Reading This Could Help You Sleep: Caffeine in Your Body
Get the Lead Out
So Much Coffee, So Little Time
Teacher Materials

Mathematics Contained in these Three Lessons

"Reading This Could Help You Sleep: Caffeine in Your Body" is an introduction to exponential functions of the form $y = a\,b^t$ at the Intermediate Algebra level, with emphasis on the meaning of these functions and their graphs. The concept of half-life is introduced. A conditional function (a piecewise-defined function) is used.

"Get the Lead Out" extends the study of exponential functions and can be used to introduce the use of logarithms to "un-do" exponential expressions in solving equations.

"So Much Coffee, So Little Time" can be used at the Intermediate Algebra level to help students see the value of "solving systems of linear equations". It shows connections between solving linear equations and exponential functions, and continues the study of exponential functions, this time of the form $y = a\,b^t + c$. End behavior of these functions with horizontal asymptotes is included.

Set-up

Each students should have a graphing calculator. Alternatively, each group of 4 students should work at one computer.

Organization

These three lessons are interdependent. They should be done in the ordered listed above in the title to the Teacher Materials. You can do the first without doing the second or the third, or you can do the first and second without doing the third. We recommend that you not try to use the second or third unless you have done the one(s) that precede it. Each unit can be used at a time that is appropriate for the course, that is, they do not need to be covered on consecutive class days.

Answers to Problems and Teaching Suggestions for Reading This Could Help You Sleep: Caffeine in Your Body

1. a. $390 \times 0.87 = 339.3$
$390 \times 0.87^2 = 295.2$
$390 \times 0.87^3 = 256.8$

b. $390 \times 0.87^{24} = 13.8\,\text{mg}$

Reflection Point: Students should have attempted problem 1 for homework. You can begin class by discussing the reading and the students' answers. When you discuss this

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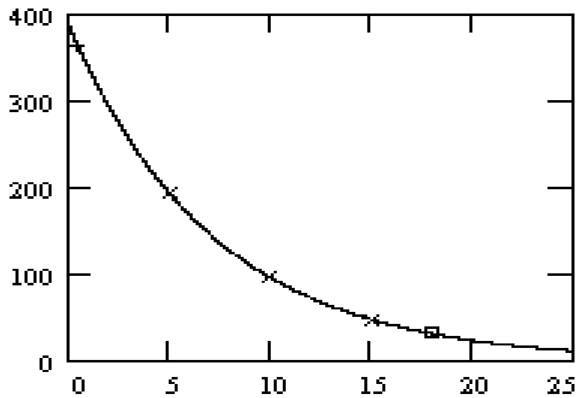
homework problem in class, have students describe how they actually computed the answers for part a. Be sure that all students see the relationship between multiplying again by 0.87 and increasing the exponent on 0.87 by 1. Having different students explain their reasoning for problem 1b in class should help all students understand the process. After the discussion, hand out **Eliminating Caffeine from the Body**, page 3 of this activity, and have the students work on this page in groups of 3 or 4.

2. a. $y = f(t) = 390(0.87)^t$
b.



c. As t gets large, $y \to 0$.
d. This point is at about $(18, 32)$ and is shown with a box in the graph.
e. When $t = \frac{1}{2}$, $y \approx 364$. This point is shown with a + on the graph.

Students can get this result approximately from their graphs. You might wish to remind students that an exponent of $\frac{1}{2}$ *indicates a square root. If so, they can calculate* $390\sqrt{0.87}$ *and compare with their reading from the graph.*

f. When $y = 195$, $t \approx 5$ hours.
g. After two half-lives, $y = 97.5$, which is $390(\frac{1}{2})^2$; after three half-lives, $y = 48.75$, half of the previous result. $t \approx 10$ hours is two half-lives; three half-lives is approximately 15 hours.

Students can get these results approximately from their graphs. Help them recognize that after each 5 hour period (approximately), the amount of caffeine remaining in the body is about half what it was at the beginning of the 5-hour period. Studying the graph with this in mind will help to prepare students for later work with half-life. These three points are marked with $\times$ *on the graph.*

Homework

Problem 3 is intended to extend students' familiarity with exponential functions. Each person's table and graph will differ from the others, of course, but we suppose most will have more than one drink during the day containing caffeine. In this case, most will have to write a conditional function. For a person who consumes130 mg at 8:00 am, 40 mg at 9:30 am, and 165 mg at 1:00 pm, the conditional function could be written as

$$f(t)=\begin{cases}130(0.87)^t & 0\le t<1.5\\ 145.5(0.87)^{t-1.5} & 1.5\le t<5\\ 254.4(0.87)^{t-5} & 5\le t<24\end{cases}$$

or as

$$f(t)=\begin{cases}130(0.87)^t & 0\le t<1.5\\ 130(0.87)^t+40(0.87)^{t-1.5} & 1.5\le t<5\\ 130(0.87)^t+40(0.87)^{t-1.5}+165(0.87)^{t-5} & 5\le t<24\end{cases}$$

where t measures time in hours after 8 am. Since the person in our table had her first cup of coffee at 8 am, she measures t in hours since 8 am. Thus, after 9:30 am, she will need to use $t-1.5$ as an exponent to indicate hours that have passed since her $9:30$ am Dr. Pepper and later, $t-5$ as an exponent to indicate hours that have passed since her 1pm espresso. Note that in the first form of the function, the total amount of caffeine in the person present at time $t=1.5$ and $t=5$ was computed and used to start the next exponential. In the second form of the function, the caffeine in each drink is considered as being eliminated exponentially and independently from the caffeine in the other drinks.

To graph such a function on a graphing calculator, set the calculator to DOT mode (as opposed to CONNECTED) to avoid lines joining endpoints. The function can be graphed several different ways. For example, students could graph

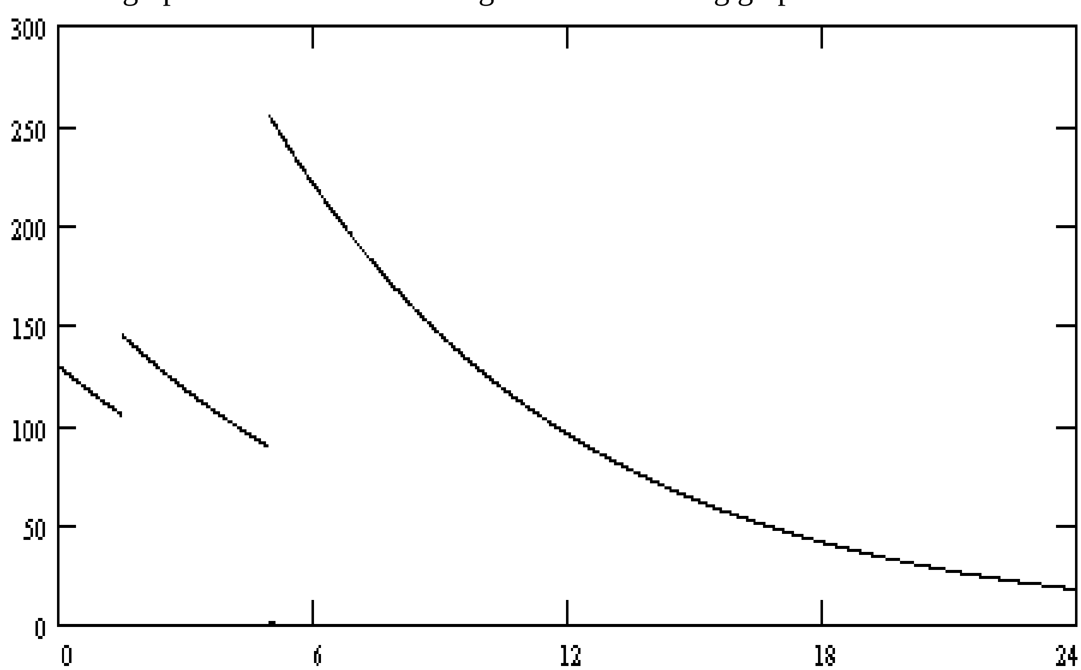
$Y_1=130(.87)^x(x<1.5)$
$Y_2=145.5(.87)^{x-1.5}(x\ge 1.5)(x<5)$
$Y_3=254.4(.87)^{x-5}\,(x\ge 5)$

or

$Y_1=130(.87)^x(x<1.5)+145.5(.87)^{x-1.5}(x\ge\ 1.5)(x<5)+254.4(.87)^{x-5}\,(x\ge 5)$

or $Y_1=130(.87)^x+40(.87)^{x-1.5}(x\ge 1.5)+165(.87)^{x-5}(x\ge 5)$

with the window set for $x\in[0,24]$ and $y\in[0,Y\max]$ with Ymax depending on the student. The graph should look something like the following graph.



Answers to Problems and Teaching Suggestions for **Get the Lead Out**

1. a. 0.9850, 0.9702, 0.9557, 0.9413 mg, respectively
b. 7 days
c. $f(t) = (0.985)^t$

Reflection Point: It is assumed students will have worked problem 1 for homework. If you discuss this homework problem in class, ask students how they did the problems. You might then note that the answer to part b is the approximate solution to the equation $(0.985)^t = 0.9$ and discuss how you could use this equation to find t; that is, take the log of both sides. If you plan to teach solution of exponential equations "by hand", now is a good time to help students recognize the need for an inverse for exponential functions with which to do it. The equations can be solved using a graphing utility, of course, but if students try that method first, they will be ready to recognize that the search for an appropriate window for a solution to these equations may not be as fast or as easy as a logarithmic approach.

After discussing problem 1, you should hand out **Get the Lead Out** Classroom materials and have students work on problems 2 and 3 in groups of 3 or 4. You can also have them work on problem 4 on page 3 of this activity, or you can assign that problem for homework.

2. a. 1.9700, 1.9404, 1.9113 mg, respectively

b. $f(t) = 2(0.985)^t$. After 46 days (or 45.862), the amount of lead in her body will be reduced to 1 mg.

If students have any difficulty answering part b, ask them to explain how they worked part a. Remind them about the caffeine activity. Help them generalize for any t *and then recognize that the function is exactly that generalization. The determination of half-life can be found from the graph or by taking the log of both sides of the equation* $0.985^t = 0.5$. *Discussion to review the term half-life would be helpful here.*

c. Another 46 days.

Here you might also show students another form of the function based on the half-life idea: $f(t) = 2(\frac{1}{2})^{\frac{t}{45.862}}$. *It might help them understand this if they use their calculator to see that*

$$(\tfrac{1}{2})^{\frac{1}{45.862}} = 0.985.$$

d. The half-life is 45.862 days. This means that every 45.862 days, half of the lead is eliminated from the blood.

e. To a level of 0.4 mg: 106.5 days. To a level of 0.2 mg: another 46 days, or 45.862 - thus about 152 days.

For the 0.4 mg question, students might use their graphing calculators or they might calculate using logarithms. To answer the 0.2 mg question, they have many choices, but we hope they will use the half-life idea of part d.

3. $1\% = 0.01$; one hundredth of one percent is $0.01 \times 0.01 = 0.0001$ *Be aware of the potential for error here. Misunderstanding of this decimal will lead students far astray on the rest of the problem!*

a. $f(t) = 40(0.9999)^t$

b. 18.989 or 19 years; 37.9788 or 38 years

c. 18.989 or 19 years

Reflection point: This is a good time to ascertain that students understand half-life.

4. a. If 0.02% of the cadmium is removed from the body each day, it will take about 3465 days, or 9.49 years. The easiest way to find this is to solve $(0.9998)^t = \frac{1}{2}$. If 0.004% of the cadmium is removed from the body each day, it will take about 17328 days, or 47.47 years, which can be found by solving $(0.99996)^t = \frac{1}{2}$. So the half-life for cadmium is between 9.5 and 47.5 years, depending on the person.

Answers to Problems and Teaching Suggestions for **So Much Coffee, So Little Time: More on Caffeine in the Body**

This activity can be used to help students see the value of "solving systems of linear equations" at the Intermediate Algebra level. It shows connections between solving linear equations and exponential functions; that is, we need to use a number of different mathematical concepts to analyze real problems.

1. After n cups, t hours after the first cup, there will be $f(t)$ grams of caffeine in the body.

n cups	t hours	$f(t)$ mg of caffeine in body
1	0	100
2	1	187
3	2	262.7
4	3	328.5

2. Solve $\begin{cases} A + C = 100 \\ 0.87A + C = 187 \end{cases}$ to get $A = -669.23$ and $C = 769.23.$ Graph $f(t) = -669.23(0.87)^t + 769.23.$

It will be seen from the graph that this function is monotone increasing and its end behavior is to approach the line $y = 769.23$, *although students may not recognize immediately that the asymptote is exactly the constant term of the function. From this information, students can conclude that no one will die of an overdose of caffeine, which would happen at a level of about 5000 mg, by drinking 1 cup of coffee per hour.*

Reflection point: Students should have attempted problems 1 and 2 for homework. At the beginning of class, initiate a whole-class discussion to be sure students understand the form of this exponential function, $f(t) = Ab^t + C$, why this one (with $b = 0.87$) does not go off to infinity, and how they might have known what its horizontal asymptote would be from looking at the equation. One result of this example is that the amount of caffeine stabilizes at an equilibrium value of 769.23 mg in the body. Discuss this result with students in relation to medicines which are given periodically; that is, that the amount of medicine in the body stabilizes at a level deemed appropriate by the physician.

Pass out **Smokin'** and have the students work on problems 3, 4, and 5 in groups.

3. a. Solve $\begin{cases} A + C = 0 \\ 0.96A + C = 2.7 \end{cases}$ to get $A = -67.5$ and $C = 67.5$. Thus, the function is $y = -67.5(0.96)^t + 67.5$.

b. $y = -67.5(0.96)^{30} + 67.5 = 47.7$ mg.

c. $y = 67.5$

Assure that students see this result geometrically (on their graphs) and algebraically (from studying the function). It would also be valuable to see it numerically from an electronically generated table.

4. a. The amount absorbed each year is 0.001 $\times$ 20 $\times$ 365 $\times$ 0.1=0.73, so $f(1) = 0.73$. Also, $f(t) = A(0.93)^t + C$, so students need to solve the equations

$$\begin{cases} 0 = A + C \\ 0.73 = 0.93A + C \end{cases}$$

giving $A = -10.43$ and $C = 10.43$, so $f(t) = -10.43(0.93)^t + 10.43$ and $f(30) = 9.25$ mg of cadmium.

b. The amount absorbed each year is 0.002 $\times$ 20 $\times$ 365 $\times$ 0.4=5.84, so $f(1) = 5.84$. Also, $f(t) = A(0.986)^t + C$, so students need to solve the equations

$$\begin{cases} 0 = A + C \\ 5.84 = 0.986A + C \end{cases}$$

giving $A = -417.14$ and $C = 417.14$ so $f(t) = -417.14(0.93)^t + 417.14$ and $f(30) = 143.87$ mg of cadmium.

5. 47.7 (our result in problem 3) is about 1.6 times the "average." In order to assess the relationship between our result that the smoker receives 47.7 mg <u>more</u> cadmium as a result of smoking and the "average" amount of 30 mg, we would need to know what proportion of the 50 year old population has smoked for 30 years and what is the average number of cigarettes smoked per day. We would also need to know for how long and at what ages other fractions of this population smoked. That the added amount due to cigarettes is actually between about 10 mg and about 150 mg makes it even harder to assess the validity of this statement. There is no right or wrong answer to this problem. Hopefully it will cause students to think more deeply about the math and about what additional information might be needed before making a statement like that given in this problem.