Macro 1: Dynamic Programming 1

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DP Warm up: Cake eating problem

$$(*)\ \max f_1(y_1) + f_2(y_2)\ s.t.\ y_1 + y_2 \leq 100, y_1 \geq 0, y_2 \geq 0$$

1. $v_1^*(x) \equiv \max f_1(y_1) + f_2(y_2)\ s.t.\ y_1 + y_2 \leq x, y_1 \geq 0, y_2 \geq 0$
2. $v_2(x) = \max f_2(y_2)\ s.t.\ 0 \leq y_2 \leq x$
3. $v_1(x) = \max f_1(y_1) + v_2(x - y_1)\ s.t.\ 0 \leq y_1 \leq x$
4. Claim: If f_1, f_2 are continuous, then (a) $v_1^*(x) = v_1(x)$, (b) $(y_1^*, y_2^*) = (y_1(100), y_2(100 - y_1(100)))$.

Terminology: (v_1, v_2) are value functions and v_1^* is the optimal value function. Equation 2-3 is Bellman's equation. Functions $(y_1(x), y_2(x))$ solving eq 2-3 are called decision rules. Decision rules in eq 2-3 that achieve $v_1^*(x)$ are optimal decision rules.

The warm-up problem displays many common features of the dynamic programming approach.

- Step 1: generalize the initial problem.
- Step 2: break initial problem into perhaps many subproblems of lower dimensionality
- Step 3: solve "backwards" sequentially from the "last period" or last stage.
- The value function v_1 solving Bellman's equation is, under some regularity conditions, the optimal value function v_1^*.
- Decisions (y_1^*, y_2^*) solving the original problem can be obtained from the "decision rules" solving Bellman's equation: $(y_1^*, y_2^*) = (y_1(100), y_2(100 - y_1(100)))$

Proof of Claim $v_1^*(x) = v_1(x)$: Hueristic Argument

$$v_1^*(x) \equiv \max f_1(y_1) + f_2(y_2) \ s.t. \ y_1 + y_2 \leq x, y_1 \geq 0, y_2 \geq 0$$

$$v_1^*(x) = \max_{0 \leq y_1 \leq x} [\max_{0 \leq y_2 \leq x - y_1} f_1(y_1) + f_2(y_2)]$$

$$v_1^*(x) = \max_{0 \leq y_1 \leq x} [f_1(y_1) + v_2(x - y_1)] = v_1(x)$$

Dynamic Programming: Finite Horizons

- Objective: $\sum_{t=1}^{T} u_t(x_t, y_t)$
- Return Function: $u_t(x, y)$
- States and Controls: $x \in X_t$ and $y \in Y_t(x)$
- Law of Motion: $x_{t+1} = g_t(x_t, y_t)$
- Optimal Value Function:
  $v_t^*(x_t) = \max_{(y_s,..,y_T)} \sum_{s=t}^{T} u_s(x_s, y_s) \ s.t. \ y_s \in Y_s(x_s), x_{s+1} = g_s(x_s, y_s)$
- Bellman's eqn:
  $v_t(x) = \max_{y \in Y_t(x)} u_t(x, y) + v_{t+1}(g_t(x, y)) \ for \ t < T$
  $v_t(x) = \max_{y \in Y_t(x)} u_t(x, y) \ for \ t = T$

Dynamic Programming: Finite Horizons

- We work with additive objectives, although DP methods apply to recursive objectives. Additive objectives are common in the consumption, investment, growth, business cycle, finance, public economics, IO and labor literatures.
- Recursive Utility: $U_t(c_t, ..., c_T) = W(c_t, U_{t+1}(c_{t+1}, ..., c_T))$
- Example: $W(c, y) = u(c) + \beta y$ and $U_T(c_T) = u(c_T)$ produces $U_t(c_t, .., c_T) = \sum_{s=t}^{T} \beta^{s-t} u(c_s)$

Two DP formulations for a problem

$$\max \sum_{t=1}^{T} \beta^{t-1} u(c_t) \ s.t. \ \sum_{t=1}^{T} p_t c_t \leq x$$

Formulation 1:

- $y = c$ and $Y_t(x) = \{c : p_t c \leq x, c \geq 0\}$
- $u_t(x, y) = \beta^{t-1} u(y)$
- $x' = g_t(x, y) = x - p_t y$
- $v_t(x) = \max_{y \in Y_t(x)} u_t(x, y) + v_{t+1}(g_t(x, y)) \ for \ t < T$

Two DP formulations for a problem

$$\max \sum_{t=1}^{T} \beta^{t-1} u(c_t) \ s.t. \ \sum_{t=1}^{T} p_t c_t \leq x$$

Formulation 2:

- $y = c$ and $Y_t(x) = \{c : p_t c \leq x, c \geq 0\}$
- $u_t(x, y) = u(y)$
- $x' = g_t(x, y) = x - p_t y$
- $v_t(x) = \max_{y \in Y_t(x)} u_t(x, y) + \beta v_{t+1}(g_t(x, y)) \ for \ t < T$
- This formulation is commonly used in the literature. Both result in $v_1^*(x) = v_1(x)$

Dynamic Programming: Infinite Horizons

Some problems are naturally formulated in infinite horizon models (optimal growth, business cycles, asset pricing ...) and some are easier to solve with an infinite horizon compared to a long but finite horizon.

Several new technical issues arise:

- When is $\sum_{t=1}^{\infty} u_t(x_t, y_t)$ bounded?
- What does Bellman's equation (BE) look like? How many solutions to BE are there? Do solutions to BE coincide with the optimal value function? Can one approximate them?

Dynamic Programming: Infinite Horizons

When is $\sum_{t=1}^{\infty} u_t(x_t, y_t)$ bounded?

Easy: $u_t(x, y) = \beta^{t-1} u(x, y), \beta \in (0, 1)$ and $u(x, y)$ is bounded.
Bounded means bounded above and below.

What does BE look like?

- $v(x) = \max_{y \in Y(x)} u(x, y) + \beta v(g(x, y))$
- Note: $Y(x), g(x, y)$ are now restricted to be time invariant.
- How many solutions are there?
- Answer: exactly 1 but this will require an investment in some mathematical tools - a fixed point theorem applied to BE.

One popular period utility function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

For different choices of σ it is either unbounded below, unbounded above or both.

We will assume $u(x, y)$ is bounded mainly because the theory of dynamic programming with bounded returns is easier ... but still not easy. It is common to develop side arguments to use return functions that are not bounded.

An Existence Theorem

$$T(v)(x) \equiv sup_{y \in Y(x)} u(x, y) + \beta v(g(x, y))$$

We will show that the contraction mapping theorem (CMT) delivers the existence of a unique function v s.t. $v = T(v)$.

Some (Condensed) Steps:

- T maps a function v into a function $T(v)$.
- T is a contraction map on a complete metric space.
- $(B(X), d)$ defines our complete metric space where ...
- $B(X) = \{v : v : X \rightarrow R \text{ } and \text{ } sup_{x \in X}|v(x)|$ is finite $\}$
- $d(v, w) = sup_{x \in X}|v(x) - w(x)|$ is our metric
- CMT: $\Rightarrow \exists! \; v \in B(X) \; s.t. \; v = T(v)$

Metric spaces

Def: A metric space is a set X and $d : X \times X \to R$ s.t.

- (positivity) $d(x, y) \geq 0, \forall x, y \in X \ and \ d(x, y) = 0 \ iff x = y$
- (symmetry) $d(x, y) = d(y, x), \forall x, y \in X$
- (triangle) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$

Def: A metric space (X, d) is complete if every Cauchy sequence $\{x_t\}_{t=1}^{\infty}$ in X converges to a point $x \in X$.

Comment: A metric gives a measure of distance between any points in the space. The metric $d(v, w) = sup_{x \in X} |v(x) - w(x)|$ is "the pointwise maximum gap between v and w".

In our application, $B(X)$ is the "set" X in the definition. Obvious notational conflict!

Contraction Mapping Theorem (CMT)

Def: Let (X, d) be a metric space and $T : X \to X$. T is a contraction map if $\exists\ k\ s.t.\ 0 \leq k < 1\ s.t.$

$$d(T(x), T(y)) \leq k\ d(x, y), \forall (x, y) \in X$$

Notation: $T^n(x) = T^{n-1}(T(x))$

CMT: Let (X, d) be a complete metric space and $T : X \to X$ a contraction map with modulus k. Then (1) there exists a unique $\bar{x} \in X\ s.t.\ T(\bar{x}) = \bar{x}$ and (2) $\forall x \in X, d(T^n(x), \bar{x}) \leq k^n d(x, \bar{x})$.

Is T a contraction map?

$$T(v)(x) \equiv sup_{y \in Y(x)} u(x,y) + \beta v(g(x,y))$$

THM: (Blackwell's Sufficient Conditions)
Let $X \subset R^n$ and $B(X)$ be the set of bd functions w/ sup norm. If $T : B(X) \to B(X)$ satisfies (1)-(2) then T is a contraction:

(1) (monotonicity) $\forall v, w \in B(X), v \leq w \Rightarrow T(v) \leq T(w)$
(2) (discounting) $\exists \beta \in (0,1)\ s.t.\ \forall v \in B(X), \forall x \in X, \forall a \geq 0$

$$T(v+a)(x) \leq T(v)(x) + \beta a$$

Comment: It is easy to see that in our application both conditions are easy to verify. They hold as the "discount" factor β governing preferences serves the role of the "beta" in condition (2).

Does all the machinery deliver anything besides existence?

Answer: Yes.

- It also delivers a method to approximate solutions to BE.
- Start with any guess $v \in B(X)$. Apply T to v repeatedly. CMT says that you converge geometrically (with convergence factor β^n) to the unique fixed point. Stokey and Lucas (1989) provide some theorems that tell you that decision rules (THE THINGS WE REALLY CARE ABOUT) will also converge to optimal decision rules.
- It allows one to work out properties of decision rule $y(x)$ and value function $v(x)$ (are they increasing, continuous, concave and differentiable?) using basic finite-dimensional optimization theory and comparative statics tools. The heavy lifting (i.e. existence) is already done.

Infinite Horizon DP: An Example

Optimal Growth Problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \; s.t. \; c_t + k_{t+1} \leq F(k_t, 1), c_t, k_{t+1} \geq 0$$

$$v(k) = \max u(F(k,1) - k') + \beta v(k') \; s.t. \; 0 \leq k' \leq F(k,1)$$

Assume: $u(c) = \log(c), F(k,1) = k^{\alpha}$
Method: Guess and Verify / Method of Undetermined Coefficients

Infinite Horizon DP: An Example

Guess: $v(k) = a_1 + a_2 \log(k)$
FOC: $u'(F(k,1) - k') = \beta v'(k') \Rightarrow k' = \frac{\beta a_2 F(k,1)}{1+\beta a_2}$

Verify:

$$v(k) = \log(\frac{F(k,1)}{1+\beta a_2}) + \beta[a_1 + a_2 \log(\frac{\beta a_2 F(k,1)}{1+\beta a_2})]$$

$$a_1 = \log(1/(1+\beta a_2)) + \beta[a_1 + a_2 \log(\frac{\beta a_2}{1+\beta a_2})]$$

$$a_2 = (1+\beta a_2)\alpha \Rightarrow a_2 = \frac{\alpha}{1-\beta\alpha}$$

Upshot: v solving BE is increasing, concave and differentiable.

Properties of Value Functions and Decision Rules

We already know conditions under which solutions to BE exist.
What are the properties of $(v(x), y(x))$ that solve BE?

When is $v(x)$ (i) increasing, (ii) continuous, (iii) concave and (iv) differentiable in x?
When is $y(x)$ (i) single valued ,(ii) increasing and (iii) continuous in x?

Here are a few standard (partial) results.

Properties: v is increasing in x

Basic Result:

$$h(x) = \max f(x, y) \ s.t. \ y \in Y(x)$$

Thm: *If $h(x)$ exists, $f(x, y)$ is increasing in x, and $Y(x)$ is increasing in x (i.e. $x \geq_X x' \Rightarrow Y(x') \subset Y(x)$) then $h(x)$ is monotone in x based on the partial order $\geq_X$.*

Proof: Let $x \geq_X x'$ and let (y, y') denote the maximizers for (x, x'). Then the following hold:

$$h(x) \geq f(x, y') \geq f(x', y') = h(x') \ \|$$

NOTE: $\geq_X$ is a partial order if it is reflexive, antisymmetric and transitive.

Properties: v is increasing in x

Application:

$$(BE)\ v_j(x) = \max u_j(x,y) + \beta v_{j+1}(y)\ s.t.\ y \in Y_j(x)$$

Claim: If $u_j(x,y)$ is increasing in x and $Y_j(x)$ is increasing in x, then $v_j(x)$ is increasing in x.

Sketch of Proof: Objective $u_j(x,y) + \beta v_{j+1}(y)$ is increasing in x by construction. Thus, apply the previous result. $\|$

Discuss application to infinite horizon problems and to problems where $g(x,y) \neq y$.

Properties: v is continuous in x

Basic Result

$$h(x) = \max f(x, y) \; s.t. \; y \in \Gamma(x)$$

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$$

Thm: (Thm of the Maximum) *Let* $X \subset R^l, Y \subset R^m, f : X \times Y \to R$ *is continuous,* $\Gamma : X \to Y$ *is a compact-valued and continuous correspondence, then (i)* $h(x)$ *is continuous and (ii)* $G(x)$ *is a nonempty, compact-valued and uhc correspondence.*

[Micro 1 handles ALL the machinery behind the Thm of the Max.]

Properties: v is continuous in x

Application:

$$(BE)\ v_j(x) = \max u_j(x,y) + \beta v_{j+1}(g_j(x,y))\ s.t.\ y \in Y_j(x)$$

Assume: u_j, g_j are continuous and Y_j is a compact-valued and continuous correspondence.

All assumptions of the Thm of the Max hold in the last period as $v_{J+1} = 0$. Thus, v_J is continuous in x. Work backwards from the last period. Only need objective is continuous. This holds as sum of cont. fns. is continuous and as compostion of continuous fns is continuous.

Properties: v is concave in x

$$(BE)\ v_j(x) = \max u_j(x,y) + \beta v_{j+1}(y)\ s.t.\ y \in Y_j(x)$$

Thm: *If u_j is concave in (x,y) and v_{j+1} is concave and the graph of the constraint set (i.e. $\{(x,y) : x \in X, y \in Y_j(x)\}$) is a convex set, then $v_j = T(v_{j+1})$ is concave.*
Sketch: Let $(x_1, x_2) \in X$. Let $y_1 \in Y_j(x_1), y_2 \in Y_j(x_2)$ achieve the maximum. Let $x_\theta = \theta x_1 + (1-\theta)x_2$, $y_\theta = \theta y_1 + (1-\theta)y_2$.

$$T(v_{j+1})(x_\theta) \geq u_j(x_\theta, y_\theta) + \beta v_{j+1}(y_\theta)$$

$$\geq \theta[u_j(x_1,y_1) + \beta v_{j+1}(y_1)] + (1-\theta)[u_j(x_2,y_2) + \beta v_{j+1}(y_2)]$$

$$= \theta T(v_{j+1})(x_1) + (1-\theta)T(v_{j+1})(x_2)\ \|$$

Properties: v is differentiable in x

$$(BE)\ v(x) = \max u(x,y) + \beta v(y)\ s.t.\ y \in Y(x)$$

EX: Growth Model

$$(BE)\ v(x) = \max u(F(x,1) - y) + \beta v(y)\ s.t.\ 0 \leq y \leq F(x,1)$$

The key issue is to prove when v is differentiable and NOT what the derivative is provided that it is differentiable.

Properties: v is differentiable in x

THM: (Benveniste Scheinkman) *Let $X \subset R^l$ be convex set, $v : X \to R$ be concave, $x_0 \in intX$, and let D be a neighborhood of x_0.*
If there is a concave, differentiable fn $W : D \to R, W(x_0) = v(x_0)$ and $W(x) \leq v(x), \forall x \in D$, then $v(x)$ is differentiable at x_0 and $V_i(x_0) = W_i(x_0)$ for $i = 1, ..., l$.

Comment: This result looks (perhaps) like it does not help very much as it involves a good guess of a "W" function. However, in practice it is often easy to guess a good W function.
Comment: The theorem has compelling graphical intuition ... as concavity is almost differentiability.

Properties: v is differentiable in x

EX: Growth Model

$$(BE)\ v(x) = \max u(F(x,1) - y) + \beta v(y)\ s.t.\ 0 \leq y \leq F(x,1)$$

Step 1: $W(x) \equiv u(F(x,1) - y_0) + \beta v(y_0), \forall x \in D(x_0)$, where y_0 is best choice at x_0 and y_0 is interior to the choice set.

Step 2: Verify $W(x) \leq v(x), \forall x \in D(x_0), W(x_0) = v(x_0)$ and $W'(x_0) = u'(F(x_0,1) - y_0)F_1(x_0,1)$

Step 3: Apply previous result that v is concave.

Step 4: Apply B-S Thm to get v is diff at x_0.