Macro I

Homework 8 - Balanced Growth

1. Consider a version of the Diamond growth model. The model is an overlapping generations model where agents live two period lifetimes. Agents have 1 unit of available time to work when young and 0 units when old. For agents born in period t preferences are given by $U(c_{1,t}, c_{2,t+1})$. There is a constant returns production function $F(k_t, l_t A_t)$, where (k_t, l_t) are capital and labor inputs. Technology is denoted A_t, where $A_{t+1} = A_t g_A^t$ and $g_A \geq 1$ is an exogenous growth rate. There is no growth in the population. Feasibility requires that in any period t the feasibility condition $c_{1,t} + c_{2,t} + k_{t+1} + g_t = F(k_t, l_t A_t)$ holds. There is a government that imposes a constant tax rate τ on labor income that is used to finance government purchases g_t.

(a) Define a competitive equlbrium for this model.

(b) Define what it means for a competitive equilibrium to display balanced growth. [Hint: This involves looking for equilibria in which any equilibrium variable x_t has the property that $x_t = \bar{x} g_x^t, \forall t = 0, 1, ...$ for some level $\bar{x}$ and growth rate g_x.]

(c) Calculate the growth rates that must hold for each variable in a balanced-growth competitive equilibrium. Express the "level" of capital in a balanced-growth competitive equilibrium in terms of the primitives of the model. How does the tax rate τ impact this "level"?

Use the following functional forms in your answer to part (c).

$$F(k, lA) = k^\alpha (lA)^{1-\alpha}, U(c_1, c_2) = \log c_1 + \beta \log c_2$$

2. Consider a model of embodied technological change. To take advantage of technological change in this model one needs to buy new capital. The quality of new capital at time t is given by q_t. One unit of investment expenditiures at time t, measured in units of consumption, is converted into q_t units of capital next period.

Technology:

$c_t + i_t = y_t = F(k_t, 1)$ and F is CRS

$k_{t+1} = k_t(1 - \delta) + q_t i_t$

$q_t \equiv q^t$, where $q > 1$

A consumer has one unit of labor that it supplies each period. The consumer solves problem (P1), where (W_t, R_t) are rental rates:

Problem (P1) $\qquad \max \sum_{t=0}^{\infty} \beta^t u(c_t)$

$$c_t + i_t = W_t + k_t R_t \text{ and } k_{t+1} = k_t(1-\delta) + i_t q_t \text{ and } c_t, k_{t+1} \geq 0$$

(a) Define a competitive equilibrium in this framework.

(b) Define what it means for a competititve equilibrium to display balanced growth.

(c) What restrictions on the growth rates of output, consumption, investment, ... must hold in an equilibrium that displays balanced growth, given $F(k,1) = Ak^{\alpha}, \alpha \in (0,1)$?

(d) What restriction on the functional form u is potentially consistent with a competitive equilibrium displaying balanced growth?

(e) Suppose that a government in this model imposes a proportional tax $\tau > 0$ on the rental payments to capital and that the proceeds are returned as a lump-sum transfer. This implies that the net payment to capital is now $k_t R_t(1-\tau)$. How does this affect the level and growth rate of the balanced-growth path for capital? Explain.

3. Consider an overlapping generations model where agents live J periods. The technology for the economy is given below. Aggregate variables are determined by sums across the agents alive in the model period. All generations carry the same number (mass) of agents. Thus, $C_t = \sum_j c_{j,t-j+1}, K_t = \sum_j k_{j,t-j+1}, L_t = \sum_j h_{j,t-j+1} l_{j,t-j+1}$ denote aggregate variables. The notational convention is that agent choices have two subscripts, where the first subscript j refers to the agents age and the second subscript refers to the birth year $t-j+1$. Here birth year is specified by current year t less age j plus one so that by convention an agent is age 1 in the birth year. All agents of the same age at a given time are assumed to be identical.

Technology:

$C_t + I_t = Y_t = F(K_t, L_t A_t)$ and F is CRS and $F_1 > 0, F_{11} < 0$

$A_{t+1} = A_t \gamma$ for $\gamma > 1$.

$K_{t+1} = K_t(1-\delta) + I_t$

Assume that agents born in model period t (i.e. $j = 1$ at time t) solve P1:

$(P1) \quad \max \sum_{j=1}^{J} \beta^{j-1} u(c_{j,t}) \quad s.t.$

$c_{j,t} + k_{j+1,t} \leq W_{t+j-1} h_{j,t} l_{j,t} + k_{j,t}(1 + R_{t+j-1} - \delta), \forall j$

$h_{j+1,t} = H(h_{j,t}, s_{j,t})$ and $l_{j,t} + s_{j,t} = 1, \forall j$

$k_{1,t} = 0$ and $h_{1,t} = h_1$

Define the set of time periods to be $\mathbf{T} = \{.., -1, 0, 1, ...\}$.

Definition: A competitive equilibrium is $(c_{j,t}, s_{j,t}, l_{j,t}, h_{j,t}, k_{j,t}), \forall j = 1, ..., J, \forall t \in \mathbf{T}$ and $(W_t, R_t), \forall t \in \mathbf{T}$ such that

- Agents born at any time $t \in \mathbf{T}$ solve P1.
- $W_t = F_2(K_t, L_t A_t) A_t$ and $R_t = F_1(K_t, L_t A_t), \forall t \in \mathbf{T}$
- $C_t + K_{t+1} = F(K_t, L_t A_t) + K_t(1 - \delta), \forall t \in \mathbf{T}$

(a) If a competitive equilibrium displays balanced growth, then what must the growth rate of output, wages and the rental rate of capital equal? Explain.

(b) Suppose that an equilibrium displays balanced growth. What functional forms for u or H are potentially consistent with balanced growth in that they are not ruled out? Explain.