Nearly Lipschitzian Divergence Free Transport Propagates Neither Continuity Nor BV Regularity

F. Colombini, T. Luo, J. Rauch

Abstract
We give examples of divergence free vector fields
\[ a(x, y) \in \cap_{1 \leq p < \infty} W^{1,p}(\mathbb{R}^2). \]
For such fields the Cauchy problem for the linear transport equation
\[ \frac{\partial u}{\partial t} + a_1(x, y) \frac{\partial u}{\partial x} + a_2(x, y) \frac{\partial u}{\partial y} = 0, \quad \text{div } a := \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} = 0, \quad (1) \]
has unique bounded solutions for \( u_0 \in L^\infty(\mathbb{R}^d) \). The first example has nonuniqueness in the Cauchy problem for the ordinary differential equation defining characteristics. In addition, there are smooth initial data \( u_0 \in C_0^\infty(\mathbb{R}^2) \) so that the unique bounded solution is not continuous on any neighborhood of the origin.

The second example is a field of similar regularity and initial data in \( W^{1,1} \subset BV \) so that for no \( t > 0 \) is it true that \( u(t, \cdot) \) is of bounded variation.

1 \( C^\infty \) propagates to discontinuous
Suppose that \( a \in (W^{0,\infty} \cap W^{1,1})(\mathbb{R}^d) \) is a bounded divergence free vector field on \( d \) dimensional Euclidean space. For arbitrary bounded initial data

*Partially supported by the US National Science Foundation grant NSF-DMS-0104096
$u_0 \in L^\infty(\mathbb{R}^d)$ there is a unique solution $u \in L^\infty([0, \infty[ \times \mathbb{R}^d)$ of the initial value problem
\[ \partial_t u + \mathbf{a} \cdot \nabla u = 0, \quad u(0, \cdot) = u_0(\cdot). \]
(see e.g. [DL]). In the planar case, $\mathbf{a} \in L^\infty(\mathbb{R}^2)$ suffices for uniqueness of bounded solutions of divergence free fields. The proof in [CR] uses a method introduced in [CL]. In general dimension, a sequence of papers ([DL] [Li], [CL1], [CL2], [CL3], [Am]) end with Ambrosio’s recent proof of uniqueness when $\mathbf{a}$ is of bounded variation. In higher dimensions there are examples of nonuniqueness for nearly $BV$ fields ([Ai], [CLR], [Del-2]).

If $\mathbf{a} \in W^{1,\infty}(\mathbb{R}^d)$ is uniformly Lipschitzian, then the transport propagates all Hölder regularity in the sense that if $u_0 := u|_{t=0} \in C^\alpha(\mathbb{R}^d)$ with $0 \leq \alpha \leq 1$, then for all $T > 0$, the solution $u$ is also Hölder, $u \in C^\alpha([0, T] \times \mathbb{R}^d)$.

A formal interpolation between propagation of $W^{0,\infty}(\mathbb{R}^2)$ when $\mathbf{a} \in W^0(\mathbb{R}^2)$ and propagation of $W^{1,\infty}(\mathbb{R}^2)$ when $\mathbf{a} \in W^{1,\infty}(\mathbb{R}^2)$ suggests that if $\mathbf{a} \in W^{0,\infty}(\mathbb{R}^2)$ and $u_0 \in W^{0,\infty}(\mathbb{R}^2)$ then the solution belongs to $C^\alpha$. Nothing of the sort is true. We present an example in dimension $d = 2$ of a field in all the Hölder spaces and an initial datum which is smooth and of compact support so that the solution is not continuous on any neighborhood of the origin.

The field in that example also has the property that the characteristics, defined by solving the ordinary differential equations
\[ \frac{dx}{dt} = \mathbf{a}_1(x(t), y(t)), \quad \frac{dy}{dt} = \mathbf{a}_2(x(t), y(t)), \]
have nonunique solutions. The divergence free hypothesis shows that the flow preserves volumes. Nonuniqueness is an extreme form of length distortion, an interval of length zero is distorted to an interval of finite length. This is consistent with volume preservation. The length distortion explains the lack of propagation of regularity. Formally, if $\Phi_t$ is the flow generated by $\mathbf{a}$ then one thinks of the solution as the composition $u_0(\Phi_{-t}(x))$. In order for this to propagate Hölder regularity one needs $\Phi_{-t}$ to be Lipschitzian. This is guaranteed when $\mathbf{a}$ is Lipschitzian.

In the one dimensional case, $d = 1$, length distortion and volume distortion are equivalent and are often controlled by one sided inequalities on the derivative $\partial a(t, x)/\partial x$. There is an extensive literature going back at least to Oleinik’s uniqueness proof [O] showing that compression is good for uniqueness while rarefaction is good for existence while bad for uniqueness.
Figure 1: Positive octant phase portrait

([B0], [BJ], [PP], [PR]). In the Oleinik proof the entropy condition controls the possible stretching of lengths. Our example is incompressible. It shows that length distortion and not volume distortion is key in these earlier works.

A simple explicit $C^\alpha$ field exhibiting nonuniqueness of characteristics and therefore infinite length distortion is the following.

**Example.** In the positive quadrant \{ $x > 0$ and $y > 0$ \} consider the divergence free double shear

$$-y^\alpha \partial_x - x^\alpha \partial_y.$$  \hfill (3)

Characteristics satisfy

$$\frac{dx}{dt} = -y^\alpha, \quad \frac{dy}{dt} = -x^\alpha.$$  

Therefore,

$$\frac{d}{dt} \left( x^{1+\alpha} - y^{1+\alpha} \right) = 0.$$  

The phase portrait is sketched in Figure 1.

The line $x = y$ is invariant. The solution with initial value $(x(0), y(0)) = (b, b)$ with $b > 0$ is given by

$$x(t) = y(t) = \left( b^{1-\alpha} - (1 - \alpha) t \right)^{-\frac{1}{1-\alpha}}.$$  

This curve reaches the origin at the time

$$t_*(b) := \frac{b^{1-\alpha}}{1-\alpha}.$$  

3
Through the point \( (t, (b, 0, 0) \) pass this backward characteristic and also the characteristic \( x = y = 0 \). If these backward paths hit \( t = 0 \) at points where \( u_0 \) takes distinct values, the requirement that \( u \) be constant on characteristics yields incompatible values. This is the heart of the following construction.

In the next definition note that \( s(\log s)^2 \) is strictly increasing for \( 0 \leq s < e^{-2} \).

**Definition.** Suppose that \( 0 \leq f \in C(\mathbb{R}) \) vanishes for \( s \leq 0 \), is nondecreasing and uniformly bounded, and for \( 0 \leq s \leq e^{-2}/2 \)

\[
f(s) = s (\log s)^2.
\]

Define the bounded divergence free field

\[
a_1(x, y) \partial_x + a_2(x, y) \partial_y := -f(y) \partial_x - f(x) \partial_y. \tag{4}
\]

Then \( a \) belongs to all the Hölder spaces \( C^\alpha(\mathbb{R}^2) \) with \( 0 < \alpha < 1 \), to \( W^{1,p}(\mathbb{R}^d) \) for all \( 1 \leq p < \infty \), and even more \( \nabla a \in BMO(\mathbb{R}^2) \).

**Theorem 1.** Suppose \( a(x, y) \) is the vector field in the definition. Suppose that \( u_0 \in C^\infty_0(\mathbb{R}^2) \) vanishes when both \( x \) and \( y \) are nonpositive, and is strictly positive when \( x \) and \( y \) are strictly positive and small. Then the unique solution \( u \in L^\infty([0, \infty] \times \mathbb{R}^2) \) of the transport equation with these initial data is not continuous on a neighborhood of the origin in \([0, \infty] \times \mathbb{R}^2 \).

**Proof.** Supposing that \( u \) is a solution which is continuous on a neighborhood of the origin in \([0, \infty] \times \mathbb{R}^2 \) we derive a contradiction.

The characteristic beginning at \( (b, b) \) with \( 0 < b \leq e^{-2}/2 \) is equal to \( (x(t), x(t)) \) where

\[
\frac{dx}{dt} = -x (\log x)^2, \quad x(0) = b.
\]

Then

\[
\frac{d}{dt} \frac{1}{\log x} = -\frac{1}{(\log x)^2} \frac{1}{x} \frac{dx}{dt} = 1.
\]

Thus

\[
\frac{1}{\log x(t)} = t + \frac{1}{\log b}, \quad \log x(t) = \frac{\log b}{t \log b + 1}.
\]
The path arrives at the origin at the finite time
\[ t^*(b) := \frac{-1}{\log b}. \]

The method of characteristics in the form of the next lemma is needed. The proof is standard.

**Lemma 2.** Suppose that \( \gamma(t) = (t, x(t)) \) with \( x : [0, c] \to \mathbb{R}^2 \) is an integral curve of \( \partial_t + a_1 \partial_x + a_2 \partial_y \) with the property that \( a \) is uniformly Lipschitzian on a neighborhood of \( x([0, c]) \). If \( u \) is a continuous solution of (1) on a neighborhood of \( \gamma([0, c]) \) in \( 0 \leq t \leq c \) then \( u \) is constant on \( \gamma([0, c]) \).

This lemma is applied to the characteristic beginning at \( (b, b) \) near the origin in the positive quadrant. The characteristic arrives at the origin at the small time \( t^*(b) \). The Lemma with \( c = t^*(b) - \epsilon \) implies that
\[
    u\left(t^*(b) - \epsilon, x(t^*(b) - \epsilon), x(t^*(b) - \epsilon)\right) = u_0(b, b).
\]
Passing to the limit \( \epsilon \to 0 \) using the continuity of \( u \) yields for \( b \) small
\[
    u(t^*(b), 0, 0) = u_0(b, b) > 0. \tag{5}
\]

On the other hand, the field \( a \) vanishes in the quadrant where both \( x \) and \( y \) are negative. The Lemma implies that \( u \) is independent of time in that quadrant and therefore that \( u(t, x, y) \) vanishes when both \( x \) and \( y \) are negative. Since \( u \) is continuous near the origin it follows that for \( t \) small
\[
    u(t, 0, 0) = 0.
\]

For \( b \) small this contradicts the conclusion (5) and the proof is complete. \( \blacksquare \)

**Remark** The solution \( u \) is continuous at the point \( (0, 0, 0) \) with \( u(0, 0, 0) = 0 \). In fact, the values of \( u \) in a small neighborhood of \( (0, 0, 0) \) are determined by the values of \( u_0 \) on a small neighborhood of \( (0, 0) \). By continuity of \( u_0 \) these values differ little from 0.

## 2 Bounded variation is not propagated

We give a simple example for which BV regularity is not propagated. Suppose that \( g(s) \in C_0^0(\mathbb{R}) \) with
\[
    g(s) = -s \log |s| + s
\]
on a neighborhood of $s = 0$. Then near the origin $g' = -\log |s|$. Define the divergence free bounded field

$$b := g(y) \partial_x.$$ 

The flow of this field and its inverse are shears given explicitly by

$$\Phi_t(x, y) = (x + tg(y), y), \quad \Phi_{-t}(x, y) = (x - tg(y), y).$$

The solution of the associated linear transport equation with initial value $u_0$ is given by

$$u(t, x, y) = u_0(\Phi_{-t}(x, y)) = u_0(x - tg(y), y).$$

Then

$$\partial_y u = -tg'(y) \frac{\partial u_0(x - tg(y), y)}{\partial x} + \frac{\partial u_0(x - tg(y), y)}{\partial y}. \quad (6)$$

For $u_0 \in W^{1, 1}$,

$$\left\| \frac{\partial u_0(x - tg(y), y)}{\partial x} \right\|_{L^1(\mathbb{R}^2_x)} \text{ and } \left\| \frac{\partial u_0(x - tg(y), y)}{\partial y} \right\|_{L^1(\mathbb{R}^2_y)}$$

are independent of $t$. The strategy for small $y$ is to take advantage of the large factor $tg'(y) \sim t |\log|y||$ in the first summand.

**Theorem 3.** Suppose that $u_0(x, y) \in (L^\infty \cap W^{1, 1})(\mathbb{R}^2)$ satisfies $u_0(x, y) = \cos (r^{-1}(\log r)^{-2})$ on a neighborhood of $(0, 0)$. Then for all $t > 0$ the unique bounded solution of the initial value problem

$$u_t + g(y) u_x = 0, \quad u(0, x, y) = u_0(x, y)$$

satisfies $u(t, \cdot) \notin BV(\mathbb{R}_x^2)$. 

**Proof.** Near the origin

$$\frac{\partial u_0}{\partial r} \sim \frac{1}{r^2 (\log r)^2}$$

is just barely $L^1$. Since

$$\frac{\partial u_0}{\partial x} = \frac{\partial u_0}{\partial r} \frac{\partial r}{\partial x}$$

and the second factor is bounded away from zero when the argument of $(x, y)$ is bounded away from $\pm \pi/2$ it follows that $\partial_x u_0$ is also borderline $L^1$. 

6
Fix $t > 0$ and introduce the change of coordinates

$$(x, y) := (x - tg(y), y),$$

with associated polar coordinates $(r, \theta)$. This change preserves area and $y = y$.

The expression (6) is valid in $y \neq 0$. Therefore if $u(t, \cdot)$ belongs to $BV(\mathbb{R}^2_{x,y})$ it follows that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_y u(t, x, y)| \, dx \, dy < \infty.$$

The second summand in (6) belongs to $L^1(\mathbb{R}^2_{x,y})$ for all $t$ with norm independent of $t$. To complete the proof it suffices to show that for $\phi \in [0, \pi/2]$ and $0 < \epsilon << 1$

$$\infty = \int_{\epsilon}^{\phi} \int_{0}^{\epsilon} \left| \log |y| \frac{1}{r^2 \log^2 r} \right| r \, dr \, d\theta,$$

since the $L^1$ norm of the first summand in (6) is at least as large as a positive multiple of the right hand side.

On the circle of radius $r$ in the $(x, y)$ plane one has $|y| \leq r$, so $|\log |y|| \geq |\log |r||$. Therefore the integrand is bounded below by $\frac{1}{r \log r}$ which is not integrable.

Acknowledgements The research of J. Rauch was partially supported by the U.S. National Science Foundation under grant DMS-0104096. T. Luo's research was partially supported by the Start up fund of Georgetown University. JR thanks the Universities of Nice and Pise, and FC the University of Michigan for their hospitality during 2002-2003.

3 References


[CR] F. Colombini, J. Rauch, Uniqueness in the Cauchy problem for transport in $\mathbb{R}^2$ and $\mathbb{R}^{1+2}$, preprint.


Ferruccio COLOMBINI
Dipartimento di Matemactia
Università di Pisa
Pisa, Italia
colombini@dm.unipi.it
Tao LUO
Department of Mathematics
Georgetown University
Washington DC, 20057, USA
tl48@georgetown.edu

Jeffrey RAUCH
University of Michigan
Ann Arbor 48104 MI, USA
rauch@umich.edu