# Light's Speed in Water

Teacher Materials

## Mathematics Contained in the Lesson

This activity has students combine a number of concepts they have previously learned. In particular, they use the Pythagorean theorem and the distance/rate/time relationship to develop a function that gives the time it takes for light to travel from one point under water to another point above the water. They must then minimize this function, using graphical techniques. Finally, they explore this function as one parameter changes (the speed of light in water) minimizing this function for a number of different parameter values. The speed of light in water is approximated by the parameter value that gives a minimum closest to the observed value.

You can discuss such concepts as domain and range and give students experience in using their calculator to graph functions, finding an appropriate viewing-window, and using the minimum feature of their calculator.

This activity can be used in a precalculus course in a unit on functions or in a unit on functions involving radicals. It gives students an advanced modeling experience that draws on the knowledge of a number of ideas, helping to develop their "mathematical maturity." In addition, it gives them experience in solving maximum/minimum problems, good preparation for subsequent work in calculus.

The activity is intended as one lesson. It is possible that it will take your class longer than one class session to complete it, so plan for two consecutive class meetings.

## Our Setup

This activity is somewhat complex to run in class, but we have had quite a bit of success with it, as well as a bit of fun. In this section, we will describe how we have run the "experiment." We had a number of failures before we found this approach.

We use clear plastic hamster cages that are about 14 cm high, 20 cm long, and 14 cm wide. These can be found in pet stores for under $5.00. (If you can find a clear plastic container of about that size with perpendicular sides, get that one!) We make an indelible mark on the bottom, **inside** end of the cage, as in figure 1a. We make a line on the outside of the container perpendicular to the long edge, even with the indelible mark, as in the side view of the container seen in figure 1b. It is crucial that this line makes a 90 degree angle with the base. We make a final horizontal mark exactly 12 cm above the indelible mark, on the side of the cage, also as in figure 1b. When we run the experiment in class, this last mark makes it easy to fill the container with water so that the indelible mark is exactly 12 cm below the water level. The "12 cm" depth of water is not important. You can make your mark a convenient height above the base, as long as it is measured accurately. (If you are going to assign the optional problem 6, you may want to mark other heights in your set-up.) This mark should be "near" the top of the container so that students can easily make measurements later in this demonstration, but far enough



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below the top of the container so that you don't spill the water when moving or emptying the container. We usually bring the container to class empty and fill it from a pitcher once we have it in the proper position.

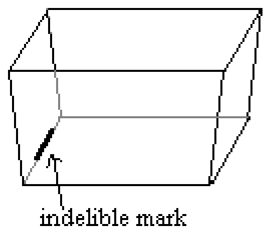


figure 1a

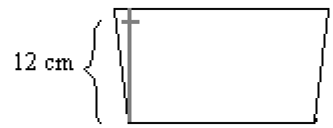


figure 1b

On a table in a location in the classroom that is visible to all students, set up the container of water. Figure 2 demonstrates our classroom setup. We note several things about this setup. First, we put a piece of wood next to the container in which we have drilled a hole for holding a wooden dowel, so that the dowel is exactly 50 cm from the indelible mark. (50 cm works well, but you could make your own choice for this distance.) This saves us from having to measure distances each time we set up this procedure. Having the dowel set perpendicularly in the wood makes student measurements more accurate later. We make a mark on the dowel exactly 20 cm above the level of the water. (Again, you can pick whatever height is convenient for you. Don't make this distance too large, or it will be difficult for some students to make observations from that mark.) Finally, we tape a string across the top of the container, parallel to the end of the container beginning at the top of the line perpendicular to the long edge. This facilitates students' measuring the horizontal distance that points are removed from the indelible mark on the bottom of the container.

With all the materials previously prepared, it only takes a few minutes to actually set this model up in a classroom.

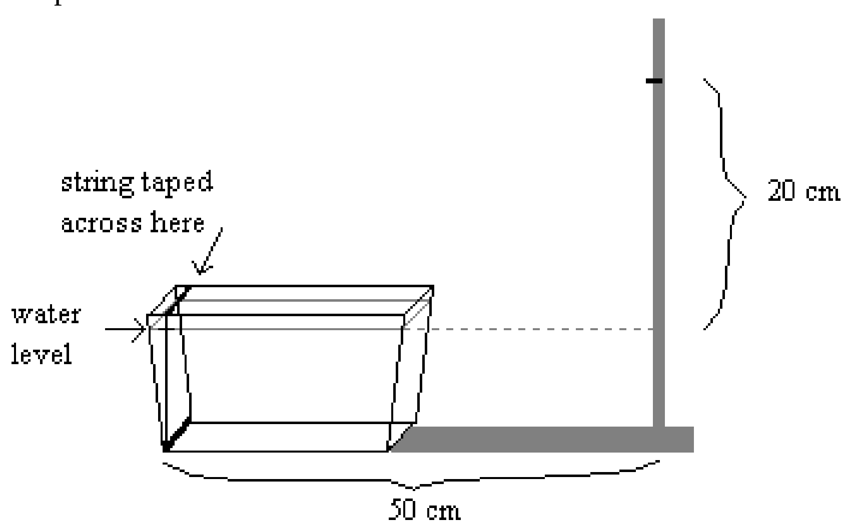


figure 2

**Organization**

Students should read the Reading Assignment before coming to class. Have the container of water set up in front of the class, as described above. Write the three distances 1) depth of water, 2) distance of dowel from indelible mark, and 3) height of mark on dowel above water level, on the board for easy recall by the students. You should also record the speed of light in a vacuum, 30 cm/nanosecond, on the board. Be sure students are aware that this is quite a good approximation by comparing their answer to problem 1 (186,411 miles per second) with the more accurate speed of light, 186,281 miles per second. Discuss why this is a reasonable approximation to use for the speed of light in air. (Light travels a little slower in air than in a vacuum, but the gases in the air do not obstruct the speed of light very much. Air varies from one space to another and even from one time to another in the same space; but the speed of light in no air, in a vacuum, is a reasonable approximation in relatively clean air.)

In this opening discussion, get your students to discuss reasonable guesses for the speed of light in water, then get them to agree on one particular guess. The lesson is probably more effective if this guess is not too good, so you might actually suggest an initial "guess" of 15 cm per nanosecond, half the speed of light in air.

Be sure students understand the basic principle: that the path light travels between any two given points is the path that takes it the least time to traverse. Discuss with students why a straight path may not be the quickest path. Display one possible path from the mark on the bottom of the container to the mark on the dowel using bright colored yarn taped to the container and dowel as seen by the heavy gray line in figure 3.

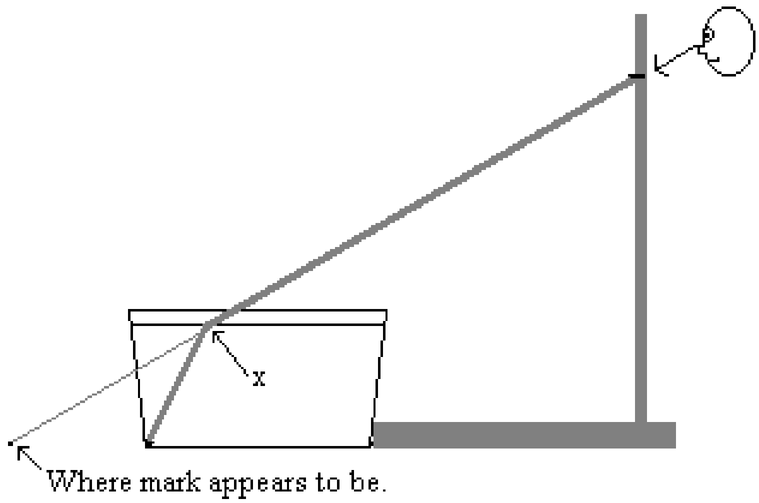


figure 3

Make sure that students understand that a person standing where we have indicated in figure 3 would see the mark on the dowel, point $x$, and the mark on the bottom of the container coincide, even though they aren't on a straight line. You could have several students actually do this. (One trick is to stand away from the dowel so that your eye can bring both marks into focus.) Also discuss where the person would "think" the mark on the bottom of the container was located. You might indicate this with another piece of yarn that continues straight along the line from the eye to the water, as indicated by the narrow gray line in figure 3.

**Beginning the lesson**

At this point, have the class pick a "reasonable" path for the light traveling from mark $A$ to mark $B$; that is, pick a "reasonable" value for the distance $x$ from the end of the container. Students should begin working problem 2, in groups of 3 or 4. For problem 2, success depends on students making a good sketch. For this question, they will use the agreed-upon guess for the speed of light in water in their calculations.

Some groups may have difficulty with part a. If so, ask them questions, such as, how do you compute the time it takes to travel 100 miles if your speed is 50 miles per hour. You might also have to ask them how they can find the length of a hypotenuse of a right triangle.

As groups complete part a, give them a new value for $x$, a different value for each group, so that they can work part b. When all groups have finished part a, bring the class back together to see if they agree on the answer. If they don't, have a discussion to see if the class can come to an agreement, possibly letting different groups display their "formula", with their reasons for that formula. Some groups may work the problem wrong, some may make a calculation error, some may differ because of the accuracy they used in their calculations. Encourage them to use the full accuracy of their calculators. If they learn to use their calculators well, this is actually easier.

As groups finish part b, have them record the results in a table on the board similar to table 1 in their materials. If some groups finish parts b and c early, you can have them repeat these parts with another value or go on to part d. We have observed that many groups, when repeatedly given more values for $x$ in part b, find the formula for $T(x)$ without being asked because they observed the pattern. The pattern is particularly easy to observe when using a graphing calculator because students will keep changing one number in their expression.

As groups complete question 2e, have them go to the water set-up and observe if their calculated answer is the correct (actual) distance, by measuring the distance of the actual exit point. This can be found by laying a ruler across the top of the container, with zero at the string that is taped across the top. One person puts a pencil point near the water at the point where they computed the light should exit, using the ruler to measure. Another person looks at the mark from the observation point you have marked on the dowel. See figure 4.

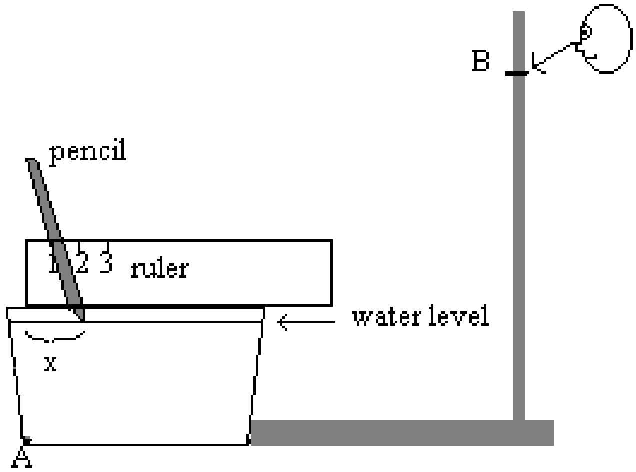


figure 4

The observer has a view similar to figure 5a; that is, the pencil point is nowhere near mark $A$. He or she then tells the person with the pencil whether to move the point forward or backward, until the pencil point lines up with marks $A$ and $B$, as in figure 5b. Looking down from above, the person with the pencil, or a third person, should read off the correct value for $x$ from the ruler. **It is crucial that the pencil point be <u>almost</u> touching the water when the measurement is taken.**

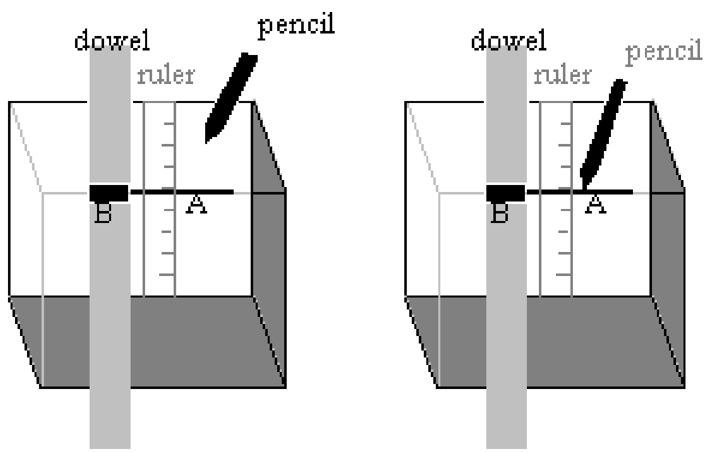


figure 5a figure 5b

When a group has measured the correct value for $x$, record their value in a separate table. If the measured value does not agree with the value they computed in part e, give them another value for $v$, the speed of light in water, or (better) let them choose their own guess, and proceed to problem 3. Groups can measure and proceed to problem 3 at different times.

Bring the class back together for a discussion once all groups have completed problem 2 and estimated the correct $x$-value by measuring. Most likely, the values each group measured for $x$ are slightly different. From the different measured values, have the class determine a most likely candidate and a range. For example, they might say that $x$ is about 8.3 cm, but is almost certainly between 7.9 and 8.6. You could take an average and find a standard deviation to do this, but that is not necessary; this is an opportunity, though, to discuss measurement error and the related statistical issues. Have a discussion on how this value compares to the predicted $x$-value from part 2 and why they differ.

At this point, if any groups have computed an $x$-value for another value of $v$, the "guessed" speed of light in water, have them record their predicted $x$-value on the board. From the known values, let the class decide on other values to try for $v$ and let each group work with at least one. Some groups will see the pattern and will be able, in a matter of just a few minutes, to compute a value for $x$ for their $v$-value. Encourage them to keep narrowing in on the correct value.

When the class has enough predicted values of $x$ to compare with their measured "actual value of $x$", they can state what they think is the correct value for the speed of light in water. By looking at the results, they should be able to give an interval which they are fairly confident includes the correct speed of light in water.

**Answer:** The speed of light in water is 22.5 cm per nanosecond.

Question 6 is optional. It permits students to consolidate what they learned in the lesson. You could use it to help students who you believe need to rethink the entire process, or you could use it to test understanding.

You can omit the last section **A Light Under the Water**. Its purpose is to explain why light may reflect back into the water instead of exiting into the air. The students will find that, no matter how far away the point $P_i$ is from mark $A$, the exit point $x_i$ is always less than some relatively small constant, $c$. Thus, the answer to problem 7 is that the light cannot exit, because that point is never the best exit point. The only thing left is that the light must reflect back into the water. It also says that if a person was at that point under water, and was looking up, that person would see the entire outside world in a circle with a radius of $c$.

You could use this section to apply some trigonometry, by computing the angle $\theta$ in figure 6.

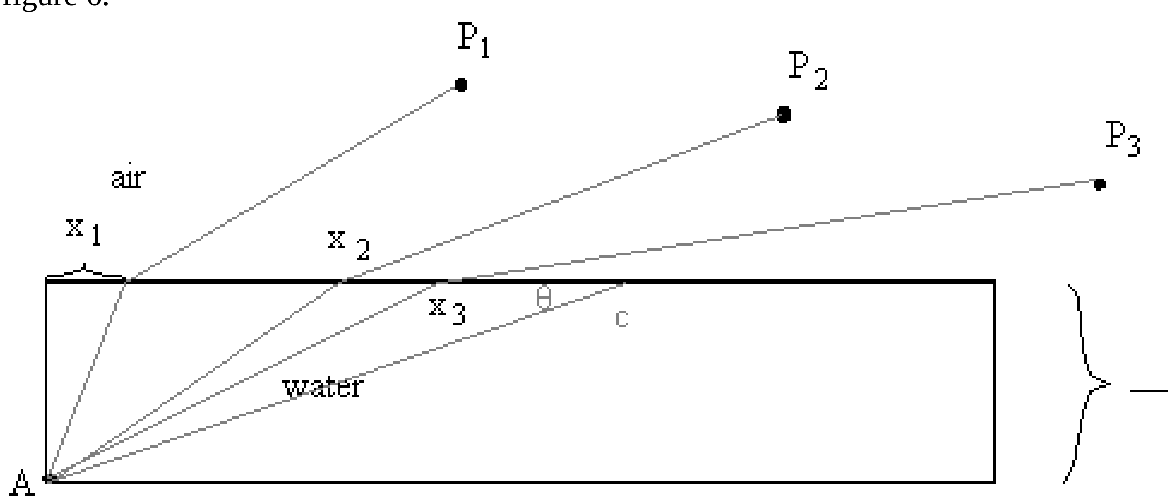


figure 6

As an aside that your class might find interesting, a mathematician has fairly recently used these principles to determine how to cut diamonds so that a maximum amount of light is reflected to the observer. This approach is now widely used to cut diamonds so that they have a much greater radiance then they traditionally had.

**Answers to problems**
Since your container might be a different size than ours, we will give you formulas to compute the answers using the variables in figure 7 and $v$ for the speed of light in water.

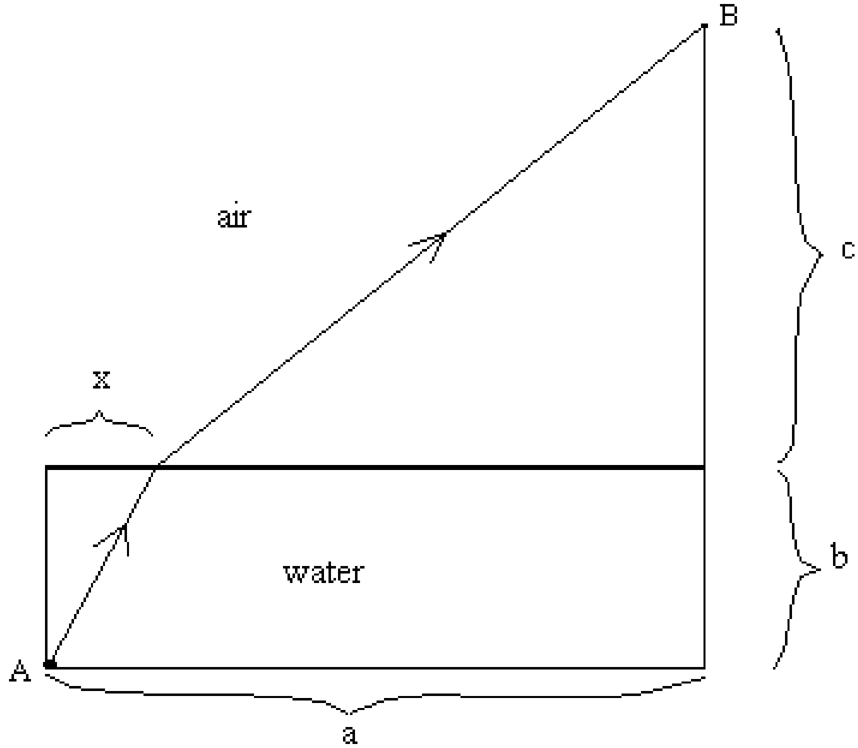


figure 7

**1.** 186,281 miles per second.

**2. a.**

$$\text{time} = \frac{\sqrt{x^2+b^2}}{v} + \frac{\sqrt{(a-x)^2+c^2}}{30}$$

where $x$ is the given value.

**b.** This answer is given by the same formula as in part a, using the new value for $x$.

**c.** The $x$-value that resulted in the smallest time.

**d.**

$$T(x) = \frac{\sqrt{x^2+b^2}}{v} + \frac{\sqrt{(a-x)^2+c^2}}{30}$$

**e.** Again, this depends. Students should become skilled at changing their window, especially the vertical portion, so they can better "see" where the minimum occurs. They can also do this by watching the $y$-values change on the screen as they use trace. Finally, they can use the "min" feature of their calculator, although we would recommend they find the minimum by using TRACE the first time or two they do this to develop their understanding of the concept of minimum.

**f.** Light probably does not take their "possible path" because the assumed speed for light in water was probably wrong. Students should determine whether the path they chose is wrong because light's speed in water is faster than they guessed or slower than they guessed.

**3.** They just minimize the same function $T(x)$ given above, but using the new value for $v$. Some students will recognize they only have to change one number in their function and graph again. Note that when $v$ is increased, the $y$-values become a lot smaller, so the $y$Min and $y$Max should be changed to see an appropriate portion of the graph. $y$Min and $y$Max need to be "close" to each other so that the curved shape of the function can be seen. If the width of the interval [$y$Min, $y$Max] is too large, part of the curve will look like a horizontal line.

**4.** It is interesting to see how close a class can get to the exact measured $x$-value. This must be balanced with the additional time it takes. The correct speed is $v = 22.5$ cm per nanosecond. If measurements are fairly accurate, you can expect the class to get a value for $v$ between 22 and 23 cm per nanosecond.

**5.** This could be homework. Make sure students reflect on the mathematics they have used and how this allowed them to compute something that would seem next to impossible to compute.

**6.** If students do not seem to have a confident understanding of the work they have done in this lesson when they return to class after they have completed problems 2-5, you might wish to assign this problem. It need not be assigned to every member of the class if only some need the experience. It could also be used as a group test at a later date.

**7.**
$$x_i \leq \frac{vb}{\sqrt{900 - v^2}}$$

**8.** The light reflects back into the water for any point more horizontally removed that $vb/\sqrt{900 - v^2}$ from mark $A$. Angle $\theta$ in figure 6 is

$$\theta = \arctan\left(\frac{\sqrt{900 - v^2}}{v}\right) = 41.4 \text{ degrees}$$