Macro I

Homework 7 - One-Sector Growth Model

Consider the following decision problem and equilibrium concept.

$$v(k,K;\tau) = \max_{k' \geq 0} u(k[1 + (R(K) - \delta)(1 - \tau)] + W(K) - k') + \beta v(k', H(K;\tau);\tau)$$

Definition: A recursive competitive equilibrium is (i) decision rules $(c(k,K;\tau)$ and $k(k,K;\tau)$, (ii) government decision $G(K;\tau)$, (iii) a law of motion $H(K;\tau)$ and (iv) factor price functions $(W(K), R(K))$ such that

1. $(c(k,K;\tau), k(k,K;\tau))$ solve Bellman's Equation, given W, R, H.
2. $W(K) = F_2(K,1)$ and $R(K) = F_1(K,1)$ hold for all $K > 0$
3. Feasibility: $c(K,K;\tau) + k(K,K;\tau) + G(K;\tau) = F(K,1) + K(1-\delta)$ holds for all $K > 0$, $\tau \in (0,1)$
4. Govt. Budget: $G(K;\tau) = \tau K(R(K) - \delta)$ holds for all $K > 0$, $\tau \in (0,1)$
5. Law of Motion: $H(K;\tau) = k(K,K;\tau)$ holds for all $K > 0$, $\tau \in (0,1)$

Assume $u(c) = \log c$ and $Y = F(K,L) = AK^{\alpha}L^{1-\alpha}$ and $L = 1$. The model economy is defined by the parameters $(\beta, A, \alpha, \delta)$.

1. Suppose that we observe this economy in steady state and (i) the investment-output ratio is $I/Y = 0.2$, (ii) the capital-output ratio is $K/Y = 3.0$, (iii) the payment to capital as a ratio to output is 0.3 and (iv) the tax rate is $\tau = 0.2$.

What model parameters $(\beta, A, \alpha, \delta)$ are consistent with this data? [Hint: A is not uniquely determined so normalize it so that $A = 1$]. Calculate steady state (Y^*, K^*, c^*) based on these parameter values.

2. Suppose that you want to predict how the economy will respond to a decrease in the tax rate on capital income from the initial value of $\tau = 0.2$ to a permanently lower value $\tau = 0.1$. This policy change occurs by surprise at time $t = 1$. It is understood that government spending will adjust to always equal the tax revenue produced by the new tax rate according to equilibrium condition 4.

It is clear from the definition of equilibrium that knowledge of $H(K,\tau)$ and the initial capital level K^* from problem 1 will be sufficient, given the new value of τ.

(a) Simulate $\{K_t, Y_t, W_t\}_{t=1}^{T}$ for $T = 100$, given $K_1 = K^*$. Do so by following steps 1-4 below. Roughly, these steps involve approximating H with a first order Taylor series approximation of the decision rule for consumption.

(b) Plot $\bar{H}(K,\tau)$ on a 45 degree line diagram for $\tau = 0.2$ and $\tau = 0.1$.

Step 1 Find $(\beta, A, \alpha, \delta)$ based on problem 1.

Step 2 Approximate $H(K,\tau)$ with $\bar{H}(K,\tau)$

$$\bar{H}(K,\tau) = W(K) + K[1 + (R(K) - \delta)(1-\tau)] - \bar{c}(K,\tau)$$

$$\bar{c}(K,\tau) = \hat{c}(K^*,\tau^*) + (\hat{c}_1(K^*,\tau^*), \hat{c}_2(K^*,\tau^*)) \cdot (K - K^*, \tau - \tau^*)$$

$$\hat{c}(K^*,\tau^*) = c^*$$

Step 3 Figure out $(\hat{c}_1(K^*,\tau^*), \hat{c}_2(K^*,\tau^*))$

Step 4 Simulate $\{K_t, Y_t, W_t\}_{t=1}^{T}$ using $\bar{H}(K,\tau)$ and $\tau = 0.1$.

Some notes to carry out Step 3 are contained on the next page!!

How to Carry Out Step 3:

1. An equilbrium decision rule $c(K, K, \tau)$ satisfies $(*)$:

$$(*) \quad u'(c(K,K,\tau)) - \beta u'(c(H(K,\tau), H(K,\tau), \tau))[1 + (R(H(K,\tau)) - \delta)(1-\tau)] = 0$$

$$H(K,\tau) = W(K) + K[1 + (R(K) - \delta)(1-\tau)] - c(K,K,\tau)$$

2. Rewrite $(*)$ using $\hat{c}(K,\tau) = c(K,K,\tau)$ so that we no longer distinguish between little k and big K

$$(**) \quad u'(\hat{c}(K,\tau)) - \beta u'(\hat{c}(H(K,\tau),\tau))[1 + (R(H(K,\tau)) - \delta)(1-\tau)] = 0$$

$$H(K,\tau) = W(K) + K[1 + (R(K) - \delta)(1-\tau)] - \hat{c}(K,\tau)$$

3. Differentiate $(**)$ with respect to (K,τ) and evaluate at (K^*, τ^*). Use compact notation:

$$(***) \quad u''\hat{c}_1 - \beta u''[1 + (F_1 - \delta)(1-\tau)]\hat{c}_1 H_1 - \beta u' F_{11} H_1 (1-\tau) = 0$$

$$u''\hat{c}_2 - \beta u''[1 + (F_1 - \delta)(1-\tau)](\hat{c}_1 H_2 + \hat{c}_2) - \beta u'[F_{11} H_2 (1-\tau) - (F_1 - \delta)] = 0$$

$$H_1 = F_{12} + [1 + (F_1 - \delta)(1-\tau)] - \hat{c}_1 + K F_{11}(1-\tau)$$

$$H_2 = -K(F_1 - \delta) - \hat{c}_2$$

4. Solve for $(\hat{c}_1(K^*,\tau^*), \hat{c}_2(K^*,\tau^*))$ in $(***)$. Note that these derivatives are the only unknowns in this system. This holds because the other variables in the system are known at the steady-state values (c^*, K^*, τ^*).