Macro 1: Dynamic Programming 2

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DP Problems with Risk

Strategy: Consider three classic problems: income fluctuation, optimal (stochastic) growth and search. Learn about methods and theoretical results in each literature at the same time.

income-fluctuation problem

$$\max E[\sum_{t=0}^{\infty} \beta^t u(c_t)] \; s.t. \; c_t + a_{t+1} \leq e_t + a_t(1+r), c_t, a_{t+1} \geq 0$$

stochastic optimal growth problem.

$$\max E[\sum_{t=0}^{\infty} \beta^t u(c_t)] \; s.t. \; c_t + k_{t+1} \leq z_t F(k_t, 1) + k_t(1-\delta), c_t, k_{t+1} \geq 0$$

Basic Issues and Themes:

- Tree structure of events
- Mathematical representation of $E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$
- Mathematical representation of (c_t, a_{t+1}) or (c_t, k_{t+1})
- Bellman's equation with risk
- Why we will (typically) assume that shocks are Markovian
- State variables will now play two roles: define choice set AND describe probabilistic views of future events
- Connection between decision rules and allocations (c_t) in original problem

Tree Structure of Events:

- Income-fluctuation problem
- Histories: $E^t = \{(e_0, ..., e_t) : e_j \in E, \forall j = 0, ..., t\}$.
- E is finite set
- Specific History: $e^t = (e_0, ..., e_t) \in E^t$
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- optimal growth problem
- Histories: $Z^t = \{(z_0, ..., z_t) : z_j \in Z, \forall j = 0, ..., t\}$.
- Z is finite set
- Specific History: $z^t = (z_0, ..., z_t) \in Z^t$

Representation of $E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$ and decisions:

- Probability: $P(e^t)$ for $e^t \in E^t$
- Math: We will focus on context where the number of histories is finite or countable. This makes theories of integration boil down to simple summation and avoids mathematical technicalities (probability and measure theory).
- $E[\sum_{t=0}^{\infty} \beta^t u(c_t)] = \sum_{t=0}^{\infty} \sum_{e^t \in E^t} \beta^t u(c(e^t)) P(e^t)$
- Consumption: $c_t : E^t \to R_+$
- Asset Holding: $a_{t+1} : E^t \to R_+$

Income Fluctuation Problem

Bellman's Eqn:

$$v(a,e) = \max_{(c,a') \in \Gamma(a,e)} u(c) + \beta E[v(a',e')|e]$$

$$\Gamma(a,e) = \{(c,a') : c + a' \leq a(1+r) + e, c, a' \geq 0\}$$

Decision Rules solving BE: $(c(a,e), a(a,e))$
Consumption: choices are produced by repeated application of decision rules and known initial conditions (a_0, e_0)

$$c_0(e^0) = c(a_0, e_0), c_1(e^1) = c(a(a_0, e_0), e_1)$$

$$c_2(e^2) = c(a(a(a_0, e_0), e_1), e_2)$$

Optimal Growth Problem

Bellman's Eqn:

$$v(k,z) = \max_{(c,k')\in\Gamma(k,z)} u(c) + \beta E[v(k',z')|z]$$

$$\Gamma(k,z) = \{(c,k') : c + k' \leq zF(k,1) + k(1-\delta), c, k' \geq 0\}$$

Decision Rules solving BE: $(c(k,z), g(k,z))$
Consumption: choices along a branch of the event tree are produced by repeated application of decision rules

$$c_0(z^0) = c(k_0, z_0), c_1(z^1) = c(g(k_0, z_0), z_1)$$

Income Fluctuation Problem

Two Versions of Bellman's Eqn:

$$v(a, e^t) = \max_{(c,a') \in \Gamma(a,e^t)} u(c) + \beta E[v(a', e^{t+1})|e^t]$$

$$\Gamma(a, e^t) = \{(c, a') : c + a' \leq a(1 + r) + e_t, c, a' \geq 0\}$$

$$v(a, e) = \max_{(c,a') \in \Gamma(a,e)} u(c) + \beta E[v(a', e')|e]$$

$$\Gamma(a, e) = \{(c, a') : c + a' \leq a(1 + r) + e, c, a' \geq 0\}$$

Latter holds w/ Markov shocks. It is computationally tractable. Former is not. We will only consider Markov shocks.

Income Fluctuation Problem: Euler eqn

$\max E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$ $s.t.$ $c_t + a_{t+1} \leq e_t + a_t(1+r), c_t, a_{t+1} \geq 0$

Claim: Assume u is differentiable. If $(c_0(e^0), c_1(e^1), ...)$ is a solution and is interior at e^t (i.e. $c_t(e^t), a_{t+1}(e^t) > 0$ and $c(e^t, e_{t+1}) > 0, \forall e_{t+1}$), then (*) holds at e^t:

$$(*)\ u'(c_t(e^t)) = \beta(1+r)E[u'(c_{t+1}(e^{t+1}))|e^t]$$

Proof: Construct deviation that decreases consumption in history e^t by x and increases it by $x(1+r)$ in histories (e^t, e_{t+1}) for each e_{t+1} without any other changes. It's a feasible deviation for small x. At a solution, the marginal gain evaluated at $x = 0$ must be zero:

$$-\beta^t u'(c(e^t))P(e^t) + \beta^{t+1}(1+r) \sum_{e_{t+1} \in E} u'(c(e^t, e_{t+1}))P(e^t, e_{t+1}) = 0$$

Income Fluctuation Problem

A1: u is cont and bd., A2: u is inc. and concave, A3: u is diff

$$v(a,e) = \max_{(c,a')\in\Gamma(a,e)} u(c) + \beta E[v(a',e')|e]$$

$$\Gamma(a,e) = \{(c,a') : c + a' \leq a(1+r) + e, c, a' \geq 0\}$$

Claim: Assume A1-3. Define $T(v)$ w/ RHS of BE. Then

1. $T : C^b(R_+ \times E) \to C^b(R_+ \times E)$
2. $\exists!\ v \in C^b(R_+ \times E)$ solving BE
3. v concave in a implies $T(v)$ concave in a.
4. v is differentiable in a and $v_1(a,e) = u'(c(a,e))(1+r)$
5. $(c(a,e), a(a,e))$ are increasing in a.

Income Fluctuation Problem

Brief Sketch of Proof of Claim:

1. Apply Thm of Max.
2. Apply CMT. Verify T is contraction map using Blackwell.
3. Standard argument for concavity. Nothing new.
4. Apply B-S Thm using "w" fn below
   $w(a, e) \equiv u(a(1 + r) + e - a^*) + \beta E[v(a^*, e')|e]$
5. Graph $u'(a(1 + r) + e - a')$ and $\beta E[v_1(a', e')|e]$ as fn of a'.
   Note: only current return fn graph shifts as a increases.

Income Fluctuation Problem

Alternative Derivation of Euler Eqn: based on K-T conditions and v differentiable.

$$L = u(a(1+r)+e-a')+\beta E[v(a',e')|e]+\lambda_1[a(1+r)+e-a']+\lambda_2[a'-0]$$

1. $-u'(c) + \beta E[v_1(a',e')|e] - \lambda_1 + \lambda_2 = 0$
2. $\lambda_1[a(1+r) + e - a'] = 0$
3. $\lambda_2[a' - 0] = 0$

Note: $u'(0) = \infty$ implies $\lambda_1 = 0$ and $a(1+r) + e - a' \neq 0$. This also implies that the following holds for $a' = a(a,e)$:

$$u'(c(a,e)) \geq \beta E[v_1(a',e')|e] = \beta(1+r)E[u'(c(a',e'))|e]$$

Income Fluctuation Problem

Some History:

1. 1950's: Friedman, Modigliani and Brumberg state permanent-income and life-cycle hypothesis
2. Original Motivation: explanation for cross-sectional and long-run consumption-savings facts
3. 1970s: Theorists recast this work as income fluctuation problem and employ DP methods
4. 1980s: Euler eqns lead to empirical work on estimating preference parameters AND test of theory.
5. Extreme View: financial markets can move resources forward and backward over time BUT not across histories!
6. Currently: Leading view of consumption and savings behavior

Income Fluctuation Problem

Using the Euler eqn: Application $u(c) = c^{1-\gamma}/(1-\gamma)$

- $u'(c_t) = \beta(1+r)E[u'(c_{t+1})]$
- $1 = \beta(1+r)E[\frac{u'(c_{t+1})}{u'(c_t)}] = \beta(1+r)E[(c_{t+1}/c_t)^{-\gamma}]$
- $0 = \log(\beta(1+r)) + \log(E[(c_{t+1}/c_t)^{-\gamma}])$
- $(\frac{c_{t+1}}{c_t})^{-\gamma} \sim LN(\mu, \sigma^2)$ implies $E[(\frac{c_{t+1}}{c_t})^{-\gamma}] = exp(\mu + \sigma^2/2)$
- $0 = \log(\beta(1+r)) + \log(exp(E[\log((c_{t+1}/c_t)^{-\gamma})] + (1/2)Var(\log((c_{t+1}/c_t)^{-\gamma}))$
- $E[\log \frac{c_{t+1}}{c_t}] = \frac{1}{\gamma} \log \beta(1+r) + \frac{1}{2}\gamma Var(\log(c_{t+1}/c_t))$
- Result: Mean log growth rate increasing in discount factor, interest rate **and** the variance of consp growth. Second term sometimes labeled a precautionary savings effect.

Income Fluctuation Problem

Using the Euler eqn for estimation and testing

- $u'(c_t) = \beta(1+r)E_t[u'(c_{t+1})]$ and $u(c) = c^{1-\gamma}/(1-\gamma)$
- $\epsilon_{t+1} \equiv c_t^{-\gamma} - \beta(1+r_{t+1})c_{t+1}^{-\gamma}$
- $E[\epsilon_{t+1}] = E[c_t^{-\gamma} - \beta(1+r_{t+1})c_{t+1}^{-\gamma}] = 0$
- Use "moment condition" above to estimate (β, γ)
- Given data $\{c_t, r_t\}_{t=1}^{T+1}$, employ the estimator $(\hat{\beta}, \hat{\gamma}) \equiv argmin_{(\beta,\gamma)} \sum_{t=1}^{T} \epsilon_{t+1}^2$
- See Hansen and Singleton (1981, 1982) for estimation and a test.
- Note: can estimate some parameters (β, γ) without specific parametric assumptions on "other" model parameters (e.g. shock process driving earnings and the interest rate).

Income Fluctuation Problem

When do agents save more as a measure of earnings risk increases?

$\max u(c_1) + \beta E_\theta[u(c_2)]$ $s.t.$ $c_1 + a_2 \leq a_1(1+r) + e_1,\ c_2 = a_2(1+r) + e_2,\ e_2 \sim F_\theta$

Theorem: Assume u is st. increasing, st. concave and u' is convex, then $a_2(a_1, e_1; \theta)$ is increasing in risk θ, for any (a_1, e_1).

Definition: Let F_θ and $F_{\theta'}$ be distributions indexed by θ, θ'. F_θ is riskier than $F_{\theta'}$ provided condtion (*) holds for all concave functions f

$$(*) \quad \int f(x) dF_{\theta'} \geq \int f(x) dF_\theta$$

Comment: sometimes called second-order stochastic dominance or the Rothschild-Stiglitz (1971) definition of increasing risk.

Income Fluctuation Problem

Sketch of Proof of Theorem:

Step 1: Graph two terms in Euler eqn as a function of a_2.

$$u'(a_1(1+r)+e_1-a_2) \geq \beta(1+r)E_\theta[u'(a_2(1+r)+e_2)]$$

Step 2: Argue $E_\theta[u'(a_2(1+r)+e_2)] \geq E_{\theta'}[u'(a_2(1+r)+e_2)]$. Why? Set $f = u'(...)$. Use u' convex. Apply definition using θ is riskier than θ'

Step 3: "intersection" point of the graph of the Euler equation in Step 1 is further to the right when risk is greater. This implies the result.

Income Fluctuation Problem

When do agents save more as a measure of earnings risk increases?

$\max u(c_1) + \beta E_\theta[u(c_2)]$ $s.t.$ $c_1 + a_2 \leq a_1(1+r) + e_1,\ c_2 = a_2(1+r) + e_2$

Theorem suggests that if u is locally quadratic (about the relevant region) then riskier earnings do not lead to greater precautionary savings. u locally quadratic implies that u' is locally linear.

Theorem can be extended to models with many periods. Need to show that the value function inherits a key property ($v_1(a, e)$ convex in a) from the utility function based on backwards induction.

Search Problem

Problem: Agent receives a wage offer w. If accept, utility is $\sum_{t=0}^{\infty} \beta^t u(w)$. If reject, current utility is $u(0)$ and draw w' out of a distribution F tomorrow and repeat accept/reject choice.

Two Versions of BE:

$$(**)\ v(w) = \max\{\frac{u(w)}{1-\beta}, u(0) + \beta E[v(w')]\}$$

$$(*)\ v(w) = \max_{y \in \{accept, reject\}} u(w, y) + \beta E[v(w')|w, y]$$

Both lead to the same equation. The former is "the standard formulation in the literature". The latter is equivalent with the right choice of $u(w, y)$ and the conditional expectation.

Argue that $(*)$ produces $(**)$

$$(*)\ v(w) = \max_{y \in \{accept, reject\}} u(w,y) + \beta E[v(w')|w,y]$$

1. $u(w,y) = u(w)$ if $y = accept$ and $u(w,y) = u(0)$ if $y = reject$
2. $v(w) = \max\{u(w) + \beta v(w), u(0) + \beta E[v(w')]\}$ holds after defining the conditional expectation in a relevant way.
3. CMT implies that $\exists!$ bd, weakly increasing fn v solving (*).
4. At any accepted wage w, $v(w) = u(w) + \beta v(w)$ and thus $v(w) = \frac{u(w)}{1-\beta}$.
5. Conclude: $v(w) = \max\{\frac{u(w)}{1-\beta}, u(0) + \beta E[v(w')]\}$

Search Problem

$$v(w) = T(v)(w) \equiv \max\{\sum_{t=0}^{\infty} \beta^t u(w), u(0) + \beta E[v(w')]\}$$

$$v(w; b) = T_b(v)(w) \equiv \max\{\sum_{t=0}^{\infty} \beta^t u(w), u(b) + \beta E[v(w'; b)]\}$$

Claim: If u is bd, then

1. T and T_b are monotone and discount
2. $\exists! v^* \in B(R_+, R)$ solving $v^* = T(v^*)$ or $v^* = T_b(v^*)$
3. v^* is increasing in w
4. accept decision based on a threshold wage
5. unemploy benefit $b_2 > b_1$ implies $v^*(\cdot; b_2) \geq v^*(\cdot; b_1)$ and threshold wage that is accepted is larger for b_2 than b_1.

Search Problem

Prove Claim 5: $v^*(w;b)$ is increasing in b for any w:

Step 1: Let $b_2 > b_1$. Let $v^*(w;b)$ be solution to BE.
Step 2: Define $v_n(w;b_2) = T^n_{b_2}(v^*(\cdot;b_1))$ for $n = 1, 2, ...$
Step 3: Clearly, $v^*(\cdot;b_1) \leq v_1(\cdot;b_2) \leq v_2(\cdot;b_2) \leq \cdots \leq v^*(\cdot;b_2)$ holds as T_{b_2} is monotone.
Step 4: Result: $v^*(\cdot;b_2) \geq v^*(\cdot;b_1)$
Step 5: Step 4 implies reservation wage increases with b

The monotonicity of the solution to BE in b implies that reservation wage is also montone in b. Proof is by picture once establish that v^* is monotone in b.

Career-Job Search Problem

Objective: $E[\sum_{t=0}^{\infty} \beta^t(\theta_t + \epsilon_t)]$

Each period an agent has a career-job pair (θ, ϵ). Each period the agent has three options.

Option 1: stick with (θ, ϵ). Period payoff is $\theta + \epsilon$. Start next period with (θ, ϵ).
Option 2: keep career θ but draw new job. Period payoff $\theta + E[\epsilon']$. Start next period with (θ, ϵ').
Option 3: draw new career and job. Period payoff $E[\theta' + \epsilon']$. Start next period with (θ', ϵ').

Career-Job Search Problem

BE

$v(\theta,\epsilon) = T(v)(\theta,\epsilon) \equiv \max\{T_1(v)(\theta,\epsilon), T_2(v)(\theta,\epsilon), T_3(v)(\theta,\epsilon)\}$

$T_1(v)(\theta,\epsilon) = \theta + \epsilon + \beta v(\theta,\epsilon)$
$T_2(v)(\theta,\epsilon) = \theta + E[\epsilon'] + \beta E[v(\theta,\epsilon')]$
$T_3(v)(\theta,\epsilon) = E[\theta' + \epsilon'] + \beta E[v(\theta',\epsilon')]$

Show: v solving BE is increasing in both arguments.
Main steps.
Step 1: $\exists!$ solution to BE
Step 2: v increasing in (θ,ϵ) implies $T(v)$ is also.
Step 3: $B^{bd,inc}$ is closed subset of B^{bd}

Step 1 use Blackwell and CMT. Step 2 use T_1, T_2, T_3 each map inc fn into inc fn and, thus, T also does so. Step 3 prove this then use Corollary 1 to CMT in SL (1989) to get result.