

Macro I

Homework 4- Consumption

1. A consumer lives for  $J$  periods and has preferences over consumption given by the utility function  $U(c_1, \dots, c_J) = \sum_{j=1}^J \beta^j u(c_j)$ . Suppose that the consumer can borrow and lend at a fixed interest rate  $r$  subject to the restriction that the consumer cannot have a negative asset position at death (i.e.  $a_{J+1} \geq 0$ ). The period budget constraint for an agent is  $c_j + a_{j+1} \leq a_j(1+r) + e_j$ .

(a) Formulate this problem as a dynamic programming problem. Be clear on states, controls and the form of Bellman's equation.

(b) For the special case where  $\beta = 1$  and  $r = 0$  calculate (i.e. write down) the optimal decision rules and value function each period. Indicate any assumptions on the period utility function that you need to justify your answer.

(c) Suppose that  $u(c) = c^{1-\sigma}/(1-\sigma)$  for  $\sigma \neq 1$  and  $u(c) = \log c$  when  $\sigma = 1$ . Characterize the determinants of the growth rate in consumption across periods.

[Translation: State and prove a Theorem of the form "Theorem: If  $\{c_j\}_{j=1}^J$  solves the decision problem and blah holds, then  $c_{j+1}/c_j = \text{blah}$ ". ]

2. A standard result is the equivalence between budget sets stated in terms of a present value condition or in terms of a sequence of budget restrictions. Assume that earnings are risky. The functions below are functions of earnings histories. For example,  $c_j : E^j \rightarrow R$  and  $a_{j+1} : E^j \rightarrow R$  where  $E^j = \{e^j = (e_1, e_2, \dots, e_j) : e_k \in E \text{ for } k = 1, \dots, j\}$  and  $E \subset R_+$ .

Prove that  $\Gamma_1(a_1) = \Gamma_2(a_1)$ .

$$\Gamma_1(a_1) = \{(c_1, \dots, c_J) : \exists \{a_j\}_{j=2}^{J+1} \text{ s.t. (1) holds } \forall j, \forall e^j \text{ and } a_{J+1} \geq 0\}$$

$$\Gamma_2(a_1) = \{(c_1, \dots, c_J) : (1') \text{ holds and } c_j(e^j) \geq 0, \forall j, \forall e^j \in E^j\}$$

$$(1) \quad c_j(e_1, \dots, e_j) + a_{j+1}(e_1, \dots, e_j) \leq a_j(e_1, \dots, e_{j-1})(1+r) + e_j, c_j(e_1, \dots, e_j) \geq 0$$

$$(1') \quad \sum_{j=1}^J \frac{c_j(e_1, \dots, e_j)}{(1+r)^{j-1}} \leq \sum_{j=1}^J \frac{e_j}{(1+r)^{j-1}} + a_1(1+r), \forall (e_1, \dots, e_J) \in E^J$$

3. Consider the problem below where earnings  $e_j$  are independently distributed according to a distribution function  $F_j$  in period  $j$ . How does the

consumption allocation that solves this problem change when a government lump-sum taxes the agent  $T_2$  in period 2 and makes a lump-sum transfer of  $T_2(1+r)^{J-2}$  to the agent in period  $J$ ? Explain carefully. Assume that  $T_2$  is less than the smallest endowment shock possible in period 2.

$$\max E[\sum_{j=1}^J \beta^{j-1} u(c_j)]$$

$$c_j + a_{j+1} \leq a_j(1+r) + e_j \text{ and } c_j \geq 0 \text{ and } a_{J+1} = 0$$

4. An agent maximizes  $E[\sum_{j=1}^J \beta^{j-1} u(c_j)]$  subject to  $c_j + a_{j+1} \leq a_j(1+r) + e_j$  and  $c_j, a_{j+1} \geq 0, \forall j$ .

Assume that  $u(c) = \log(c), \beta = 1.0, r = 0, J = 10$  and  $e_j$  equals 1, 2, 3 with equal probability each model period. Also assume that earnings are independently drawn each period, initial asset holding is zero (i.e.  $a_1 = 0$ ) and the initial earnings is  $e_1 = 2$ .

(a) Write a computer program to compute optimal decision rules to this problem. Include all your computer programs with your problem set.

(b) Use a random number generator to simulate 1000 histories of earnings shocks  $(e_1, \dots, e_J)$  over  $J = 10$  periods. Each history starts with  $e_1 = 2$ . For each history, calculate the realized asset holding profile. Then calculate the sample average of asset holding at each age across these 1000 histories. Plot the sample averages at each age and take this as an approximation of  $E[a_j]$  for  $j = 1, \dots, J$ .

(c) Comment on your approximation of  $E[a_j]$  in light of your answer to problem 1(b) and 1(c). Specifically, in the model for problem 1 what does theory predict for the asset profile when earnings are constant at  $e_j = 2$  each period and earnings are NOT random?

Hint:

1. Modify the code for the finite dynamic programming problem from homework 3. Allow for two state variables  $x = (a, e)$ . This implies using two DO LOOPS over state variable.

2. Compute the return function once at the beginning of the code. Represent it with an array  $U(NA, NE, NA)$  where  $NA$  and  $NE$  are the discrete number of asset levels allowed and the discrete number of earnings levels allowed. An element of the return function is  $u(a(1+r) + e - a')$ . Penalize choices  $a'$  that lead to negative consumption - the same trick as employed in homework 3. Represent the value function with an array  $V(NA, NE, NJ)$ , where  $NJ$  denotes the number of time periods.

3. A standard random number generator produces random numbers which are uniformly distributed on the interval  $[0, 1]$ . A draw of a  $J$  period history of iid random variables on  $[0, 1]$  can then be transformed into earnings shocks on the set  $E = \{1, 2, 3\}$  by a simple cutoff rule.