules mentioned. Three alternatives to Axiom 3 are Axioms 3a, 3b, and 3c.

AXIOM 3a: (Locality) For any \( n \), the scoring rule \( S \) satisfies the condition that \( S_i(\mathbf{r}) \) relies solely on the assessment \( \mathbf{r}_i \) made for event \( i \).

AXIOM 3b: (Neutrality) For any \( p, \mathbf{r} \in \Delta_n \), the expected loss of \( p \) given \( \mathbf{r} \) is equal to the expected loss of \( \mathbf{r} \) given \( p \), that is,

\[
L(p, \mathbf{r}) = L(\mathbf{r}, p).
\]

AXIOM 3c: (Loss as a Function of Discrepancy) For any assessment \( p, \mathbf{r} \in \Delta_n \), the expected loss can be expressed as a function of the discrepancy \( p - \mathbf{r} \), that is, \( \exists \) a function \( H \) such that

\[
L(p, \mathbf{r}) = H(p - \mathbf{r}).
\]
THEOREM 2 Other Characterization Theorems.

(a) [Shuford et al.] A scoring rule satisfies Axioms 1, 2, and 3a if and only if it is a positive affine transformation of the logarithmic scoring rule.

(b) [Savage] A scoring rule satisfies Axioms 1, 2, and, 3b if and only if it is a positive affine transformation of a quadratic scoring rule.

(c) [Savage] Axiom 3b can be replaced by Axiom 3c in (b).

In order to characterize the logarithmic scoring rule, we can use the notion of locality. What this requires is that the score should depend solely on the probability of the event that occurs and nothing else. The intuitive appeal of this axiom to some statisticians is that the locality principle is somewhat analogous to the likelihood principle, in the sense that only the assessment for the event that occurs should matter. Also, the expected score for this rule is closely related to Shannon’s measure of entropy. A practical problem with this rule, however, is the potential to attain a score of $-\infty$. In instances where scores are aggregated over time or analyzed over a series of experiments, a scoring system that generates $-\infty$ may be undesirable.

For the characterization of the quadratic rule, Selten (1998) uses four axioms – invariance, elongation invariance, strict propriety, and neutrality.

Selten’s Elongation Invariance Axiom states that if an impossible outcome is added then the score should remain the same. The inclusion of this axiom was used to provide a solution to the case when an unexpected outcome suddenly occurs. In this article, we did not consider this because we can always create an event $E_{n+1}$ from the start which represents cases that are not covered by events 1 to $n$. The inclusion of this in $\Omega$ is viable because countability is retained and the definition of a probability measure would force it to be assigned a coherent probability of 0.

On the other hand, the neutrality property was discovered by Savage (1971) together with the discrepancy property
It is only fair to require that this loss function is ‘neutral’ in the sense that it treats both theories equally. If $p$ is wrong and $q$ is right then $p$ should be considered to be as far from the truth as $q$ in the opposite case that $q$ is wrong and $p$ is right.

A scoring rule should not be prejudiced in favor of one of both theories in contest between $p$ and $q$. Therefore, the neutrality axiom is a natural requirement to be imposed on a reasonable scoring rule. (Selten, 1998, p. 54)

In some cases, such an argument could hold. But sometimes the treatment of the two theories may not be equal. Consider the case where there are three possible outcomes and an assessor has two assessments in mind, namely, $(1, 0, 0)$ and $(1/3, 1/3, 1/3)$. It seems plausible that for some instances a greater loss should be given to one compared to the other given that he knows which one is actually right. Suppose he knows decisively that event 1 is going to occur and he chooses to report an ‘ignorant’ assessment, then in some cases, it should be reasonable to accept that this could incur a different loss than the case where he had the ‘ignorant’ belief and chose to provide the categorical forecast.

The proportionality axiom provides an alternative representation of losses in the sense that the loss that one should incur should be dependent on the quality of information that one has. Choosing to lie when one has valuable (in this case, sharp) information should incur greater loss in utility compared to when one has less precise information to begin with.

In truth, the neutrality axiom is not too different from the proportionality axiom. They differ only in terms of a constant determined by the two assessments $p$ and $r$. If $||p|| = ||r||$ then the proportionality axiom becomes equivalent to the neutrality axiom. But for the spherical rule, this could only happen when the two predictions are equally sharp meaning that the two theories generate the same expected gain when the assessor decides to report truthfully. This reinforces the idea that
under the spherical rule expected losses are only equal in relative terms.

6. ENDING REMARKS

The main focus of this article was to study a commonly cited scoring rule, the spherical rule. Detailed studies and characterizations are available for the logarithmic and quadratic scoring rules. In response, we have tried to do the same for the spherical scoring rule by presenting some of its properties and providing new geometric interpretations. Moreover, we have found that a comparable axiomatic characterization exists for this rule. Through a set of three reasonable axioms, namely invariance, strict propriety, and proportionality, we have shown that scoring rules that satisfy these conditions are related to the spherical scoring rule through some positive affine transformation.

Several authors have proposed that certain scoring rules be the default for problems related to probability elicitation or evaluation because of the fact that certain rules satisfy certain properties (such as locality and neutrality) which they consider to be desirable. However, there is no compelling argument yet that has been made to say that only one rule should be used all the time, since different scenarios may make certain properties more desirable than others. This is in no way an attempt to convince people that the spherical scoring rule is `superior’ to other rules. It is an attempt to better understand the different scoring rules commonly available to aid decision makers in the selection of an appropriate scoring rule. Though the choice of a scoring rule can play an important role in the elicitation and assessment process of decision making, it must be remembered that the effective selection of a rule is only one part of the process. Other components of this process definitely should also be considered and taken into account.
ACKNOWLEDGMENTS

This article has benefitted from the helpful comments provided by Bob Clemen, Casey Lichtendahl, Bob Nau, Luca Rigotti, and Bob Winkler.

APPENDIX: PROOFS

Proof of lemma 1 It is easy to verify algebraically that Axioms 1 and 3 follow. Axiom 2 can immediately be established through Hölder’s Inequality. What we want to show is that it is the only standardized rule that satisfies all three.

Let $S$ be a standardized scoring rule (WLOG w.r.t. event 1) that satisfies the said axioms. By Axiom 1, it follows that for all $i = 1, \ldots, n$

$$S_i(e_{jn}) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

(5)

Next, for every $p \in \Delta_n$ and every $e_{in}$, Axiom 3 implies that the following relationship holds for all $i$,

$$\|e_{in}\| \cdot L(e_{in}, p) = \|p\| \cdot L(p, e_{in})$$

$$\mathbb{E}_p S(p) - \mathbb{E}_p S(e_{in}) = \|p\| \cdot \mathbb{E}_{e_{in}} S(e_{in}) - \mathbb{E}_{e_{in}} S(p).$$

But from (5), we have

$$\mathbb{E}_p S(p) - p_i(1 - p_i) \cdot 0 = \|p\| \cdot [1 - S_i(p)]$$

$$S_i(p) = 1 - \frac{\mathbb{E}_p S(p)}{\|p\|} + \frac{p_i}{\|p\|}. \quad \text{(6)}$$

Using this expression in the formula for $\mathbb{E}_p S(p)$ yields:

$$\mathbb{E}_p S(p) = \sum_{i=1}^{n} p_i S_i(p)$$

$$\mathbb{E}_p S(p) = \sum_{i=1}^{n} p_i \left(1 - \frac{\mathbb{E}_p S(p)}{\|p\|} + \frac{p_i}{\|p\|}\right)$$
\[ \mathbb{E}_p S(p) = 1 - \frac{\mathbb{E}_p S(p)}{||p||} + \frac{||p||^2}{||p||} \]

\[ (||p|| + 1) \mathbb{E}_p S(p) = ||p|| + ||p||^2 \]

\[ \mathbb{E}_p S(p) = ||p||. \]

Plugging this back to (6), we have:

\[ S_i(p) = 1 - \frac{\mathbb{E}_p S(p)}{||p||} + \frac{p_i}{||p||} = \frac{p_i}{||p||}, \]

which is the spherical scoring rule.

Now, we present two lemmas to help us prove Theorem 1.

**Lemma 2** A scoring rule \( S \) satisfies Axioms 1 – 3 if and only if a positive affine transformation of \( S \), \( S^* = aS + b \ (a > 0) \), also satisfies all three same axioms.

**Proof of lemma 2** (A1) The following statements are equivalent: \( S_i(p) = S_{\theta(i)}(\theta(p)) \iff aS_i(p) + b = aS_{\theta(i)}(\theta(p)) + b \iff S^*_i(p) = S^*_{\theta(i)}(\theta(p)) \). (A2) See Winkler and Murphy (1968). (A3) Let \( L^*(r, p) \) denote the loss function for \( S^* \). Then we have:

\[ L^*(r, p) = \sum p_j \left[ \frac{a - p_j}{||p||} + b - \frac{r_j}{||r||} - b \right] = a \left[ \sum \frac{p_j^2}{||p||} - \frac{p_j r_j}{||r||} \right] = a \cdot L(r, p). \]

The conclusion immediately follows.

**Lemma 3** A scoring rule \( S \) that satisfies Axioms 1 – 3 is a positive affine transformation of a standardized scoring rule.

**Proof of lemma 3** What we are going to show is that a scoring rule satisfying the three axioms must have a bounded range of values (that is, \( |S(e_{jn})| < \infty \) for all \( j \)), say the interval \([a, b]\). If so, \( S \) must be a positive affine transformation of some standardized scoring rule, \( S' \). In particular, the transformation is \( S' = \frac{1}{b-a} S - \frac{a}{b-a} \).
First, we note that by Axiom 2, for any \( p \neq r \)
\[
\mathbb{E}_p(S(p)) > \mathbb{E}_p(S(r)) \\
\mathbb{E}_p(S(p)) > -\infty.
\] (7)

In particular, it follows from (7) that for every \( i \),
\[
\mathbb{E}_{e_{in}}(S(e_{in})) > -\infty \Rightarrow S_i(e_{in}) > -\infty.
\] (8)

Also, since we want to maximize expected score, strict propriety prevents \( S_i(r) \) from being unbounded above (whenever \( p_i \neq 0 \)), that is for any two forecasts \( P, R \in \mathcal{P} \),
\[
\mathbb{E}_p(S(r)) < \infty.
\] (9)

Else, honest reporting will never happen as long as \( r \) is reported regardless if \( p \) represents the true beliefs of the assessor. Hence, \( |S_i(e_{in})| < \infty \).

The only thing left to show is \( S_i(e_{jn}) > -\infty \) for \( j \neq i \). We do this by contradiction. Suppose that \( S_i(e_{jn}) = -\infty \) for a particular \( j \neq i \). Consider the assessment \( q \) such that \( q_j = \alpha \in (0, 1) \) and \( q_i = 1 - \alpha \). Then, we have
\[
L(e_{jn}, q) = \mathbb{E}_q(S(q)) - \mathbb{E}_q(S(e_{jn})) \\
= \mathbb{E}_q(S(q)) - [(1 - \alpha)S_i(e_{jn}) + \alpha S_j(e_{jn})].
\]

But we know from (7) and (9) that \( |\mathbb{E}_q(S(q))| < \infty \) and \( |S_j(e_{jn})| < \infty \), this means that \( L(e_{jn}, q) = +\infty \). Then by Axiom 3,
\[
||q||L(q, e_{jn}) = L(e_{jn}, q) \\
L(q, e_{jn}) = +\infty \\
\mathbb{E}_{e_{jn}}(S(e_{jn})) - \mathbb{E}_{e_{jn}}(S(q)) = +\infty \\
S_j(q) = -\infty.
\]

This implies \( \mathbb{E}_q(S(q)) = -\infty \). This is a contradiction since Axiom 2 prevents honest reporting from getting a score of \(-\infty\).

\[\square\]

Proof of theorem 1 By Lemma 3, any scoring rule satisfying Axioms 1–3 is a positive affine transformation of a standardized scoring rule satisfying the same three axioms. But
from Lemma 2, it has been established that any positive affine transformation of a scoring rule, say S, which satisfies the three axioms imply that S must also satisfy Axioms 1–3. Lastly, from Lemma 1, we know that the only standardized scoring rule that satisfies these three axioms is the spherical scoring rule. Hence, the conclusion holds.

Proof of theorem 2 (a) Refer to Shuford et al.(1966), (b) Refer to Selten (1998), (c) Result was shown by Savage (1971).

NOTES

1. By this condition, we consider primarily in this article the discrete version of scoring rules since we would like to give some geometric interpretations in \( \mathbb{R}^n \). The presentation and notations, however, are given such that they could be extended to the continuous case (e.g., see Matheson and Winkler, 1976). Note however that with the addition of some additional restrictions and conditions are necessary such as \( \mathcal{P} \)-integrability and convexity on the class of measures.

2. The quadratic rule in the form presented in Section 2 is not standardized but can be standardized through a positive affine transformation. The logarithmic rule, however, cannot be standardized since its range is not bounded.

3. Selten calls invariance as symmetry and strict propriety as incentive compatibility. While Savage (1971) calls Selten’s neutrality as symmetry.

REFERENCES


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