Heterogeneous Agent Models: I

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Introduction

- Early heterogeneous-agent models integrated the income-fluctuation problem into general equilibrium models.
- A key feature of these models is that agents receive idiosyncratic (more on this later) shocks and that asset markets are incomplete by assumption. The asset markets are like those in the income-fluctuation problem.
- Some of the notable theoretical properties of models in this class arise not from heterogeneity alone but from the shock structure and the lack of complete markets.
- As these are general equilibrium models they are potentially useful for policy analysis

Model Ingredients

- Preferences: $E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$
- Agent-specific (idiosyncratic) shocks: $z_t \in Z \subset R$
- Asset(s): $a_t \in A = [\underline{a}, \infty)$
- Agent's state: $x = (a, z) \in X = A \times Z$
- Distribution of agents: ψ_t describes the distribution of agents at time t over the state space X.
- Math: Typically the distribution will be given by a probabillity measure. This is not essential but it is useful for theoretical results. When the distribution is represented by a probability measure then the "mass" of agents equals 1.

Probability Space

- The triple $(X, \mathcal{X}, \psi)$ is called a probability space when X is a set, $\mathcal{X}$ is a σ-field of X and ψ is a probability measure on $\mathcal{X}$.
- A class $\mathcal{X}$ of subsets of X is a σ-field if it is a field and if it is closed under countable unions of sets in $\mathcal{X}$.
- A set function $\psi : \mathcal{X} \to [0, 1]$ is a probability measure if (i) $\psi(B) \in [0, 1], \forall B \in \mathcal{X}$, (ii) $\psi(X) = 1$ and $\psi(\emptyset) = 0$ and (iii) $\psi(\cup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} \psi(B_k)$ for any disjoint sequence of sets $B_k \in \mathcal{X}$.
- Interpretation: $\psi(B)$ is the mass or fraction of agents with a state x in the set B. In applications, ψ will contain all the information of a distribution function considered in a begining stats course.
- Reference: Billingsley (1986) Probability and Measure.

Example 1: Exchange Economy

- Narrative: Consider a situation where agents live forever.
- Each period each agent receives an endowment $e(z) > 0$ of a perishable good. The shock process z is a finite Markov process with transition matrix $\pi(z'|z) > 0, \forall z, z' \in Z$.
- An agent starts each period with a credit balance $a \in A = [\underline{a}, \infty)$
- An agent's state is $x = (a, z) \in X = A \times Z$.
- Each period, agents choose consumption $c \geq 0$ and **next period** credit balances $a' \geq \underline{a}$ to maximize utility.
- The price of one unit of (next period) credit balances is q.

Example 1: Exchange Economy

Definition: A stationary equilibrium is $(c(x), a(x), q, \psi)$ such that

1. $(c(x), a(x))$ solve P1, given q.
2. $\int_X c(x)d\psi = \int_X e(z)d\psi$ and $\int_X a(x)d\psi = 0$
3. $\psi(B) = \int_X P(x, B)d\psi$ for all $B \in \mathcal{X}$.

$$\mathbf{P1} \quad \mathrm{v(x) = \max_{(c,a')} u(c) + \beta E[v(a', z')|x]}$$

$$c + a'q \leq a + e(z) \ and \ c \geq 0, a' \geq \underline{a}$$

$$P(x, B) \equiv \sum_{z' \in Z} \pi(z'|z) 1_{\{(a(x), z') \in B\}}$$

P is a transition function describing the probability of going from state x to the set B next period.

Example 1: Exchange Economy

Definition: A recursive equilibrium is $(c(x,\psi), a(x,\psi), q(\psi), \Gamma(\psi))$ such that

1. $(c(x,\psi), a(x,\psi))$ solve P1, given (q,Γ)
2. $\int_X c(x,\psi)d\psi = \int_X e(z)d\psi$ and $\int_X a(x,\psi)d\psi = 0$
3. $\psi'(B) = \Gamma(\psi)(B) \equiv \int_X P(x,\psi,B)d\psi$ for all $B \in \mathcal{X}$.

$$\mathbf{P1} \quad \mathrm{v}(\mathrm{x},\psi) = \max_{(\mathrm{c},\mathrm{a}')} \mathrm{u}(\mathrm{c}) + \beta \mathrm{E}[\mathrm{v}(\mathrm{a}',\mathrm{z}',\psi')|\mathrm{x}]$$

$$c + a'q(\psi) \leq a + e(z) \ and \ c \geq 0, a' \geq \underline{a}$$

$$\psi' = \Gamma(\psi)$$

Discussion

One can try to learn about properties of stationary or recursive equilibria by computing equilibria in example economies or by proving theorems. We will do both shortly.

First, we introduce a similar economy but with a neoclassical production function.

Example 2: Production Economy

- Narrative: Consider a situation where agents live forever.
- Each period each agent receives a labor endowment $e(z) > 0$. The shock process z is a finite Markov process with transition matrix $\pi(z'|z) > 0, \forall z, z' \in Z$.
- An agent's state is $x = (a, z) \in X = [0, \infty) \times Z$.
- Each period, each agent chooses consumption $c \geq 0$ and **next period** capital $a' \geq 0$ to maximize utility, given wage and interest rate (w, r).
- There is an aggregate production function $F(K, L)$ and capital depreciates at rate δ.

Example 2: Production Economy

$K \equiv \int_X a d\psi$ and $L \equiv \int_X e(z) d\psi$

Definition: A stationary equilibrium is $(c(x), a(x), w, r, \psi)$ such that

1. $(c(x), a(x))$ solve P1, given (w, r).
2. $w = F_2(K, L), r = F_1(K, L) - \delta$
3. $\int_X c(x) d\psi + \int_X a(x) d\psi = F(K, L) + K(1 - \delta)$
4. $\psi(B) = \int_X P(x, B) d\psi$ for all $B \in \mathcal{X}$.

$$\mathbf{P1} \quad \mathrm{v(x) = \max_{(c,a')} u(c) + \beta E[v(a', z')|x]}$$

$$c + a' \leq a(1 + r) + we(z) \ and \ c \geq 0, a' \geq 0$$

$$P(x, B) \equiv \sum \pi(z'|z) 1_{\{(a(x), z') \in B\}}$$

Example 2: Production Economy

$K(\psi) \equiv \int_X a d\psi$ and $L(\psi) \equiv \int_X e(z) d\psi$

Definition: A recursive equilibrium is $(c(x,\psi), a(x,\psi), w(\psi), r(\psi), \Gamma(\psi))$ such that

1. $(c(x,\psi), a(x,\psi))$ solve P1, given (w, r, Γ)
2. $w(\psi) = F_2(K(\psi), L(\psi)), r(\psi) = F_1(K(\psi), L(\psi)) - \delta$
3. $\int_X c(x,\psi)d\psi + \int_X a(x,\psi)d\psi = F(K(\psi), L(\psi)) + K(\psi)(1-\delta)$
4. $\psi'(B) = \Gamma(\psi)(B) \equiv \int_X P(x,\psi,B)d\psi$ for all $B \in \mathcal{X}$.

$$\mathbf{P1} \quad \mathrm{v}(\mathrm{x},\psi) = \max_{(\mathrm{c},\mathrm{a}')} \mathrm{u}(\mathrm{c}) + \beta \mathrm{E}[\mathrm{v}(\mathrm{a}', \mathrm{z}', \psi')|\mathrm{x}]$$

$$c + a' \leq a(1 + r(\psi)) + w(\psi)e(z) \ and \ c \geq 0, a' \geq 0$$

$$\psi' = \Gamma(\psi)$$

Discussion

1. The exchange and production economy just considered have idiosyncratic risk but not any source of aggregate risk.
2. One might say that the economy displayed aggregate risk if the aggregate endowment or labor endowment was random over time. The economies under consideration have individual agents experiencing labor endowment risk but the aggregate labor endowment evolves deterministically.
3. We will restrict attention to distributions ψ where the distribution of the z component are already at the stationary fractions implied by $\pi(z'|z)$. The definition of recursive equilibrium was stated somewhat informally - conditions 1-4 did not specify the class of ψ for which the conditions hold.

Compute a Stationary Equilibrium: Algorithm 1

Algorithm 1:

1. Normalize $L = 1$. Guess $K \in [\underline{K}, \bar{K}]$.
2. Set $w = F_2(K, 1), r = F_1(K, 1) - \delta$
3. Iterate on Bellman's eqn (or Euler equation) backwards until approximate convergence to compute decision rules $(c(x; w, r), a(x; w, r))$.
4. From an arbitrary state $x_0 = (a_0, z_0) \in X$, simulate a very long sequence $\{z_t\}_{t=0}^{T}$ of shocks using the transition matrix $\pi(z'|z)$. Calculate $\{x_t\}_{t=0}^{T}$ where $x_{t+1} = (a(x_t; w, r), z_{t+1})$.
5. Set $K' = (1/(T+1)) \sum_{t=0}^{T} a(x_t; w, r)$.
6. If $K' = K$ up to computational tolerance, then (w, r, K, L) are stationary equilibrium values. Otherwise, modify guess in step 1 and repeat, keeping the shock sequence unchanged.

Compute a Stationary Equilibrium: Algorithm 2

Algorithm 2:

1. Normalize $L = 1$. Guess $K \in [\underline{K}, \bar{K}]$.
2. Set $w = F_2(K, 1), r = F_1(K, 1) - \delta$
3. Iterate on Bellman's eqn (or Euler equation) backwards until approximate convergence to compute decision rules $(c(x; w, r), a(x; w, r))$.
4. Guess arbitrary prob meas ψ_1 on X. Iterate on $\psi_{t+1}(B) = \int_X P(x, B) d\psi_t$ for "rich class of sets B" until convergence. Use $P(x, B) = \sum_{z' \in Z} \pi(z'|z) 1_{\{(a(x;w,r),z') \in B\}}$. Call ψ^* the approx converged measure.
5. Compute $K' = \int_X a d\psi^*$.
6. If $K' = K$ up to computational tolerance, then (w, r, K, L) are stationary equilibrium values. Otherwise, modify guess in step 1 and repeat.

A Specific Model

Model Parameters: preferences, technology and endowments

1. $u(c) = c^{1-\sigma}/(1-\sigma)$, where $(\beta, \sigma) = (0.96, 1.5)$
2. $F(K, L) = AK^{\alpha}L^{1-\alpha}$ where $(A, \alpha, \delta) = (1.0, 0.36, 0.1)$.
3. $Z = \{z_1, z_2\} = \{0.8, 1.2\}$ and $e(z) = z, \forall z \in Z$
4. $\pi(z_1|z_1) = \pi(z_2|z_2) = 0.5$

The economy features iid labor endowment shocks. Aggregate labor endowment will be constant at 1 over time.

Computed Results

Apply Algorithm 1: Results

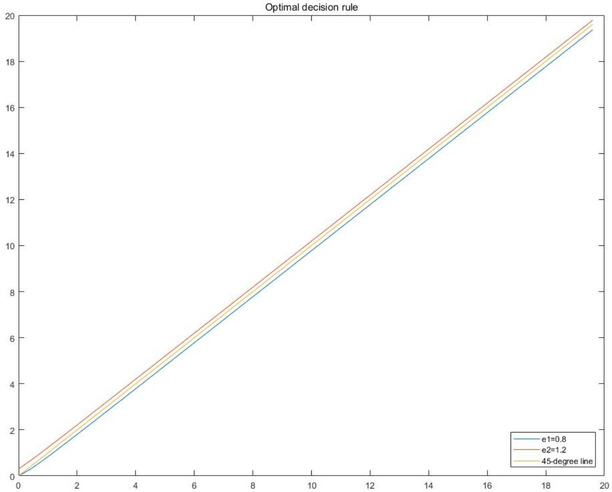
$$K = 4.316, r = 0.0412, \beta(1+r) = 0.999552 < 1$$

Computational Details:
Uniform Grid $[0, 40]$ with 101 grid points.
Simulation: $T = 100,000$

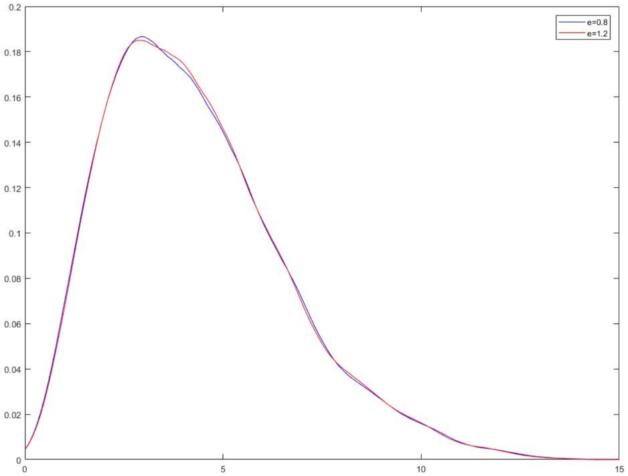
Graph:

Decision Rule for asset holding
Distribution (density) of asset holding
Marginal Propensity to Consume
Consumption and asset holding realizations

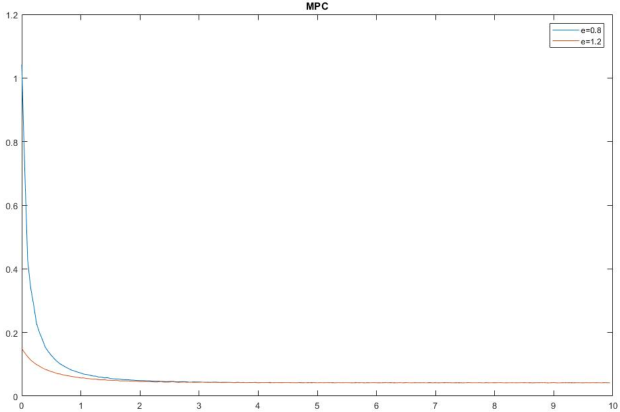
Asset Decision Rule



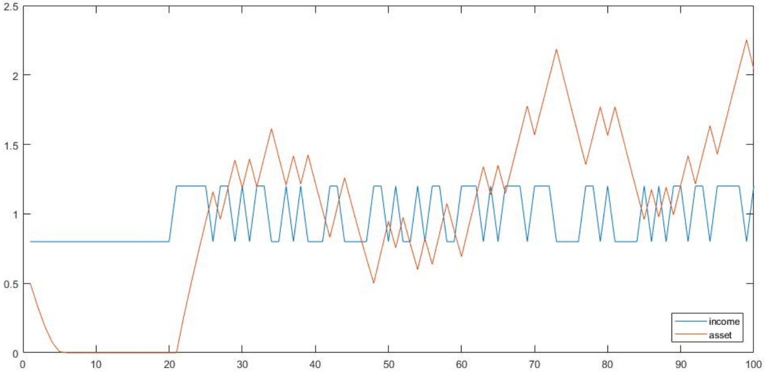
Distribution



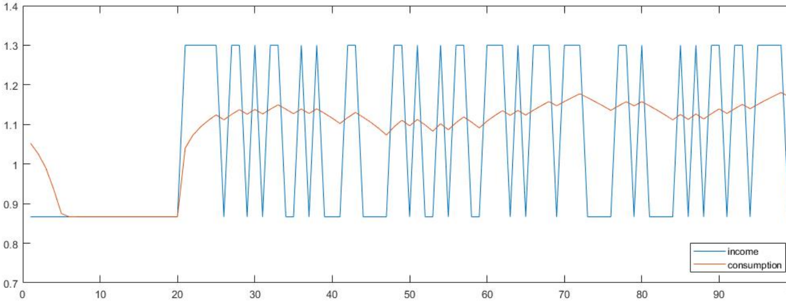
Marginal Propensity to Consume



Realizations: Assets



Realizations: Consumption



Computed Results

Findings of Simulations:

(1) consumption is smoother than labor income,
(2) capital sometimes hits the lower bound,
(3) decision rule for capital has a flat spot,
(4) st positive fraction of agents are at the corner in a steady state.

Points (2)-(4) hold not just in a specific model economy. We can prove all of these more generally and the literature does so. We state and prove one central result for these models.

Computed Results

Compare two economies one with no idiosyncratic risk and one with idiosyncratic risk, keeping preferences, aggregate labor and production technology the same.

Results: In a stationary equilibrium computed with Algorithm 1

1. Idiosyncratic Risk: $\beta(1+r) = \beta(1+F_1(K,1)-\delta) < 1$
2. No Idiosyncratic Risk: $\beta(1+r) = \beta(1+F_1(K,1)-\delta) = 1$
3. Capital stock is strictly larger with idiosyncratic risk than without.

Huggett (1997, Theorem 1)

Theorem: Consider the growth model with $\pi(z'|z) > 0, \forall z, z' \in Z$.
In a stationary equilibrium $\beta(1+r) < 1$.
Main Steps in the Proof:

1. nec cond: $u'(c(x)) \geq \beta(1+r)E[u'(c(x'))|x] =$ if $a(x) > 0$
2. Integrate and simplify
   $\int_X u'(c(x))d\psi \geq \beta(1+r)\int_X E[u'(c(x'))|x]d\psi$
   $\int_X E[u'(c(x'))|x]d\psi = \int_X u'(c(x))d\psi^*$, where
   $\psi^* = \int_X P(x,\cdot)d\psi$ by Stokey and Lucas (1989, Thm 8.3)
   equilbrium condition 4 implies $\psi^* = \psi$
   This implies $1 \geq \beta(1+r)$
3. Huggett (1997, Lemma 1) shows $1 = \beta(1+r)$ is not possible.

Huggett (1997, Lemma 1)

Lemma 1: $1 = \beta(1+r)$ is not possible.
Main Steps/Ideas in Proof:

1. Suppose bwoc that $1 = \beta(1+r)$.
2. Show that there is a set $B = [0, a) \times z$, where $\psi(B) > 0$ and $u'(c(x)) > \beta(1+r)E[u'(c(x'))|x], \forall x \in B$.

   Intuition: $1 = \beta(1+r)$ implies that agents draw down assets to zero when hit by enough bad shocks. See Fig.
3. Step 1-2 imply first line below. The second line follows by previous reasoning in Theorem 1:
   $\int_X u'(c(x))d\psi > \int_X E[u'(c(x'))|x]d\psi$
   $\int_X u'(c(x))d\psi > \int_X E[u'(c(x'))|x]d\psi = \int_X u'(c(x))d\psi^* = \int_X u'(c(x))d\psi$. This is a Contradiction.

Overview: Some Model Properties

1. These models have incomplete insurance by assumption. Consumption falls when the agent receives low labor endowment. This is financed by reducing risk-free assets. Incomplete insurance for some risks holds empirically - see Cochrane (1991, JPE) among others.
2. Marginal propensities to consume differ across agents. Agents up against the borrowing constraint or near to it have higher MPCs. Johnson, Parker and Souleles (2006, AER) provide empirical support for MPC differences.
3. A one-time wealth redistribution produces non-monotonic transitional dynamics. Why? Marginal propensities differ - see Huggett (1997). Similar models without idiosyncratic risk do not produce such transitional dynamics.

Overview: Some More Model Properties

1. There is more steady state capital with idiosyncratic risk than without. This is equivalent to $\beta(1+r) < 1$ - see Huggett (1997, Thm 1). This holds when u is concave.
2. The model with credit but no physical capital features in stationary equlibrium $q > \beta$ where $q = \beta$ would hold absent idiosyncratic risk - see Huggett (1993). Since $q = 1/(1+r)$ the credit model features a low risk-free rate.
3. Early quantitative work (e.g. Aiyagari (1994)) found that the model concentrates much less wealth in the upper tail compared to US data. Open question: Is earnings risk, preference shocks or rate of return risk quantitatively key for producing US wealth upper tail?

Some Questions Addressed in the Recent Literature

There has been a substantial increase in top-end earnings and income inequality in the US over the last 30 years or so. There has also been an increase in US wealth inequality. What has produced the increase in wealth inequality?

A quantitative question:

How much of the observed increase in US wealth inequality can be explained by the observed increase in earnings inequality based on the Aiyagari-Huggett model.

The question above is a classic quantitative question of the form: How much of fact X can be explained by theory Y when calibrated to match fact Z. Discipline comes from measuring "X" and "Z".

Closely Related Literature

These notes have focused on the most basic heterogenoeus-agent models.

1. Many natural questions involve idiosyncrtic and aggregate sources of risk. Krusell and Smith (1998) developed some computational methods to approximate equilibria in models which are generalizations of the models in these notes.
2. While these notes focus on models of consumer heterogeneity, models with firm heterogeneity are important for many reasons. Much of the modeling language and computational techniques from the heterogeneous-agent literature have also been applied to firm heterogeneity. Hopenhayn and Rogerson (1993) is an early example.