Macro 1: Asset Pricing

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October, 2016

Agenda

We develop theory for various asset prices (stock prices, prices of disount bonds and so on). At the simplest level we develop necessary conditions for agents to hold a specific asset. Such conditions have been used for estimation and testing of theory. We also develop general equilibrium based theory where asset prices depend on economic primitives (i.e. preferences, endowments and so on). Such frameworks have been useful for general theoretical understanding as well as for applied work.

The historical development of the literature on general equilibrium based asset pricing started with exchange economies. We follow this historical arc.

Narrow Agenda

Topic 1 - Necessary conditions (recursive and non-recursive formulation of decision problems)

Topic 2 - Lucas asset pricing model (price claims to a tree and other assets)

Topic 3 - Mehra-Prescott exercise based on the Lucas model

Topic 4 - Hansen-Jaganathan bounds - restrictions on stochastic discount factors implied by prices and payouts.

Topic 5 - Price a specific non-traded asset (future earnings)

Develop Necessary Conditions

Exogenous: p_t^k, d_t^k, y_t

$$\max E[\sum_{t=0}^{T} \beta^t u(c_t)] \ s.t.$$

- $c_t + \sum_k p_t^k q_{t+1}^k \leq y_t + \sum_k q_t^k (p_t^k + d_t^k)$
- $c_t \geq 0, \forall t = 1, ..., T$ and $q_{T+1}^k = 0, \forall k$

FONC: (restated various ways, use $c_t > 0$)

- $u'(c_t) p_t^k = \beta E_t[u'(c_{t+1})(p_{t+1}^k + d_{t+1}^k)], \forall t, \forall k$
- $p_t^k = E_t[\frac{\beta u'(c_{t+1})}{u'(c_t)}(p_{t+1}^k + d_{t+1}^k)]$ or
  $p_t^k = E_t[m_{t,t+1}(p_{t+1}^k + d_{t+1}^k)]$
- $1 = E_t[\frac{\beta u'(c_{t+1})}{u'(c_t)}(\frac{p_{t+1}^k + d_{t+1}^k}{p_t^k})]$ or $1 = E_t[m_{t,t+1} R_{t+1}^k]$

Terminology:

$m_{t,t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$ is called a stochastic discount factor

$R^k_{t+1} = (\frac{p^k_{t+1}+d^k_{t+1}}{p^k_t})$ is called a gross return on asset k

Note: When the utility function is generalized, then necessary conditions $1 = E_t[m_{t,t+1}R^k_{t+1}]$, or stated in a related form, hold for agents who are off corners in the holding of asset k.

Additive separability aids empirical work by limiting the potential empirical determinants of a realization of the (unobserved) stochastic discount factor.

Obviously, $m_{t,t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$ are random variables even though the notation hides this

Reformulate Recursively

$x = (q^1, ..., q^K, z)$

z is Markov

prices, dividends and endowments are functions of z

$v_t(x) = \max u(c) + \beta E[v_{t+1}(x')|x]$ s.t.

- $c + \sum_k p^k(z) q^k_{t+1} \leq y(z) + \sum_k q^k(p^k(z) + d^k(z))$
- $c \geq 0$ and $q^k_{t+1} = 0, \forall k$ when $t = T$

FONC: if $(c, (q^1_{t+1}, .., q^K_{t+1}))$ solve RHS of BE, given x , then

$$u'(c)p^k(z) = \beta E[\frac{\partial \ v_{t+1}(q^1_{t+1}, .., q^K_{t+1}, z')}{\partial \ q^k_{t+1}}|z], \forall k$$

$$\frac{\partial \ v_t(q^1, .., q^K, z)}{\partial \ q^k} = u'(c)(p^k(z) + d^k(z))$$

Lucas (1978): The Economy

Agents: 1 agent
Preferences: $E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$
Endowments: y_t follows a finite Markov chain $F(y'|y)$
Primitives: (β, u, F)

Story: One agent has a tree which produces y_t apples at time t. The apple harvest is Markovian. The agent likes apples. Apples are the only good in the economy. Apples cannot be stored.
Question: In this economy, what is “a” or “the” price of the tree?
Answer: The tree price is a function mapping the current harvest into a price with a key property: when the agent owns all of the tree, then the price persuades the agent that it is best to continue to hold all the shares in the tree.

A Two-Period Lucas Warmup w/o Risk

1. What is the price p of the tree giving y_2 apples tomorrow?
2. Max $u(c_1) + \beta u(c_2)$ $s.t.$ $c_1 + z_2 p \leq (p + y_1),\ c_2 = z_2 y_2$
3. Slope of indifference curve at $(c_1, c_2) = (y_1, y_2)$:

$$\frac{d}{dc_1} c_2(c_1) = -\frac{u'(c_1)}{\beta u'(c_2)} = -\frac{u'(y_1)}{\beta u'(y_2)}$$

4. Slope of budget line: $-\frac{y_2}{p}$
5. Equate slopes and impose $z_2 = 1$ as equil condition:

$$\frac{y_2}{p} = \frac{u'(y_1)}{\beta u'(y_2)} \Rightarrow p = \frac{\beta u'(y_2)}{u'(y_1)} y_2$$

6. A unique tangent to the indiff curve implies a unique equilibrium tree price.

Lucas (1978): Definition of Equilibrium

$$(P1)\ v(z,y) = \max u(c) + \beta E[v(z',y')|y]\ s.t.$$

$$c + p(y)z' \leq z(p(y)+y)\ and\ c, z' \geq 0$$

Definition: A recursive equilibrium is $(c(z,y), s(z,y), p(y))$ s.t.

1. (optimization) $(c(z,y), s(z,y))$ solve P1.
2. (market clearing) $c(1,y) = y\ and\ s(1,y) = 1, \forall y \in Y$

Note: Could include $(c(z,y), s(z,y), v(z,y))$ as equilibrium elements. Implicitly, the definition above makes sense with the prior information that there is a v solving BE. Thus, $(c(z,y), s(z,y))$ solving P1 means that (i) $(c(z,y), s(z,y))$ are feasible and (ii) $v(z,y) = u(c(z,y)) + \beta E[v(s(x,y),y')|y]$ holds for all states (z,y), given v.

Lucas (1978): Analysis

1. Prop: For any $p(y), \exists$ a unique bd. cont. v solving P1.
2. Necessary condition for solution to P1:
   $u'(c(z,y))p(y) = \beta E[v_1(s(z,y),y')|y]$
   $u'(c(z,y))p(y) = \beta E[u'(c(s(z,y),y')(p(y')+y')|y]$
3. Impose equilibrium condition 2:

$$u'(y)p(y) = \beta E[u'(y')(p(y')+y')|y]$$

$$(*)\ f(y) = T(f)(y) \equiv \beta E[f(y')|y] + \beta E[u'(y')y'|y]$$

4. CMT implies $\exists!$ f solving $(*)$ and $p(y) = f(y)/u'(y)$
5. $p(y) = E[\sum_{t=1}^{\infty} \frac{\beta^t u'(y_t)}{u'(y)} y_t | y]$ solves step 3 and 4!
6. Conclusion: there is ONLY 1 equilibrium price function in the Lucas model. Non-fundamental stuff cannot matter!

Lucas (1978): Analysis

$p(y) = E[\sum_{t=1}^{\infty} \frac{\beta^t u'(y_t)}{u'(y)} y_t | y]$ where y is the time 0 endowment

An Interpretation: The equilibrium price of the tree is a sum of various products. The sum is over ALL of the distinct commodities (in the Debreu sense) from data $t = 1$ and beyond. For any distinct commodity, the product is the time-0 competitive equilibrium price ($p(y^t)$ where $y^t = (y_0, ..., y_t)$) of that commodity times the endowment of that commodity.

$$p(y) = \sum_{t=1}^{\infty} \sum_{y^t} p(y^t) y_t = \sum_{t=1}^{\infty} \sum_{y^t} (\frac{\beta^t u'(y_t) Prob(y^t | y)}{u'(y)}) y_t$$

Lucas (1978): Pricing Arbitrary Assets

Method:

1. Add any asset (in zero net supply). Roughly put, imagine an asset that has clear payoffs in goods in all future partial histories. Such an asset could be called a financial asset. The added asset DOES NOT change total apples available in the economy.
2. Define an equilibrium.
3. Carry out the analysis we just went through (derive FONC and impose equilibrium conditions).
4. Fool around until you get the price of the proposed asset stated in terms of fundamentals.

Lucas (1978): Pricing Arbitrary Assets

EX: Price a 1 period, pure-discount bond paying 1 unit tomorrow in all possible states

State: $z = (z_1, z_2)$ holdings of stock and bonds

$$(P1)\ v(z, y) = \max u(c) + \beta E[v(z_1', z_2', y')|y]\ s.t.$$

$$c + p_1(y)z_1' + p_2(y)z_2' \leq z_1(p_1(y) + y) + z_2 \ and \ c \geq 0, z_1', z_2' \geq \underline{z}$$

Definition: A recursive equilibrium is $(c(z, y), s_1(z, y), s_2(z, y), p_1(y), p_2(y))$ s.t.

1. (optimization) $(c(z, y), s_1(z, y), s_2(z, y))$ solve P1.
2. (market clearing) $c(1, 0, y) = y \ and \ s_1(1, 0, y) = 1 \ and \ s_2(1, 0, y) = 0, \forall y \in Y$

Lucas (1978): Pricing Arbitrary Assets

EX: Price a 1 period, pure-discount bond

FONC:

$$u'(c)p_2(y) = \beta E[v_2(z_1', z_2', y')|y]]$$

$$u'(c(z,y))p_2(y) = \beta E[u'(c(s_1(z,y), s_2(z,y), y'))|y]$$

FONC + equil condtions:

$$u'(y)p_2(y) = \beta E[u'(y')|y] \Rightarrow p_2(y) = E[\frac{\beta u'(y')}{u'(y)}|y]$$

Question: Within the Lucas model, when does the mean return to stock exceed the risk-free bond return?

Answer:

1. $1 = E[\frac{\beta u'(y')}{u'(y)} R^s(y, y')|y]$ *where* $R^s(y, y') = \frac{p_1(y')+y'}{p_1(y)}$
2. $1 = E[\frac{\beta u'(y')}{u'(y)} R^b(y, y')|y]$ *where* $R^b(y, y') = \frac{1}{p_2(y)}$
3. $0 = E[\frac{\beta u'(y')}{u'(y)} (R^s(y, y') - R^b(y, y'))|y]$
4. Use $E[ab] = E[a]E[b] + Cov(a, b)$ to get
   $0 = E[m|y]E[R^s - R^b|y] + Cov(m, R^s - R^b|y)$
5. $E[R^s - R^b|y] = \frac{-Cov(m, R^s - R^b|y)}{E[m|y]} = \frac{-Cov(m, R^s - R^b|y)}{p_2(y)}$
6. Need m to be random AND to covary negatively with R^s.
   EX: larger y' realizations lowers m and increases $p_1(y')$.

Lucas Model

Quantitative Question: How far does Theory X go to match observation Y, when restricted to match fact Z?

EX: Theory $X=$ Lucas model. Observation $Y = E[R^s - R^b]$. Fact $Z = Cov(g, R^s - R^b)$

- $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ *and* $g_{t+1} \equiv c_{t+1}/c_t$
- $E[R^s - R^b] = \frac{-Cov(\beta g^{-\sigma}, R^s - R^b)}{E[m]} \approx -\beta \; Cov(g^{-\sigma}, R^s - R^b)$
- $E[R^s - R^b] \approx \beta\sigma(E[g])^{-\sigma-1} \; Cov(g, R^s - R^b)$

Upshot: Answer given by difference between RHS and LHS when can measure $E[g]$ and $Cov(g, R^s - R^b)$ and together w/ restrictions on (σ, β).

Mehra-Prescott

1. US Data 1889-1978:
   $E[R^s] = 1.0698, E[R^b] = 1.0080, E[R^s - R^b] \approx .06$
2. Question: Is $E[R^s - R^b] \approx .06$ an implication of a calibrated version of the Lucas asset pricing model?
3. More specifically, if we construct a quantitative version of the Lucas model, where the endowment process matches the average growth rate, variability and serial correlation of US consumption growth and where relative risk aversion is less that 10, then does the model produce an equity premium near 6 percent?
4. Answer: $E[R^s - R^b] \leq 0.0035!$ when $\sigma \in (0, 10),\ \beta \in (0, 1)$

Mehra-Prescott

Primitives:

1. $E[\sum_{t=0}^{\infty} \beta^t u(c_t)],\ where\ u(c) = c^{1-\sigma}/(1-\sigma)$
2. $y_{t+1} = y_t x_{t+1}$, where x_t is a finite Markov chain

Equilibrium Price Functions:

$$p^s(y,x) = E[\frac{\beta u'(yx')}{u'(y)}(p^s(yx',x') + yx')|y,x]$$

$$p^b(y,x) = E[\frac{\beta u'(yx')}{u'(y)}|y,x]$$

Mehra-Prescott: Analysis

Verify: $p^s(y,x) = y\hat{p}^s(x)$ and $p^b(y,x) = \hat{p}^b(x)$ are equil price functions

$$p^s(y,x) = E[\frac{\beta u'(yx')}{u'(y)}(p^s(yx',x') + yx')|y,x]$$

$$\hat{p}^s(x) = E[\frac{\beta u'(x')}{u'(1)}(x'\hat{p}^s(x') + x')|x]$$

$$p^b(y,x) = E[\frac{\beta u'(yx')}{u'(y)}|y,x]$$

$$\hat{p}^b(x) = E[\frac{\beta u'(x')}{u'(1)}|x]$$

Mehra-Prescott: Analysis and Calibration

Construct Model Implied Mean Returns:

- $E[R^s] = \sum_x R^s(x)\pi(x)$ *and* $E[R^b] = \sum_x R^b(x)\pi(x)$
- $\pi = \pi P$ where P is transition prob matrix for x
- $R^s(x) \equiv E[\frac{x'\hat{p}^s(x')+x'}{\hat{p}^s(x)}|x]$ and $R^b(x) \equiv E[\frac{1}{\hat{p}^b(x)}|x]$

Calibrate Endowment Process:

- $x \in \{1+\mu-\delta, 1+\mu+\delta\}$
- $P = [\phi, (1-\phi); (1-\phi), \phi]$
- $(\mu, \delta, \phi) = (0.018, 0.036, 0.43)$ matches mean, SD and 1st order serial correlation of US aggregate consumption growth.

Why is model implied equity premium so small?
Answer: US agg. consumption growth is not very volatile. Thus, sdf does not move much even for high σ.

$$E[R^s - R^b] = \frac{-Cov(\beta x^{-\sigma}, R^s - R^b)}{E[\beta x^{-\sigma}]}$$

Potential Equity Premium Explanations:

- Change Preferences: habit formation or Epstein-Zin preferences
- Allow Disaster state: Reitz and later on Barro
- Add Idiosyncratic Risk Correlated w/ Agg Risk: many authors
- Add risk in long-run economic growth - Bansal and Yaron
- Account for taxation - McGrattan and Prescott
- US is Lucky - no Revolutions in last 100 years or so

Hansen-Jagannathan

Asset Pricing Theory:

- $(*)$ $q_I^i = E[x^i(w)m^j(w)|I]$
- q_I^i price of asset i given info I, $x^i(w)$ payoff of asset i in state w, $m^j(w)$ sdf of agent j in state w.
- $(**)$ $E[q(w)] = E[x(w)m(w)]$
- $(q(w), x(w))$ are $n \times 1$ vectors of random variables

Goal: Use $(**)$ and data on prices and asset payoffs to put restrictions on unobserved m.

Note: Unconditional expectation in $(**)$ gets rid of (unobserved) information set

Hansen-Jagannathan

Comments:

- Mehra-Prescott are in the applied GE tradition. They put a lot of structure on m and see what it implies for mean returns.
- Hansen and Jagannathan take a broader view of the theory. They ask what data (x and q) tell us about m with minimal restrictions (only $(**)$ and later $m > 0$) put on m.

Hansen-Jagannathan

Main Result:

They show that $(**)$ and data on asset prices and payoffs puts restrictions on the set of pairs $(E[m], SD(m))$. More specifically, for given values of $E[m]$, they calculate lower bounds on $SD(m)$ so that $(**)$ holds, given the data. Figure 1 presents the main result of the paper.

The shaded region are possible pairs not ruled out by the data, whereas the boxes are pairs produced by specific sdfs and data.

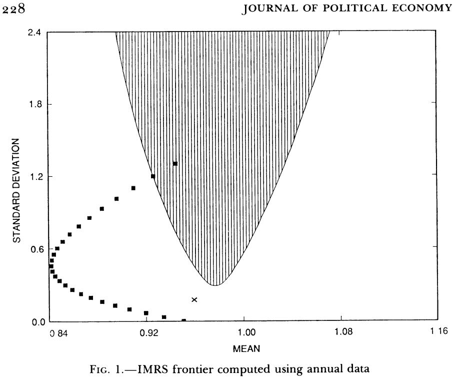

Figure: Based on annual US stock and bond data 1891 - 1985

Hansen-Jagannathan

Construct $m^*(\omega) \equiv x'(\omega)\alpha^*$ st $(**)$ holds.

$$(**)\ E[q(w)] = E[x(w)m(w)]$$

Proposition 1: Assume m satisfies $(**)$.

1. $x_1(\omega) \equiv 1 \Rightarrow E[m(\omega)] = E[m^*(\omega)] = E[q_1(\omega)]$
2. $E[x(m - m^*)] = 0$
3. $Var(m) = Var(m^*) + Var(m - m^*)$

Proof: $1. - 2.$ obvious.
3. Set $a = (m - m^*), b = m^*$ and take variances.
$Cov(m - m^*, m^*) = 0$ by 2.

$$Var(a + b) = Var(a) + Var(b) + 2\ Cov(a, b)$$

Hansen-Jagannathan

$$(**)\ E[q(w)] = E[x(w)m(w)]$$

$$m^*(\omega) \equiv x'(\omega)\alpha^*\ st\ (**)\ holds$$

Theory: Part 1 (A riskless asset is IN the data set)

Claim: Assume $x_1(\omega) \equiv 1$. If m satisfies $(**)$, then

$$(E[m], SD(m)) \in \{(a,b) : a = E[q_1], b \geq SD(m^*)\}$$

Proof: Prop1 part 3.

Hansen-Jagannathan

$$(**)\ E[q(\omega)] = E[x(\omega)m(\omega)]$$

$$m^{**}(\omega) = \hat{x}'(\omega)\alpha^*,\ where$$

$$\hat{x}(\omega) = \begin{pmatrix} 1 \\ x(\omega) \end{pmatrix}\ and\ E[\begin{pmatrix} a \\ q \end{pmatrix}] = E[\hat{x}m^{**}]$$

Theory: Part 2 (A riskless asset is NOT in the data set)

Claim: If m satisfies $(**)$ and $E[m] = a > 0$, then $SD(m) \geq SD(m^{**})$.

Proof: Prop1 part 3.

Hansen-Jagannathan

Engineering: How to construct the mean-SD bound in HJ, given data $\{q_t, x_t\}_{t=1}^{T}$ on the price and total payout to one asset?

1. $SD(m^*) = [E[(m^*)^2] - (E[m^*])^2]^{1/2}$
2. Set $E[(m^*)^2] = \frac{1}{T}\sum_t [(1, x_t) \cdot \alpha^*]^2$ and $E[m^*] = \frac{1}{T}\sum_t (1, x_t) \cdot \alpha^*$
3. Given any mean $a > 0$, construct α^* solving $E[\binom{a}{q}] = E[\hat{x}\ \hat{x}' \cdot \alpha^*]$. Thus, $\alpha^* = (E[\hat{x}\ \hat{x}'])^{-1} E[\binom{a}{q}]$.
4. Approx each term in the matrix in step 3 with a time series average (e.g. $E[x^2] = \frac{1}{T}\sum_t x_t^2$).

Pricing a Non-traded Asset

Application: pricing an individual's future earnings stream.

$P1$ $\max \sum_{t=1}^{T} \sum_{s^t \in S^t} \beta^{t-1} u(c_t(s^t)) P(s^t)$ $s.t.$

1. $c_t + \sum_k a_{t+1}^k \leq e_t + \sum_k a_t^k R_t^k$
2. $c_t \geq 0, \forall t, a_{T+1}^k \geq 0, \forall k$

$P2$ $\max \sum_{t=1}^{T} \sum_{s^t \in S^t} \beta^{t-1} u(c_t(s^t)) P(s^t)$ $s.t.$

1. $c_t + \sum_k a_{t+1}^k + z_{t+1} p_t \leq z_t(p_t + e_t) + \sum_k a_t^k R_t^k$
2. $c_t \geq 0, \forall t, a_{T+1}^k \geq 0, \forall k$ and $z_1 = 1$

Pricing a Non-traded Asset

$$(*)\ p_t(s^t) \equiv \sum_{j=t+1}^{T} \sum_{s^j \in S^j} m_{t,j}(s^j) e(s^j) \ and\ p_T(s^T) = 0, \forall s^T$$

$$m_{t,j} \equiv \frac{\beta^{j-t} u'(c(s^j))}{u'(c(s^t))} P(s^j | s^t)$$

Thm: Assume u is st. inc, concave and differentiable. Let $e_t, R_t^k, \forall k$ be positive random variables. If $(c, a^k, k = 1, .., n)$ solves P1 and c is st. positive, then $(c, a^k, k = 1, .., n, z)$ solves P2, where $z = (z_1, .., z_T), z_t = 1, \forall t$, when p satisfies (*).

Proof: Show suff conditions P1-1 to P1-3 imply suff conditions P2-1 to P2-4.

Pricing a Non-traded Asset

P1-1 $\beta^{t-1}u'(c_t(s^t))P(s^t) - \lambda(s^t) = 0$

P1-2 $-\lambda(s^t) + \sum_{s_{t+1}} \lambda(s^t, s_{t+1})R^k(s^t, s_{t+1}) = 0$

P1-3 budget constraints

P2-1 $\beta^{t-1}u'(c_t(s^t))P(s^t) - \lambda(s^t) = 0$

P2-2 $-\lambda(s^t) + \sum_{s_{t+1}} \lambda(s^t, s_{t+1})R^k(s^t, s_{t+1}) = 0$

P2-3 $-p(s^t) + \sum_{s_{t+1}} \lambda(s^t, s_{t+1})(p_t(s^t, s_{t+1}) + e(s^t, s_{t+1})) = 0$

P2-4 budget constraints

Note: P1-1 implies P2-1, P1-2 implies P2-2, P2-3 holds by definition of p and P1-3 and $z = 1$ implies that P2-4 holds.

Pricing a Non-traded Asset

The principle employed in the theorem works more generally. One could add extra constraints on asset holding (e.g. non-negativity constraints on assets) or even add in valued leisure and a labor-leisiure choice. One could also change preferences. What is important is that P1 is a concave programming problem. The Theorem does not depend on the degree of incompleteness of markets.

Interpretation: $p_t(s^t)$ are individual-specific prices NOT prices at which other people can buy a claim to your future earnings stream. These prices convince one not to sell or to buy any shares in their future earnings.