

Macro I

Homework 2- Properties of Value Functions

1. In consumer theory $V(p, y)$ and $x(p, y)$ denote the indirect utility function and the demand function.

$$V(p, y) \equiv \max_{x \in \Gamma(p, y)} u(x) \text{ where } \Gamma(p, y) = \{x \in R_+^n : px \leq y\}$$
$$x(p, y) \in \operatorname{argmax}_{x \in \Gamma(p, y)} u(x)$$

If u is strictly concave and continuous, then a standard result that follows from the Theorem of the Maximum is that $V(p, y)$ and $x(p, y)$ are continuous in (p, y) on $R_{++}^n \times R_{++}$. This relies in part on showing that the budget set is a continuous correspondence. In this problem you can assume that $V(p, y)$ is well defined over its domain. Answer the following questions:

(i) [Monotonicity of the Value function]

Prove that $V(p, y)$ is monotone increasing in y .

(ii) [Concavity of the Value function]

Prove that $V(p, y)$ is concave in y provided that the utility function $u(x)$ is increasing and concave in the commodity vector x .

(iii) [Differentiability of the Value Function]

Prove that $V(p, y)$ is differentiable in y . Be clear on any additional assumptions that are not stated in the problem that you use to prove this result. Also state what the derivative is equal to.

2. An agent maximizes expected utility $E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$. The agent receives a wage offer w in period $t = 0$. If the agent accepts this offer, the agent works forever at wage w receiving utility $\sum_{t=0}^{\infty} \beta^t u(w)$. If the agent rejects this offer, the agent receives period utility $u(0) = 0$ in period $t = 0$ and draws a wage offer from a distribution $F(w)$ next period. The decision problem repeats each period as long as the agent rejects the offer.

(i) Formulate this problem as a dynamic programming problem.

(ii) Does Bellman's equation define a contraction map? If so, then sketch the main steps of a proof. Otherwise explain why not.

(iii) Define the set of wage offers that the agent will accept.

(iv) Now assume that agents that reject offers are required to perform a service that model period that leads to some personal disutility. Specifically, agents who reject an offer now receive period utility equal to $u(0) = -1 < 0$ which is below the level $u(0) = 0$ previously associated with rejected offers. Compared to your answer in part (iii), what effect does this have on the set of wage offers that are accepted?

In your answer to part (iv), state your answer as a Theorem and then state all the main steps (i.e. key logical assertions) that prove it.

3. Consider the Bellman equation for the value of a firm that owns its capital and that takes prices $w, r > 0$ as given:

$$V(k) = \max_{(k', l)} F(k, l) - wl - [k' - (1 - \delta)k] + \frac{1}{1 + r}V(k') \quad s.t. \quad k', l \geq 0$$

Assume F is differentiable with $F_1 > 0$ and $F_{11} < 0$. Assume that there exists a concave function V solving Bellman's equation. Is V differentiable for $k > 0$? If so, what is the derivative? Explain your reasoning.

4. Consider the human capital problem of maximizing the present value of lifetime earnings by choosing time allocation over the lifetime. It is understood that (i) h denotes human capital, (ii) l and s are distinct uses of time, (iii) w and r are a wage rate and a real interest rate, (iv) whl are labor market earnings and (v) the agent lives from age $j = 1$ to age J .

Using the formulation of this problem in $P1$ or $P1'$, prove that $v_j(h)$ is concave in h for all $j = 1, \dots, J$. Sketch your proof in clear steps. Assume that $H(h, s) = h(1 - \delta) + a(hs)^\alpha$, $\delta \in (0, 1)$, $\alpha \in (0, 1)$, $a > 0$.

$$P1 \quad v_j(h) = \max_{(l, s)} whl + \frac{1}{1 + r}v_{j+1}(h') \quad s.t. \quad 0 \leq l + s \leq 1, \quad h' = H(h, s)$$

$$P1' \quad v_j(h) = \max_{h'} G(h, h') + \frac{1}{1 + r}v_{j+1}(h') \quad s.t. \quad h' \in [H(h, 0), H(h, 1)]$$

$$G(h, h') \equiv wh(1 - H(h, \cdot))^{-1}(h')$$