Macro II
Homework 4- Consumption

1. A consumer lives for \( J \) periods and has preferences over consumption given by the utility function 
\( U(c_1, \ldots, c_J) = \sum_{j=1}^{J} \beta^j u(c_j) \). Suppose that the consumer can borrow and lend at a fixed interest rate \( r \) subject to the restriction that the consumer cannot have a negative asset position at death. The period budget constraint for an agent is 
\( c_j + a_{j+1} \leq a_j(1+r) + e_j \).

(a) Formulate this problem as a dynamic programming problem. Be clear on states, controls and the form of Bellman’s equation.

(b) For the special case where \( \beta = 1 \) and \( r = 0 \) calculate (i.e. write down) the optimal decision rules and value function each period. Indicate any assumptions on the period utility function that you need to justify your answer.

(c) Suppose that \( u(c) = c^{1-\sigma}/(1-\sigma) \). Characterize the determinants of the growth rate in consumption across periods. In particular, describe how interest rates and preference parameters affect the growth rate of consumption.

(d) A standard result is the equivalence between budget sets stated in terms of a present value condition or in terms of a sequence of budget restrictions. For this problem assume that earnings are risky. Thus, the relevant functions below are functions of earnings histories.

Prove that \( \Gamma_1(a_1) = \Gamma_2(a_1) \).

\[
\Gamma_1(a_1) = \{(c_1, \ldots, c_J) : (i)c_j + a_{j+1} \leq a_j(1+r) + e_j, (ii)c_j \geq 0, (iii)a_{J+1} \geq 0\}
\]

\[
\Gamma_2(a_1) = \{(c_1, \ldots, c_J) : (i) \sum_j (c_j - e_j)/(1+r)^j \leq a_1(1+r), (ii)c_j \geq 0\}
\]

2. Write down the decision rules that solve the dynamic programming version of the problem below. It is understood that (i) preferences are quadratic (i.e. \( u(c) = -ac^2/2 + bc \) for \( a, b > 0 \)), (ii) \( (1+r) = 1 \), (iii) second period earnings \( e_2 \) takes on the value 1 and 2 with equal probabilities and (iv) \( x_1 \) describes initial resources. For simplicity, allow consumption to be negative. Compare your answer to that in problem 1b.

\[
\max E[\sum_{j=1}^{J} u(c_j)]
\]
\( c_1 + a_2 \leq x_1 \) and \( e_2 = a_2(1+r) + e_2 \)
3. Consider the problem below where earnings $e_j$ are independently distributed according to a distribution function $F_j$ in period $j$. How does the consumption allocation that solves this problem change when a government lump-sum taxes the agent $T_2$ in period 2 and makes a lump-sum transfer of $T_2(1 + r)^{J-2}$ to the agent in period $J$? Explain carefully. Assume that $T_2$ is less than the smallest endowment shock possible in period 2.

$$\max E[\sum_{j=1}^{J} \beta^{j-1} u(c_j)]$$
$$c_j + a_{j+1} \leq a_j(1 + r) + e_j \quad \forall j, \ c_j \geq 0 \quad \text{and} \quad a_{J+1} = 0$$

4. [Bonus Problem]

An agent maximizes $E[\sum_{j=1}^{J} \beta^{j-1} u(c_j)]$ subject to $c_j + a_{j+1} \leq a_j(1 + r) + e_j$ and $\forall j, c_j, a_{j+1} \geq 0$.

Assume that $u(c) = \log(c), \beta = 1, r = 0, J = 40$ and $e_j$ equals 1, 2 with equal probability each period. Also assume that earnings are iid and that initial asset holding is zero (i.e. $a_1 = 0$).

(a) Write a computer program to compute optimal decision rules to this problem.

(b) Use a random number generator to simulate 100 histories of earnings shocks over the lifetime. For each history, plot the realized profile of asset holdings that is implied by the optimal decision rules you calculate. Include the computer program with your problem set.

(c) Comment on the results of this simulation in light of your answer to problem 1(b). Specifically, in the model for problem 1(b) what does theory predict for the asset profile when earnings are constant at $e_j = 1.5$ each period?

**Hint:**

1. The computer program could be based on the finite dynamic programing methods from homework 3. One way to proceed is to allow for two state variables rather than one state variable as in homework 3. It is easy to see that the program from homework 3 is then modified by adding in one more DO LOOP which loops over this extra state variable.

2. A standard random number generator produces random numbers which are uniformly distributed on the interval $[0, 1]$. A draw of a $J$ period history of iid random variables on $[0, 1]$ can then be transformed into earnings shocks on the set $\{1, 2\}$ by a simple cutoff rule.