Neoclassical Growth Model: I

Mark Huggett[2]

[2]Georgetown

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Growth Model: Introduction

- Neoclassical Growth Model is the workhorse model in macroeconomics. It comes in two main varieties: infinitely-lived agent and overlapping generations.
- Some key features: (i) accounts for the NIPA aggregates C, I, G, (ii) accounts for factor shares, (iii) accounts for long-run growth, (iv) framework for analyzing business-cycle fluctuations (impluses and propagation mechanisms) and (v) one can address positive and normative questions within it.
- Generalization of Solow (1956) to endogenize savings decision.

Infinitely-lived agent Growth Model

1. Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$
2. Endowments: $k_0 > 0$ and 1 unit of time each model period
3. Technology: $F(k_t, l_t)$ C.R.S.

$$c_t + k_{t+1} \leq F(k_t, l_t) + k_t(1 - \delta)$$

One can rewrite this also as follows:

$$c_t + k_{t+1} - k_t(1 - \delta) \leq F(k_t, l_t)$$

Def: A competitive equilibrium is $\{c_t, k_{t+1}, l_t, w_t, R_t\}_{t=0}^{\infty}$ such that

1. $\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}$ solve P1.
2. $w_t = F_2(k_t, l_t)$ and $R_t = F_1(k_t, l_t)$ holds $\forall t$
3. $c_t + k_{t+1} = F(k_t, l_t) + k_t(1-\delta)$ holds $\forall t$

$$P1 \ \max \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t.$$

$$c_t + k_{t+1} \leq w_t l_t + R_t k_t + (1-\delta) k_t$$

$$c_t, k_{t+1} \geq 0, l_t \in [0,1], \ given \ k_0$$

Comments:

1. We assume a sequential market structure with two factor rental prices per period.
2. There is no direct requirement of profit maximization by a firm or firms. However, this is implicit in the second condition of equilibrium. Macroeconomists often simply impose competitive pricing of factor inputs to shorten the analysis. There is no explicit ownership of shares of the firm and no allocation of profits of the firm in the definition. Since firms will make zero profit in an equilibrium, abstracting from ownership and the allocation of profit shortens the analysis.
3. Sometimes we will impose different market structures: (i) time-0 AD markets and (ii) sequential markets but where the consumer owns the firm and the firm owns the capital.

Planning Problem

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t.$$

$$c_t + k_{t+1} \leq F(k_t, l_t) + k_t(1-\delta) \ and \ l_t \in [0,1], \ given \ k_0$$

1. 1st Welfare Theorem: It will be understood that within the growth model comp. equil. allocations are Pareto efficient. Thus, competitive equilibrium allocations will solve the planning problem above.
2. We will not prove the 1st Welfare Theorem. However, we will show how to recast the equilibrium concept into "the language of Debreu".

Comments:

1. In our application the commodity space is infinite dimensional. Debreu's (1959) analysis handles finite dimensional problems. Debreu (1954) shows how to prove the Welfare Theorems with finite consumers but infinite dimensional commodity space.
2. To invoke standard proofs, one would need to restate competitive equilibria in the "language of Debreu".
3. (U_i, X_i, e_i, Y_j) are the elements of an economy in the language of Debreu. In discussing competitive equilibrium there is also share ownership (i.e. θ_{ij} is the fraction of firm j owned by consumer i).

1. One agent and one firm in our application:
   $X = \{(x_1, x_2) : x_1, x_2 \geq 0\}$
   $U(x) = u(x_1) + v(x_2)$
   $e = (0, 1)$
   $Y = \{(y_1, -y_2) : 0 \leq y_1 \leq F(y_2), y_2 \geq 0\}$
2. A competitive equilibrium is (x, y, p) such that (i) $x \in \; argmax \; \{U(x) | x \in X, px \leq pe + 1 \times py\}$, (ii) $y \in \; argmax \; \{py | y \in Y\}$ and (iii) $x = y + e$.

1. One agent and one firm in our application:
   $X = \{x = \{(x_{1t}, x_{2t})\}_{t=0}^{\infty} : (x_{1t}, x_{2t}) \in R_+ \times [0, 1], \forall t \geq 0\}$
   $U(x) = \sum_{t=0}^{\infty} \beta^t u(x_{1t}, x_{2t})$
   $e = \{(e_{1t}, e_{2t})\}_{t=0}^{\infty}$ and $(e_{1t}, e_{2t}) = (0, 1), \forall t \geq 0$
   $Y = \{y = \{(z_t, -l_t)\}_{t=0}^{\infty} : \exists \{k_t\}_{t=0}^{\infty}, k_0 = k \ s.t. \ (1) \ holds\}$
   $(1) \ z_t + k_{t+1} \leq F(k_t, l_t) + k_t(1 - \delta), l_t \geq 0, \forall t \geq 0$
2. Prices: $px = \sum_{t=0}^{\infty} (p_{1t}, p_{2t}) \cdot (x_{1t}, x_{2t})$
3. A competitive equilibrium is (x, y, p) such that (i) $x \in \ argmax \ \{U(x) | x \in X, px \leq pe + 1 \times py\}$, (ii) $y \in \ argmax \ \{py | y \in Y\}$ and (iii) $x = y + e$.

1. We have considered two example economies. Both have one agent and one firm.
2. A competitive equilibrium in both economies is (x, y, p) and the three requirements of equilibria are the same in both economies.
3. What differs across the economies is that the elements (x, y, p) have very different dimensions across the two examples and that the notion of "price system" in example 2 is perhaps very different from what you are used to thinking about.
4. The equilibrium concepts used in the remainder of these slides and the notation employed will be quite different from the Debreu formalization. One reason for this is that we will be interested in computing equilibria (at least an approximation). Debreu's language is not usually the most natural way to do so.

A competitive equilibrium $\{c_t, k_{t+1}, l_t, w_t, R_t\}_{t=0}^{\infty}$ is a steady-state competitive equilibrium provided
$c_t = \bar{c}, k_t = \bar{k}, l_t = \bar{l}, w_t = \bar{w}, R_t = \bar{R}$ for all $t \geq 0$.

Analysis:

1. $u'(c_t) = \beta u'(c_{t+1})(1 + R_{t+1} - \delta)$ - FONC
2. $u'(c_t) = \beta u'(c_{t+1})(1 + F_1(k_{t+1}, l_{t+1}) - \delta)$ - FONC + equil
3. $1 = \beta(1 + F_1(\bar{k}, 1) - \delta)$ - steady state
4. $F_1(k, 1) > 0$ *and* $F_{11}(k, 1) < 0$ imply that there exists a unique steady state $\bar{k} > 0$
5. Golden Rule: $k_{GR} \equiv \ argmax \ F(k, 1) - \delta k$
   Implication: $\bar{k} < k_{GR}$ where $F_1(k_{GR}, 1) - \delta = 0$

Generalize to $u(c,l)$ rather than $u(c)$

1. $u_1(\bar{c},\bar{l}) = \beta u_1(\bar{c},\bar{l}))(1+F_1(\bar{k},\bar{l})-\delta)$ - FONC
2. $u_1(\bar{c},\bar{l})F_2(\bar{k},\bar{l}) = -u_2(\bar{c},\bar{l})$ - FONC
3. The following restrictions characterize steady states:

$$1 = \beta(1+F_1(\bar{k},\bar{l})-\delta)$$

$$u_1(\bar{c},\bar{l})F_2(\bar{k},\bar{l}) = -u_2(\bar{c},\bar{l})$$

$$\bar{c}+\delta\bar{k} = F(\bar{k},\bar{l})$$

4. An extra assumption (e.g. u additively separable or u is GHH preferences) would lead to a sharper characterization

Theorem: Assume $u(c, l) = u(c) - v(l)$, $u' > 0, u'' < 0$ *and* $u'(0) = \infty, lim_{c \to \infty} u'(c) = 0$ *and* $v' > 0, v'' > 0$. *Then there is a unique interior ss competitive equilibrium allocation* $(\bar{c}, \bar{k}, \bar{l})$.

Proof: sketch

1. Rearrange ss restrictions.

$$(1)\ 1 = \beta(1 + F_1(\bar{k}/\bar{l}, 1) - \delta)$$

$$(2)\ u'(\bar{c})F_2(\bar{k}/\bar{l}, 1) = v'(\bar{l})$$

$$(3)\ \bar{c} + \delta(\bar{k}/\bar{l})\bar{l} = F(\bar{k}/\bar{l}, 1)\bar{l}$$

2. Usual assumptions on F imply there exists a unique solution $\bar{k}/\bar{l} > 0$ to (1)

Proof: sketch (continued)

3. Substitute (3) into (2) to get (*)

$$u'(\bar{c})F_2(\bar{k}/\bar{l}, 1) = v'(\bar{l})$$

$$(*)\ u'(\bar{l}(F(\bar{k}/\bar{l}, 1) - \delta(\bar{k}/\bar{l})))\ F_2(\bar{k}/\bar{l}, 1) = v'(\bar{l})$$

4. Given unique solution $\bar{k}/\bar{l}$ to (1), LHS of (*) is st decreasing in $\bar{l}$ and RHS is st increasing in $\bar{l}$.
5. $LHS(\bar{l}) - RHS(\bar{l})$ is a continuous and st monotone decreasing fn of $\bar{l}$. Intermediate-Value Thm delivers a solution $\bar{l}$ to $LHS(\bar{l}) - RHS(\bar{l}) = 0$. It's unique by st. monotonicity.
6. Conclusion: There are unique positive values $(\bar{c}, \bar{k}, \bar{l})$ solving (1)-(3)

Perspective:

One use of the growth model is to provide insight into fiscal policy. While we know that the growth model implies that government tax and spending plans do not improve upon the market allocation, we will still pursue an analysis of proportional taxes within this model. This establishes a useful benchmark and is an important point of departure in the literature.

Def: A competitive equilibrium is $\{c_t, k_{t+1}, l_t, w_t, R_t, T_t\}_{t=0}^{\infty}$ such that

1. $\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}$ solve P1.
2. $w_t = F_2(k_t, l_t)$ and $R_t = F_1(k_t, l_t)$ holds $\forall t$
3. $c_t + k_{t+1} = F(k_t, l_t) + k_t(1-\delta)$ holds $\forall t$
4. $T_t = \tau_c c_t + \tau_w w_t l_t + \tau_k (R_t - \delta) k_t$ holds $\forall t$

$$P1 \ \max \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t.$$

$$c_t(1+\tau_c) + k_{t+1} \leq w_t l_t (1-\tau_w) + (R_t - \delta)(1-\tau_k) k_t + k_t + T_t$$

$$c_t, k_{t+1} \geq 0, l_t \in [0,1], \ given \ k_0$$

Analysis: $\tau_k > 0$ and $\tau_c, \tau_w = 0$

1. $u'(c_t) = \beta u'(c_{t+1})(1 + (R_{t+1} - \delta)(1 - \tau_k))$ - FONC
2. $1/\beta = 1 + (F_1(\bar{k}, 1) - \delta)(1 - \tau_k)$ - steady state
3. $F_1(k, 1) > 0 \; and \; F_{11}(k, 1) < 0$ imply that there exists a unique steady state $\bar{k} > 0$ and $\bar{k}$ decreases as τ_k increases.
4. $F(k, l) = Ak^{\alpha} l^{1-\alpha}$ implies

$$\bar{k} = ([\frac{1/\beta - 1}{1 - \tau_k} + \delta] \times \frac{1}{\alpha A})^{1/(\alpha - 1)}$$

How large is the steady state output effect of capital income taxation?

1. $F(k,l) = Ak^{\alpha}l^{1-\alpha}$
2. $\bar{k}(\tau_k) = ([\frac{1/\beta-1}{1-\tau_k} + \delta] \times \frac{1}{\alpha A})^{1/(\alpha-1)}$
3. $\frac{\bar{y}(\tau_k)}{\bar{y}(0)} = \frac{F(\bar{k}(\tau_k),1)}{F(\bar{k}(0),1)} = (\frac{\bar{k}(\tau_k)}{\bar{k}(0)})^{\alpha}$
4. $\frac{\bar{y}(\tau_k)}{\bar{y}(0)} = (\frac{[\frac{1/\beta-1}{1-\tau_k}+\delta]}{[1/\beta-1+\delta]})^{\frac{\alpha}{\alpha-1}} > (1-\tau_k)^{\frac{\alpha}{1-\alpha}}$ for $\delta > 0$
5. $\frac{\bar{y}(\tau_k)}{\bar{y}(0)} > (1-\tau_k)^{\frac{\alpha}{1-\alpha}} \doteq (.5)^{.3/.7} = .74$
6. When $\alpha = .3$ and $\tau = 0.5$, then output is reduced by 25 percent. Upshot: since $\alpha << 1$ capital is not too important and capital taxation has somewhat small output effects.

Analysis: $u(c,l)$ and $\tau_k, \tau_c, \tau_w > 0$

1. $u_1(c_t, l_t)\frac{1}{1+\tau_c} = \beta u_1(c_{t+1}, l_{t+1})\frac{(1+(R_{t+1}-\delta)(1-\tau_k))}{1+\tau_c}$ - FONC
2. $u_1(c_t, l_t)\frac{w_t(1-\tau_w)}{1+\tau_c} = -u_2(c_t, l_t)$ - FONC
3. In steady state the following hold:

$$1/\beta = 1 + (F_1(\bar{k}, \bar{l}) - \delta)(1 - \tau_k)$$

$$u_1(\bar{c}, \bar{l})F_2(\bar{k}, \bar{l})\frac{(1 - \tau_w)}{1 + \tau_c} = -u_2(\bar{c}, \bar{l})$$

$$\bar{c} = F(\bar{k}, \bar{l}) - \delta\bar{k}$$

4. Eq 1 implies $\bar{k}/\bar{l}$ falls as τ_k increases. $\bar{l}$ is not pinned down absent further restrictions. Effect of τ_k on SS output depends on whether $\bar{l}$ falls.

Let's analyze how the introduction of a proportional tax τ on corporate income impacts the steady state properties of the one-sector growth model. To do so, we recast the definition of equilibrium. Specifically, now the firm's problem is dynamic and we model the value of the firm when the firm owns its capital.

Definition: A steady-state equilibrium is $(c, l, b, s, k, d, w, r, p, T)$ such that

1. $(c_t, l_t, b_t, s_t) = (c, l, b, s), \forall t$ solves P1.
2. $(k_t, l_t) = (k, l), \forall t$ solves P2
3. $c + \delta k = F(k, l)$ and $s = 1$ and $b = 0$
4. $T = \tau[F(k, l) - wl - \delta k]$
5. $d = (1 - \tau)[F(k, l) - wl - \delta k]$

$$P1 \ \max \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t.$$

$$c_t + s_{t+1}p + b_{t+1} \leq wl_t + s_t(p+d) + b_t(1+r) + T$$

$$c_t, s_{t+1}, b_{t+1} \geq 0, l_t \in [0,1], \ given \ (s_0, b_0) = (1,0)$$

$$P2 \ \max \sum_{t=0}^{\infty} (\frac{1}{1+r})^t d_t \ , given \ k_0 = k$$

$$d_t \equiv [F(k_t, l_t) - wl_t - (k_{t+1} - k_t(1-\delta))] - \tau[F(k_t, l_t) - wl_t - \delta k_t]$$

Necessary Conditions for interior optimization are for all $t \geq 0$

1. $u'(c_t) = \beta u'(c_{t+1})(1+r)$
2. $u'(c_t)p = \beta u'(c_{t+1})(p+d)$
3. $F_2(k_t, l_t) = w$
4. $1 = (\frac{1}{1+r})[(1-\tau)(F_1(k_{t+1}, l_{t+1}) - \delta) + 1]$

Steady State:
$l = 1$ and $(1+r) = 1/\beta$ via 1.
$k = F_1(\cdot, 1)^{-1}([\frac{1}{\beta} - 1]/(1-\tau) + \delta)$ via 1 and 4
$w = F_2(k, 1)$ via 3
$p = \frac{\beta}{1-\beta} d = \frac{\beta}{1-\beta}(1-\tau)[F(k,1) - w - \delta k]$ via 2
$p = \frac{\beta}{1-\beta}(1-\tau)[k(F_1(k,1) - \delta)] = \frac{\beta}{1-\beta}(\frac{1}{\beta} - 1)k = k$
$T = \tau[F(k,1) - w - \delta k]$

Theorem: In a steady state in the growth model with a proportional tax τ on corporate income,

1. capital $k(\tau)$ decreases as the tax rate τ increases.
2. wage $w(\tau) = F_2(k(\tau), 1)$ decreases as the tax rate τ increases.
3. interest rate $r(\tau) = 1/\beta - 1$ is constant as the tax rate τ increases.
4. value $p(\tau) = k(\tau)$ of the firm decreases as the tax rate τ increases.

Proof: see analysis on the previous slide.

[Could generalize this Theorem to handle a labor-leisure decision.]

Motivation

It will be useful to develop the notion of a recursive competiitive equilibrium and a recursive formulation of the planning problem for the growth model. One reason for this is that to analyze these models it will often be easier to computationally approximate functions rather than infinite sequences.

The recursive notions we employ in growth theory will be closely connected to the recursive methods employed in the analysis of asset pricing models in exchange economies (e.g. Lucas (1978)).

Def: A recursive competitive equilibrium is $(c(x), g(x), w(K), R(K), G(K))$ such that

1. optimization: $(c(x), g(x))$ solve BE.
2. factor prices: $w(K) = F_2(K, 1)$ and $R(K) = 1 + F_1(K, 1) - \delta$
3. feasibility: $c(K, K) + g(K, K) = F(K, 1) + K(1 - \delta)$
4. law of motion: $G(K) = g(K, K)$

State: $x = (k, K)$

$$(BE) \;\; v(k, K) = \max \;\; u(c) + \beta v(k', K') \; s.t.$$

$$c + k' \leq w(K) + R(K)k \; and \; c, k' \geq 0 \; and \; K' = G(K)$$

Standard big K little k trick

1. We employ the state variable $x = (k, K)$. Little k is the agent's capital. Big K is economy wide capital.
2. Big K determines factor prices and helps the agent forecast the future economy wide capital. The agent knows factor prices via knowing the functions (w, R) and the level K.
3. "in equilibrium" big K and little k are equal. Thus, for the purpose of analyzing equilibria the distinction is not important. However, for the purpose of defining equilibria and clarifying the analysis the distiction is useful.
4. Absent the distinction, the agent would view future factor prices as being influenced by curent capital choices. Since we want to capture competitive behavior, the distinction is useful.

Recursive Planning Problem

1. $v(k) = \max\ u(c) + \beta v(k')\ s.t.\ c + k' \leq F(k, 1) + k(1 - \delta)$
2. Solution: Decision rule $k' = G(k)$
3. 1st Welfare Thm. implies that Law of motion in competitive equilibrium $G(k)$ coincides with the optimal decision rule $G(k)$ solving the Planning Problem.
4. Implication: can analyze competitive equilibrium by solving the planning problem.
5. All the properties of the growth model reduce to properties of $G(k)$.

Recursive Planning Problem

1. $v(k) = \max\ u(c) + \beta v(k')\ s.t.\ c + k' \leq F(k,1) + k(1-\delta)$
2. Properties: Decision rule $G(k)$ is (i) continuous, (ii) increasing, (iii) there is a unique positive k^* such that $k^* = G(k^*)$, and (vi) monotone convergence of capital to k^* induced by recursively applying $k_{t+1} = G(k_t)$ from $k_0 > 0$.
3. Properties (i)-(ii) follow standard lines from early in the semester, but (iii)-(iv) need to be developed.

Monotone Convergence: LS (1989, Ch 6)

$$v(k) = \max \; u(c) + \beta v(k') \; s.t. \; c + k' \leq F(k,1) + k(1-\delta)$$

1. $u'(F(k,1) + k(1-\delta) - G(k)) = \beta v'(G(k))$ - nec condition
2. $v'(k) = u'(F(k,1) + k(1-\delta) - G(k))(1 + F_1(k,1) - \delta)$ - BS
3. v st. concave implies $[v'(k) - v'(G(k))][k - G(k)] \leq 0, =$ iff $G(k) = k$
4. $[(1 + F_1(k,1) - \delta) - \frac{1}{\beta}][k - G(k)] \leq 0$
5. Let k^* solve $\beta(1 + F_1(k^*,1) - \delta) = 1$. Consider $k > 0$.
6. Case 1: $k < k^*$ implies $[+][k - G(k)] < 0$. Thus, $G(k) > k$
7. Case 2: $k > k^*$ implies $[-][k - G(k)] < 0$. Thus, $G(k) < k$

How fast does capital converge to steady state?

$$v(k) = \max \ u(c) + \beta v(k') \ s.t. \ c + k' \leq F(k,1) + k(1-\delta)$$

1. $k_{t+1} = G(k_t) \doteq G(k^*) + G'(k^*)(k_t - k^*)$
2. $k_{t+1} - k^* \doteq G'(k^*)(k_t - k^*)$
3. $1 - \alpha \equiv \frac{k_{t+1} - k^*}{k_t - k^*} = G'(k^*)$
4. Upshot: If $G'(k^*)$ is close to 1 then convergence is slow. Fraction α of gap closed is then small.
5. Note: Solow model with Cobb-Douglas production and capital's share near .3 implies rapid convergence. Calculations for the optimal growth model suggest similar results.

Rational Sunspot Equilibria

Idea: Contemplate an economy for which sunspot realizations are independent of shocks to preferences, endowments or technology. See if sunspot realizations impact allocations. We will do so within the growth model where the ONLY shocks are sunspot shocks.

Notation:
$s^t \equiv (s_0, ..., s_t) \in S^t = S \times \cdots \times S$ history of realizations of s_t
$s_t \in S$ finite set
$P(s^t)$ probability of "sunspot" history

A competitive equilibrium is
$\left\{c_t\left(s^t\right), k_{t+1}\left(s^t\right), l_t\left(s^t\right), w_t\left(s^t\right), R_t\left(s^t\right)\right\}_{t=0}^{\infty}$ such that:

1. $\left\{c_t\left(s^t\right), k_{t+1}\left(s^t\right), l_t\left(s^t\right)\right\}_{t=0}^{\infty}$ solve P1.
2. $w_t\left(s^t\right) = F_2(k_t\left(s^{t-1}\right), l_t\left(s^t\right)), R_t\left(s^t\right) = 1 + F_2(k_t\left(s^{t-1}\right), l_t\left(s^t\right)) - \delta$
3. Feasibility: $\forall t, \forall s^t$
   $c_t\left(s^t\right) + k_{t+1}\left(s^t\right) \leq F\left(k_t\left(s^{t-1}\right), l_t\left(s^t\right)\right) + (1-\delta)\, k_t\left(s^{t-1}\right)$
   and $l_t\left(s^t\right) \in [0,1]$

Decision Problem
P1: $\max E\left[\sum_{t=1}^{\infty} \beta^{t-1} u\left(c_t\left(s^t\right)\right)\right]$ subject to

$$c_t\left(s^t\right) + k_{t+1}\left(s^t\right) \leq w_t\left(s^t\right) l_t\left(s^t\right) + R_t(s^t) k_t\left(s^{t-1}\right),$$
$$l_t\left(s^t\right) \in [0,1]\, \forall t,\ \forall s^t$$

Theorem: *Assume* $u(c)$ *is st. increasing and st. concave and* F *is concave and CRS. Then there is no comp equilibrium in which* $c(s^t) \neq c(\hat{s}^t)$ *for some* $t \geq 0$ *and two histories* s^t *and* $\hat{s}^t \in S^t$*.*

Sketch of Proof:

1. SBWOC that there is an equilibrium w/ $c(s^t) \neq c(\hat{s}^t)$.
2. Define $\bar{c}_t = E\left[c_t\left(s^t\right)\right]$, $\bar{k}_{t+1} = E\left[k_{t+1}\left(s^t\right)\right]$, $\bar{l}_t = E\left[l_t\left(s^t\right)\right]$
3. Claim: $\left\{\bar{c}_t, \bar{k}_{t+1}, \bar{l}_t\right\}_{t=0}^{\infty}$ satisfies Feasibility. First line below takes expectations of the feasibility condition. Second line applies Jensen's inequality and uses $l(s^t) = 1$.
   $E[c_t\left(s^t\right) + k_{t+1}\left(s^t\right)] \leq$
   $E[F\left(k_t\left(s^{t-1}\right), l_t\left(s^t\right)\right) + (1-\delta)\, k_t\left(s^{t-1}\right)]$
   $\bar{c}_t + \bar{k}_{t+1} \leq E[F\left(k_t\left(s^{t-1}\right), l_t\left(s^t\right)\right)] + (1-\delta)\, \bar{k}_t \leq$
   $F(\bar{k}_t, \bar{l}_t) + (1-\delta)\bar{k}_t$
4. $E\left[\sum_{t=1}^{\infty} \beta^{t-1} u\left(\bar{c}_t\right)\right] > E\left[\sum_{t=1}^{\infty} \beta^{t-1} u\left(c_t\left(s^t\right)\right)\right]$ by Jensen's Inequality.
5. This contradicts 1st Welfare theorem.

Comments:

(1) While rational sunspot equilibria exist in some model economies, they don't matter in basic growth models.

(2) Even if they were to matter in growth models, it is not so easy to produce procyclical labor productivity, as observed in US data, with constant returns technology.

(3) Proof of Theorem used $l(s^t) = 1$. It remains to be seen if one can extend the proof to handle endogenous labor-leisure decision.