The one-sector growth model with idiosyncratic shocks: Steady states and dynamics

Mark Huggett\textsuperscript{a,b,*}

\textsuperscript{a}Department of Economics, University of Illinois, Champaign, IL 61820, USA
\textsuperscript{b}Centro de Investigacion Economica, ITAM, Mexico DF 10700, Mexico

Accepted March 1997

Abstract

This paper investigates the one-sector growth model where agents receive idiosyncratic labor endowment shocks and face a borrowing constraint. It is shown that any steady-state capital stock lies strictly above the steady state in the model without idiosyncratic shocks. In addition, the capital stock increases monotonically when it is sufficiently far below a steady state. However, near a steady state there can be non-monotonic economic dynamics.

Keywords: Growth; Idiosyncratic shocks; Steady states; Dynamics
JEL classification: E13, O41

1. Introduction

This paper characterizes the steady state and dynamic properties of the one-sector growth model with two non-standard features. The first feature is that there are a continuum of agents in the economy experiencing idiosyncratic labor endowment shocks. The endowment uncertainty is such that there is

\*Correspondence address. Instituto tecnologica Autonoma de Mexico, centro de Investigacion Economica, Av. camino Santa Teresa 930, D.F. 107000 Mexico, Mexico.

I thank seminar participants at Illinois, Waterloo, the Federal Reserve Bank of Minneapolis and the 1995 Midwest Macroeconomics meetings for comments. I also thank the referee for valuable suggestions.

0304-3932/97/$17.00 © 1997 Elsevier Science B.V. All rights reserved

\textit{Pll S0304-3932(97)00025-1}
uncertainty for individual agents but no uncertainty over the aggregate labor endowment. The second feature is that there are financial market imperfections. One of these imperfections is that agents face a borrowing constraint in that asset holding cannot be negative. The other imperfection is that there are, by assumption, no markets to insure against endowment uncertainty. Thus, individuals can only self-insure by holding non-negative quantities of a single asset—physical capital.

Becker and Foias (1987) and Hernandez (1991) have investigated the steady state and dynamic properties of this model when agents face a borrowing constraint but do not experience idiosyncratic endowment shocks. They show that the steady-state capital stock in the borrowing-constrained economy is unique and coincides with that in the complete market economy. In addition, Hernandez (1991) shows that if aggregate capital income is increasing in the level of capital, then the turnpike property holds (i.e. the capital stock converges monotonically to the steady state) regardless of the distribution of capital across agents.1 This turnpike property is a well-known property of the one-sector model with complete markets.2

Much less is known about the properties of models with idiosyncratic shocks. This is not because of a lack of importance of models of this type. Economists have been interested in such models at least since the work of Friedman (1957) on the permanent-income hypothesis. At a partial equilibrium level, the theoretical properties of this model have been developed by Schechtman and Escudero (1977), Bewley (1977), Sotomayor (1984) and others. There is also a large applied literature that is based on the Euler equations generated by these partial equilibrium models. At a general equilibrium level, the existing theoretical work has concentrated on the existence of steady states. Contributors include Laitner (1979), Bewley (1984) and Clarida (1990). Recently, there has been substantial interest in using general equilibrium models of this type to address a number of interesting quantitative questions.3 This work on quantitative questions provides additional motivation for understanding the theoretical properties of the underlying model.

1Becker and Foias (1987) examine the case in which agents discount at different rates instead of the equal discount factor case considered by Hernandez (1991). They prove that the capital stock also converges to the unique steady state and that this convergence is eventually monotonic.

2In multisector growth models, turnpike results imply convergence (not monotonic convergence) for high enough discount factors. See McKenzie (1986).

3The questions have been in the areas of asset pricing, savings, optimal debt and tax structure, business cycles, as well as income and wealth distribution. See Huggett (1993), Aiyagari (1994), Krusell and Smith (1994), Castaneda, Diaz-Gimenez and Rios-Rull (1994), Rios-Rull (1995), Aiyagari and McGrattan (1995) and references cited therein for a guide to work in these areas.
The key findings of this paper are as follows. First, in steady state the capital stock is always strictly greater than the steady-state capital stock in the complete markets economy. Thus, it is also true that in steady state the capital stock in the economy with idiosyncratic shocks is strictly greater than the steady-state capital stock in the economy without idiosyncratic shocks. This result is completely independent of the sign of the third derivative of the period utility function. This is contrary to what many economists, who are familiar with results in the partial equilibrium literature on precautionary savings (e.g. Leland, 1968; and Sandmo, 1970), might have conjectured. Second, the capital stock increases monotonically when the capital stock is below the steady state of the model economy without idiosyncratic shocks. Third, an example is provided in which near a steady state there can be interesting non-monotonic economic dynamics. Thus, the turnpike property does not hold. The example shows that the distribution of wealth can matter for the nature of economic dynamics even in one-sector models. The example makes use of the fact that the marginal propensities to save are not equated across agents in steady state and thus a redistribution of wealth can lead to a change in the aggregate capital holdings in the economy. It should be noted that idiosyncratic endowment shocks are important for this result as in the absence of idiosyncratic shocks non-monotonic dynamics are not possible (see Hernandez, 1991).

This paper is organized in five sections. Section 2 describes the economy. Section 3 characterizes steady states. Section 4 characterizes economic dynamics. Section 5 concludes.

2. The economy

2.1. The environment

The environment is composed of a continuum of infinitely lived agents. The total mass of agents is equal to 1. Each agent's preferences over consumption are given by a utility function:

\[
\mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right] \text{ where } 0 < \beta < 1.
\]

All agents have identical period utility functions \( u \). In the theorems proved in this paper, the period utility function satisfies some combination of the following assumptions:

A1 \( u : \mathbb{R}_+ \to \mathbb{R} \) is bounded and continuous.
A2 \( u' > 0, u'' < 0, \) and \( \lim_{c \to 0} u'(c) = \infty \).
A3 \( u' \) is convex.
Each agent receives a random labor endowment $e \in E$ each time period. Labor endowments are measured in efficiency units. The labor endowment follows a finite-state Markov process. Transition probabilities are given by a transition function $\pi(e'|e)$. Endowments are independent across agents. Thus, there is endowment uncertainty at the individual level but there is no uncertainty over the aggregate labor endowment. Assumption A4 lists the restrictions on endowments.

$$A4 \quad E = \{e_1, \ldots, e_n\} \quad \text{where } 0 < e_1 < \cdots < e_n \text{ and } n \geq 2.$$

$$\sum_e \pi(e'|e) = 1 \text{ for all } e \in E \text{ and } \pi(e'|e) > 0 \text{ for all } e, e' \in E.$$

The production technology is given by a total output function $f(K)$. The technology maps the capital stock (equivalently the capital—labor ratio) in a time period into an output level for that time period. The technology can be related to a standard constant returns to scale production function $F(K, L)$ as follows: $f(K) = F(K, 1) + (1 - \delta)K$. In this specification $\delta$ is the depreciation rate of capital. The technology satisfies assumption A5 which is the usual concavity condition considered in neoclassical growth theory:

$$A5 \quad f(0) = 0, f' > 0 \text{ and } f'' < 0.$$

2.2. An agent's decision problem

Each agent solves a version of the standard ‘income fluctuation problem’ (Schechtman and Escudero, 1977). In this problem, an agent faces a deterministic sequence $\{w_t, r_t\}$ of wage rates and interest rates. An agent then supplies labor inelastically and chooses capital holdings over time to maximize utility. Agents face a borrowing constraint in that each period capital holdings cannot be negative. By assumption, there are no other assets available to insure against endowment uncertainty.

This decision problem is now described in the language of dynamic programming. An agent’s position at a point in time is described by an individual state $x$. The individual state $x = (k, e)$ is the agent’s current capital holding and endowment. The individual state $x$ lies in the individual state space $X = [0, \infty) \times E$. An agent in state $x = (k, e)$ receives in period $t$ a payment of $ew_t + k(1 + r_t)$. The agent then chooses capital and consumption to solve the following dynamic programming problem. The expectation in the dynamic programming problem is over values of the agent’s idiosyncratic shocks:

$$v(x, t) = \sup_{k'} u(ew_t + k(1 + r_t) - k') + \beta E[v(k', e', t + 1)|x] \quad (1)$$

$$k' \quad \text{s.t. } 0 \leq k' \leq ew_t + k(1 + r_t).$$

If the period utility function $u$ is bounded, then the contraction mapping theorem implies that a unique, bounded solution $v$ to Eq. (1) exists.
Furthermore, Theorem 3 in Denardo (1967) says that the solution \( v \) is the optimal value function. If the utility function is also continuous, then Corollary 2 in Denardo (1967) guarantees that optimal decision rules for capital holding \( k(x, t) \) and consumption \( c(x, t) \) exist that achieve the optimal value function. Thus, we have that assumption A1 is sufficient for the existence of optimal decision rules to problem (1). Fact 1–4 below lists some properties of optimal decision rules that will be useful later.

In the proof of Fact 1–4, I use the following standard results for this type of problem. First, the value function is increasing, strictly concave and differentiable in \( k \) and \( v_t(k, e, t) = u'(c(k, e, t))(1 + r_t) \). Second, the following Euler equation is a necessary condition for utility maximization:

\[
u'(c(x, t)) \geq \beta (1 + r_t + 1) E[u'(c(k(x, t), e', t + 1)) | x] \text{ with equality if } k(x, t) > 0.
\]

**Fact.** Assume A1, A2, A4 and \( w_t, (1 + r_t) > 0 \) for all \( t \), then
1. \( c(x, t) \) and \( k(x, t) \) are continuous in \( x \).
2. \( c(k, e, t) \) is strictly increasing in \( k \) and \( c(x, t) > 0 \) for all \( x \).
3. \( k(k, e, t) \) is increasing in \( k \).
4. If \((w_t, r_t) = (w, r)\) for all \( t \) and \( \beta (1 + r) \leq 1 \), then there is an \( a > 0, e \in E \) and a set of states \( A = [0, a] \times \{e\} \) on which the Euler equation holds with strict inequality.
5. If \((w_t, r_t) = (w, r)\) for all \( t \) and \( \beta (1 + r) \leq 1 \), then for all \( k > 0 \), there is an \( e \in E \) such that \( k(k, e, t) < k \).

**Proof.** See the Appendix.

### 2.3. The firm

There is a single firm that operates the technology \( f(K) \). The firm buys capital \( K \) each period to maximize profit. Profit maximization implies that in period \( t \) the interest rate \( r_t \) satisfies Eq. (2) below. The wage rate \( W_t \) is also defined in Eq. (2):

\[
(1 + r_t) = f'(K), \quad w_t = f(K) - f'(K)K.
\]

### 2.4. Equilibrium

To state the equilibrium concept, some way of describing and keeping track of the heterogeneity in the economy is needed. At time \( t \) the distribution of

---

*The way in which this paper handles agent heterogeneity is similar to the treatment in Lucas and Prescott (1974), Foley and Hellwig (1975), Lucas (1980) and others.*
individual states across agents is described by the aggregate state \( y_t \). The aggregate state is a probability measure defined on \( X \), where \( X \) denotes the Borel subsets of \( X \). Thus, for all \( B \in X \), \( y_t(B) \) is the mass of agents whose individual states lie in \( B \) at time \( t \). Since \( y_t \) is a probability measure, the total mass of agents is equal to 1. The aggregate state \( y \) lies in the aggregate state space \( Y = \{ y : \int_X k \, dy < \infty \} \).

To describe how the aggregate state and the capital stock evolve over time, it is useful to define functions \( P(x, t, B) \) and \( K(y) \). The function \( P(x, t, B) \) is a transition function that gives the probability that an agent in individual state \( x \) at time \( t \) will have an individual state that lies in the set \( B \) next time period. The function \( K(y) \) simply gives the aggregate capital stock as a function of the aggregate state \( y \). These functions are defined below.

\[
P(x, t, B) = \pi(\{ e' \in E : (k(x, t), e') \in B \})|e, \\
K(y) = \int_X k \, dy.
\]

**Definition.** An equilibrium is a pair of functions \( c(x, t) \), \( k(x, t) \) and sequences \( \{w_t, r_t, y_t\} \) satisfying:

1. \( c(x, t) \) and \( k(x, t) \) are optimal decision rules.
2. \( \{w_t, r_t\} \) satisfy Eq. (2) for all \( t \geq 0 \).
3. Markets clear: For all \( t \geq 0 \),
   - (i) \( \int_X (c(x, t) + k(x, t)) \, dy_t = f(K(y_t)) \),
   - (ii) \( \int_X e \, dy_t = 1 \).
4. Law of motion: For all \( t \geq 0 \) and all \( B \in X \), \( y_{t+1}(B) = \int_X P(x, t, B) \, dy_t \).

### 3. Steady states

This section considers the properties of a steady state. A steady-state equilibrium is an equilibrium where in all time periods \( (w_t, r_t, y_t) = (w, r, y) \) and where \( c(x, t) \) and \( k(x, t) \) are time invariant functions of the state \( x \). Theorem 1 states that in a steady state with positive capital the capital stock satisfies the condition \( \beta f'(K(y)) < 1 \). This means that the capital stock is strictly larger than the steady-state capital stock in the economy without endowment uncertainty. It also means that the rate of return on capital will be strictly lower than the return without uncertainty. It is useful to recall from standard growth theory that the capital level solving \( \beta f'(K) = 1 \) is the steady-state capital level in the economy without endowment uncertainty.

The proof of Theorem 1 is based upon the fact that in steady state any statistic of the state of the economy must remain invariant over time. However, marginal utility averaged over the population shrinks over time when \( \beta f'(K(y)) \geq 1 \). The proof is in two steps. First, it is argued that \( \beta f'(K(y)) \leq 1 \). If this were not the case, then the Euler equation would imply that marginal utility would shrink.
over time when averaged across agents. Second, it is argued that $\beta f'(K(y)) = 1$ is not possible. The argument is nearly identical except that it is shown that in this case there would be a positive mass of agents for whom the borrowing constraint would bind and thus the Euler equation would hold with strict inequality. Therefore, marginal utility shrinks over time.

**Theorem 1.** Assume A1, A2, A4, and A5. In a positive capital steady state, $\beta f'(K(y)) < 1$.

**Proof.** First show that $\beta f'(K(y)) \leq 1$. The inequality below is a necessary condition for consumer maximization.

$$u'(c(x,t)) \geq \beta(1+r) E[u'(c(k(x,t),c',t + 1))|x].$$

Integrate both sides of the above equation using the stationary probability measure $y$. The integrals are finite as factor prices are strictly positive with a positive capital stock (assumption A5) and as Fact 2 implies that there is a strictly positive lower bound on the level of consumption in any state. The equality below follows from two facts. First, Stokey and Lucas (1989, Theorem 8.3) implies for any non-negative, measurable function $g$ that $\int_x E[g(x') | x] dy = \int_x g(x') dy^*$, where $y^*(B) = \int_x P(x,B) dy$. Second, the definition of steady state implies that $y^* = y$. The second line below follows as $c(x,t)$ is time invariant in steady state.

$$\int_x u'(c(x,t)) y(dx) \geq \beta(1+r) \int_x E[u'(c(k(x,t),c',t + 1))|x] y(dx)$$

$$\geq \beta(1+r) \int_x u'(c(x,t + 1)) y(dx), \beta(1+r) = \beta f'(K(y)).$$

Lemma 1 below completes the proof.

**Lemma 1.** Assume A1, A2, A4, and A5. In a positive capital steady state, $\beta f'(K(y)) \neq 1$.

**Proof.** See the Appendix.

**Comments:** (1) Laitner (1979, 1992), Bewley (1984) and Clarida (1990) prove for the case of i.i.d. endowment shocks that there exist steady-state equilibria where $\beta(1+r) < 1$.\(^5\) Theorem 1 above, states that only steady-state equilibria with this property are possible. One of the key steps in these existence results is

\(^5\)Bewley and Clarida actually analyse endowment economies. However, it is clear that the analysis also applies to production economies.
to argue that the capital stock in steady state is a continuous function of \((1 + r)\). Then one shows that as the interest rate approaches the level \(\beta(1 + r) = 1\) from below that steady-state capital becomes arbitrarily large. These papers use stronger restrictions on preferences than those in Theorem 1 above. Bewley (1984) and Clarida (1990) use the weakest restriction. They require that the coefficient of relative risk aversion is bounded asymptotically. This restriction implies (Schectman and Escudero, 1977, Theorem 3.9) that asset accumulation by any agent will be bounded for interest rates satisfying \(\beta(1 + r) < 1\). Sometime after early versions of this paper were available, Aiyagari (1995) developed an alternative way to prove an i.i.d. shock version of Theorem 1.

(2) It is sometimes suggested that aggregate precautionary savings arises from (i) a high time preference rate relative to the interest rate, (ii) a positive third derivative of the period utility function and (iii) a borrowing constraint. In the model of this paper, there is aggregate precautionary savings if aggregate capital in steady state is strictly larger in economies with idiosyncratic shocks than in the same economies without idiosyncratic shocks, holding aggregate labor endowment constant. Theorem 1 states that this result always holds. It is interesting to note that the result applies regardless of how much agents discount the future and regardless of the sign of the third derivative. The magnitude of the time preference rate is irrelevant to the result as the interest rate is an endogenous variable and adjusts to be always less than the time preference rate. The third derivative of the utility function has a role in determining precautionary savings in the two-period, partial-equilibrium models of Leland (1968) and Sandmo (1970) but has no role in the infinite-period, general-equilibrium model studied here. Huggett and Ospina (1997) prove for this model that aggregate precautionary savings occurs if and only if borrowing constraints are binding for a positive mass of agents. Thus, the result here is due to endogenously binding borrowing constraints.

(3) Theorem 1 can be extended to the case of a finite number of agent types differing in preferences and endowment processes. This is true as the reasoning in the theorem will hold for each agent type considered separately. Theorem 1 will also hold when discount factors differ across agent types for exactly the same reason. Theorem 1 can also be extended to the case where agents are allowed to borrow up to a negative borrowing limit. As long as the minimum earnings are sufficient to maintain asset holdings at the borrowing limit and to allow strictly positive consumption, then Theorem 1 follows. When the borrowing limit is set sufficiently negative so that it is not possible to both maintain asset holdings at the borrowing limit and afford positive consumption, then it is no longer clear that the reasoning described in the proof of Theorem 1 will work. In this case, the relevant integrals need not be bounded. Finally, it is also clear that the implication of Theorem 1 (i.e. \(\beta(1 + r) < 1\)) holds in pure endowment economies when the borrowing limit allows for positive consumption.
(4) The quantitative importance of Theorem 1 for the size of the risk-free interest rate in exchange economies is explored by Huggett (1993). The quantitative importance of Theorem 1 for the size of aggregate precautionary savings is explored by Aiyagari (1994).

4. Dynamics of capital accumulation paths

This section investigates the dynamics of capital paths. First, I briefly review existing arguments for why capital paths are monotone in the economy with borrowing constraints but without idiosyncratic shocks. I extend one of these arguments to apply to economies with idiosyncratic shocks and show that capital paths are monotone increasing when the capital stock is sufficiently far below a steady state. Second, I provide an example showing that capital paths need not be monotone near a steady state.

4.1. A partial result on monotonicity

The monotonicity result in the one-sector model with borrowing constraints but without idiosyncratic shocks is based on Euler equation arguments. The Euler equation is a necessary condition for consumer maximization which states that $u'(c_t) \geq \beta(1 + r_{t+1}) u'(c_{t+1})$, where equality holds when an agent holds strictly positive quantities of capital. The equation implies that consumption grows when capital in period $t+1$ is below the steady state and falls when capital is above the steady state. The fact that one can unambiguously 'back out' what happens to consumption allows one to prove that capital must be monotone increasing when capital is below steady state and that the opposite occurs when capital is above steady state. Actually, to prove that capital is monotone decreasing above the steady state, Hernandez (1991) assumes that the production technology is such that capital income is increasing in the level of capital.

Matters are not as simple in economies with idiosyncratic shocks. The Euler equation is then $u'(c_t) \geq \beta(1 + r_{t+1}) E[u'(c_{t+1})|e']$, where equality holds when an agent holds strictly positive quantities of capital. The expectation operator makes it unclear what will happen to either realized or expected consumption growth when $\beta(1 + r_{t+1})$ is either above or below 1. Some progress could still be made with stronger assumptions on preferences. For example, when $\beta(1 + r_{t+1})$ is greater than 1 it could still be argued that expected consumption for an individual grows if $u'$ is a convex function. This is just Jensen's inequality. This allows one to argue that with convex marginal utility (assumption A3) capital increases monotonically when $\beta(1 + r_{t+1}) > 1$. The example in the next subsection suggests that it is highly unlikely that there is any general result on the monotonicity of capital paths when $\beta(1 + r_{t+1}) < 1$. 
The next theorem formalizes the logic stated above. The theorem states that the capital stock is strictly increasing when the capital stock is below the steady state of the complete market economy or, equivalently, below the steady state of the economy without idiosyncratic shocks. The theorem also rules out aggregate fluctuations where the capital stock starts above this level and then falls below this level. Thus, any non-monotonic fluctuations in the capital stock must occur at capital levels at or above the level of capital solving $\beta f'(K) = 1$.

**Theorem 2.** Assume A1–A5.

If $\beta f'(K(y_t)) > 1$ and $K(y_t) > 0$, then $K(y_{t-1}) < K(y_t) < K(y_{t+1})$.

**Proof:** See the Appendix.

### 4.2. An example where capital paths are not monotone

This section provides an example where capital paths are not monotone. The intuition driving the example is as follows. Take an economy that is in steady state. Redistribute capital across agents without changing the aggregate capital stock. If the marginal savings propensities differ across agents in different states then the implied aggregate capital stock should not remain constant over time at the unchanged steady-state prices. There are then a couple of possibilities. One possibility is that the capital stock could monotonically converge to a different steady-state capital stock. This could happen if steady states are not unique. However, I have computed that for the example considered here there is only one steady-state capital stock. Thus, the remaining possibility is that the capital path is not monotone.

**Example.** Preferences: $u(c) = e^{1-\sigma}/(1-\sigma)$, $(\beta, \sigma) = (0.96, 1.5)$,

**Endowments:** $E = \{e_1, e_2\} = \{0.8, 1.2\}$,

$\pi(e_1|e_1) = \pi(e_2|e_2) = 0.5$,

**Technology:** $f(K) = AK^2 - (1 - \delta)K$, $(A, x, \delta) = (1, 0.36, 0.1)$

The economy has preferences and a technology of the parametric classes commonly used in applied work. The utility functions of agents are identical and homothetic. This means that with complete markets the distribution of capital across agents does not affect the dynamics of prices or economic aggregates. The technology is Cobb–Douglas and, therefore, satisfies the conditions in Hernández (1991) that guarantee that capital paths are monotone in the absence of

---

*There are some redistributions that don't make a difference. For example, there are some redistributions that are symmetric about the average capital holdings that leave the distribution unchanged.*
endowment shocks. There are two possible values of the endowment shock. The shocks are independent and identically distributed and occur with equal probability. The economy is completed by describing the initial distribution of capital across agents. The initial distribution puts 20% of the agents exactly at zero asset holdings and equal numbers of agents at all capital levels between 0 and 10.8104. Thus, the aggregate capital stock is 4.3242 which is the steady-state capital stock that I calculate for this economy. The steady-state distribution looks roughly log-normal. Therefore, the initial distribution puts less agents at high capital values and more agents near zero as compared to the steady-state distribution.

The intuition described previously for why capital paths are not monotone was based on the non-linearity of the optimal decision rule for capital. Fig. 1 graphs the steady-state optimal decision rule \( k(k, e) \) on a 45° line diagram for each of the two endowment shocks. Fig. 1 concentrates on the shape of the decision rule near the borrowing constraint. The decision rule is strongly non-linear in this region. In particular, there is a 'flat spot' at zero asset holdings. Recall that Fact 3 together with Theorem 1 established that the steady-state

![Diagram](image)

Fig. 1.
optimal decision rule always has a flat spot and thus one might expect that the non-monotonic behavior produced in the example is a general feature of this type of model economy. The fact that Fig. 1 is quite linear over much of the domain of potential asset holdings suggests that the magnitude of the departures of aggregate capital from the steady-state value will be small for this example.7

I will now discuss the properties of equilibrium paths and postpone until the next section a description of the computational methods employed to approximate equilibria. Fig. 2 describes the equilibrium path for the capital stock and, for comparison purposes, the steady-state path. Both paths are consistent with the results described in Theorems 1 and 2. Thus, the steady-state capital level lies strictly above the steady-state level in the complete market economy which for

---

7More dramatic results could easily be obtained by (1) putting a larger mass of agents at the corner of their borrowing constraint, (2) considering utility functions that are not homothetic, (3) allowing for multiple types of agents with different preferences or (4) changing the endowment process.
this example is slightly below 4.30. This is the content of Theorem 1. Also, both paths stay above the steady-state capital level in the complete market economy as Theorem 2 requires. This is illustrated by the fact that these paths stay above the shaded region at the bottom of Fig. 2.

In Fig. 2 the capital stock at first rises above and then below the steady state and thereafter slowly converges to the steady state. The dynamics of the capital stock can be thought of as depending on the initial distribution of capital and on the changes over time in the optimal decision rule for capital. For the economy considered here it turns out that the shape of the path is largely determined by the initial distribution of capital and not by changes in the decision rule over time. When the economy is simulated using the steady-state decision rule instead of the equilibrium decision rule, then one generates a capital path that is qualitatively similar to that in Fig. 2. Changes in the optimal decision rule over time serve to dampen these fluctuations considerably without changing the qualitative nature of the path. Using this intuition, capital at first increases in Fig. 2 as the marginal dissavings of agents moved to zero asset holdings are less than the marginal savings of agents experiencing an increase in asset holdings.

To close this section, I compare the present example to other examples of aggregate fluctuations that are based on financial market imperfections within the one-sector growth model. See Boldrin and Woodford (1990) for a much broader review of models of aggregate fluctuations. First, Bewley (1986) provides an example where the capital stock cycles. His example is based on there being two types of agents whose labor endowments cycle in opposite directions every other period. Second, Becker and Foias (1987) construct an example where the capital stock cycles. Their example relies on the low substitutability of capital for labor in the production technology. In their example, the total payment to capital depends sensitively on the level of capital. Hernandez (1991) proves for the case when all agents discount future utility at the same rate that such examples cannot occur when the total payment to capital is increasing in the capital stock. In the example of this paper the production technology is Cobb–Douglas and thus rules out these fluctuations when there is no idiosyncratic variation in labor endowment. Thus, the example presented here is based on endowment fluctuation as in the example in Bewley (1986). The key difference is that the role of distribution is emphasized here rather than multiple types of agents with periodically fluctuating endowments. The present example also uses the functional forms and parameter values commonly used in applied work. Thus, it is not so easy to dismiss the relevance of fluctuations produced in this way.

4.3. Computational details

The equilibrium discussed in the previous section are computed as follows. First, calculate the capital stock in a stationary equilibrium. This can be
determined by guessing different values of $K$ in the steady-state capital and then backing out factor prices from marginal products. The capital stock implied by these prices in a stationary equilibrium can then be determined by computing time-independent decision rules for capital and then integrating these decision rules with respect to the stationary distribution $\pi$ implied by the guess, $K$. The results of Hopenhayn and Prescott (1992, Theorem 2) can be applied to this problem to show that this stationary distribution is unique and that the integral can be computed up to arbitrary accuracy. Details of how to apply this theorem to the present context are provided in Huggett (1993). Steady-state capital levels are then determined by finding a fixed point of the mapping from the conjectured to the implied capital stock. When I computed this mapping, there was a unique point where the map crossed the 45° line. Thus, steady-state equilibria appears to be unique in this example. This completes the argument that there is only one steady state. The precise details of how to carry out these computations are provided in Huggett (1993).

The final step in the construction of the equilibrium is to determine the transition path from the initial capital stock level back to this steady-state level. The procedure is to assume that the economy is back in a steady state after a large number of periods, which in these calculations is 1000 periods. Next, a guess is made for the transition path for the economy. With the guess, the decision rules $k(x,t)$ can be computed by backward recursion on Bellman’s equation (1). The transition path implied by the guess can then be computed. If the implied path coincides with the guessed path and the path converges to the steady state then one has computed an equilibrium. If the guessed and implied paths do not agree then one updates the guess and repeats the process.

5. Conclusion

This paper shows that the steady-state and dynamic properties of the one-sector growth model with complete markets do not survive the addition of uninsurable, idiosyncratic shocks and a borrowing constraint. In particular, there is more capital in steady state in the model investigated here than in similar models without idiosyncratic shocks. This result does not depend on the third derivative properties of the period utility function. Instead, the result is due to endogenously binding borrowing constraints. Furthermore, there can be non-monotonic economic dynamics around a steady state. In the future it would be interesting to investigate the nature and the quantitative importance of these economic dynamics. The example provided here suggests that the economic fluctuations produced by 'shocking' the higher moments of the distribution of wealth occur at lower frequencies than business-cycle fluctuations.
6. Appendix

Fact. Assume A1, A2, A4 and \( w_n (1 + r_i) > 0 \) for all \( t \), then

1. \( c(x, t) \) and \( k(x, t) \) are continuous in \( x \).
2. \( c(k, e, t) \) is strictly increasing in \( k \) and \( c(x, t) > 0 \) for all \( x \).
3. \( k(k, e, t) \) is increasing in \( k \).
4. If \( (w_n, r_i) = (w, r) \) for all \( t \) and \( \beta (1 + r) \leq 1 \), then there is an \( a > 0, e \in E \) and a set of states \( A = [0, a) \times \{ e \} \) on which the Euler equation holds with strict inequality.
5. If \( (w_n, r_i) = (w, r) \) for all \( t \) and \( \beta (1 + r) \leq 1 \), then for all \( k > 0 \), there is an \( e \in E \) such that \( k(k, e, t) < k \).

Proof. (1) The theorem of the maximum together with the strict concavity of \( u \) and \( v \) generates the result.
2. \( c(k, e, t) \) is strictly increasing in \( k \) because \( v_1 (k, e, t) = u'(c(k, e, t))(1 + r_i) \) and \( v \) is increasing and strictly concave in \( k \). Assumption A2 guarantees that consumption is always strictly positive. Suppose, by way of contradiction, that there were values \( k_1, k_2, e \) and \( t \) such that \( k_2 > k_1 \) and \( k(k_2, e, t) < k(k_1, e, t) \). Since \( c(k, e, t) \) is strictly increasing in \( k \) it is true that

\[
\beta (1 + r_{i+1}) E[u'(c(k, e, t), e', t + 1)]|x] > \beta (1 + r_{i+1}) E[u'(c(k, e, t), e', t + 1)]|x].
\]

The Euler equation then implies that \( u'(c(k_2, e, t)) > u'(c(k_1, e, t)) \), a contradiction.
3. Fact 3 clearly follows from the continuity of \( u' \) and \( c(x, t) \) if the following holds for some \( e \in E \):

\[
u'(c(0, e, t)) \geq \beta (1 + r) E[u'(c(k, e, t), e', t + 1)]|x].
\]

Suppose, by way of contradiction, that the above inequality does not hold, then the equality below holds for all \( e \in E \). The inequality below holds as \( \beta (1 + r) \leq 1 \), \( c(k, e, t) \) is increasing in \( k \) and \( c(k, e, t) \) is time invariant when factor prices are constant.

\[
u'(c(0, e, t)) > \beta (1 + r) E[u'(c(k, e, t), e', t + 1)]|x] \leq E[u'(c(0, e', t))]|x].
\]

Given assumption A4, the above can hold only if \( c(0, e, t) \) is the same for all \( e \in E \). This then implies that \( k(0, e, t) > 0 \) for \( i \geq 2 \) and therefore for \( i \geq 2 \) the following holds:

\[
u'(c(k, e, t)) > E[u'(c(0, e', t))]|x].
\]

This contradicts the implication that \( c(0, e, t) \) is the same for all \( e \in E \).
(4) Suppose, by way of contradiction, that there exists a $k > 0$ such that for all $e \in E$ it is true that $k(k, e, t) \geq k$. Following the reasoning in Fact 3, the following must hold for all $e \in E$:

$$u'(c(k, e, t)) = \beta(1 + r) E[u'(c(k(k, e, t), e', t + 1))|x] \leq E[u'(c(k, e', t))|x].$$

Given assumption A4, this can hold only if $c(k, e, t)$ is the same for all $e \in E$. This then implies that $k(k, e, t) > k$ for $i \geq 2$ and therefore for $i \geq 2$ the following holds:

$$u'(c(k, e, t)) < E[u'(c(k, e, t))|x].$$

This contradicts the implication that $c(k, e, t)$ is the same for all $e \in E$. □

**Lemma 1.** Assume A1, A2, A4 and A5. In a positive capital steady state, $\beta'(K(y)) \neq 1$.

**Proof.** Suppose by way of contradiction that $\beta'(K(y)) = 1$. It will then be shown later that there is a set $A$ such that $y(A) > 0$ and on this set the Euler equation holds with strict inequality. It then follows from the arguments in the proof of Theorem 1 that the following equation holds. To simplify notation, the time index is dropped in this equation and throughout the proof as decision rules are time invariant in steady state.

$$\int_x u'(c(x)) \, dy > \int_x E[u'(c(k(x), e'))|x] \, dy = \int_x u'(c(x)) \, dy.$$

Therefore, the lemma follows if such a set $A$ exists. Fact 3 implies that there is a set $A = [(0, a) \times \{e\}$ on which the Euler equation holds with strict inequality. It will be argued next that, in steady state, $y(A) > 0$. Given $x$ in $X$ and a sequence $\{e_i\}$, define the sequence $\{k_i\}$ as follows:

$$k1 = k(x), \quad ki = k(k_{i-1}, e_{i-1}) \quad \text{for } i \geq 2.$$  

Fact 4 implies that for any $x$ in $X$ there is a sequence $\{e_i\}$ such that $\{k_i\}$ converges to $0$. So it is possible to go from $x$ to $A$ in $N$ or greater steps, for some value $N$ which depends on $x$. Let $P^n(x, A)$ denote such n-step probabilities. Define $\pi_{\text{min}} = \text{Minimum} \{\pi(e'|e)\}$. Assumption A4 implies that $\pi_{\text{min}} > 0$. Therefore, $P^n(x, A) \geq (\pi_{\text{min}})^n > 0$ for all $n \geq N$. Select $B = [0, k] \times E$ such that $y(B) > 0$. Fact 2 then implies that there is an $N$ such that $P^n(x, A) (\pi_{\text{min}})^n > 0$ for all $n \geq N$ and for all $x$ in $B$. Therefore, the following is true:

$$y(A) = \int_x P(x, A) \, dy = \int_x P^n(x, A) \, dy \geq (\pi_{\text{min}})^n y(B) > 0. \quad \square$$

**Theorem 2.** Assume A1–A5.

If $\beta'(K(y_i)) > 1$ and $K(y_i) > 0$, then $K(y_{i-1}) < K(y_i) < K(y_{i+1})$.  

Proof. Suppose by way of contradiction that $K(y_{t-1}) \geq K(y_t)$. Apply Lemma 2 below repeatedly to get that $\{K(y_{t-1}), K(y_t), K(y_{t+1}), \ldots\}$ is decreasing. A necessary condition for maximization is

$$u'(c(x, t-1)) \geq \beta(1 + r_t) E[u'(c(k(x, t-1), e', t))|x].$$

Integrate this expression with respect to the measure $y_{t-1}$ and apply Stokey and Lucas (1989, Theorem 8.3) to get the inequality in the first line below. Repeat the argument to get the second line below.

$$\int_x u'(c(x, t-1)) \, dy_{t-1} \geq \beta(1 + r_t) \int_x u'(c(x, t)) \, dy_t,$$

$$\int_x u'(c(x, t-1)) \, dy_{t-1} \geq \beta^{n+1} (1 + r_t) \ldots (1 + r_{t+n}) \int_x u'(c(x, t + n)) \, dy_{t+n}.$$

This implies that $\int_x u'(c(x, t + n)) \, dy_{t+n}$ tends to zero as $n$ increases because the term on the right-hand side preceding the integral grows without bound. Given any $\varepsilon > 0$, let $n$ be such that $\int_x u'(c(x, t + n)) \, dy_{t+n} < \varepsilon$. Since marginal utility is a convex function (assumption A3), Jensen’s inequality implies the line below.

$$u'(\int_x c(x, t + n) \, dy_{t+n}) \leq \int_x u'(c(x, t + n)) \, dy_{t+n} < \varepsilon.$$

This then implies that aggregate consumption becomes arbitrarily large which contradicts the fact that output is bounded (recall the implication of Lemma 2). Therefore, it is the case that $K(y_{t-1}) < K(y_t)$. Repeat the same argument to obtain the second inequality in the Theorem. □


If $\beta f'(K(y_t)) \geq 1$ and $K(y_{t-1}) \geq K(y_t)$, then $K(y_t) \geq K(y_{t+1})$.

Proof. The first inequality below is a necessary condition for consumer maximization. The second inequality follows because $\beta(1 + r_t) \geq 1$.

$$u'(c(x, t-1)) \geq \beta(1 + r_t) E[u'(c(k(x, t-1), e', t))|x]$$
$$\geq E[u'(c(k(x, t-1), e', t))|x].$$

The inequality below follows by applying Jensen’s inequality to the equation above and then integrating. The equality follows by Stokey and Lucas (1989, Theorem 8.3).

$$\int_x c(x, t-1) \, y_{t-1} \, (dx) \leq \int_x E[c(k(x, t-1), e', t)|x] \, y_{t-1} \, (dx)$$
$$= \int_x c(x, t) \, y_t \, (dx).$$
Finally, $f(K(y_{t-1})) \geq f(K(y_t))$ and the previous step yields

$$K(y_t) = f(K(y_{t-1})) - \int x c(x,t-1)y_{t-1}(dx) \geq f(K(y_t))$$

$$- \int x c(x,y_t)y_t(dx) = K(y_{t+1}). \quad \square$$

References


