Macro I - Practice Problems - Growth Models

1. Consider the infinitely-lived agent version of the growth model with valued leisure. Suppose that the government uses proportional taxes (τ_c, τ_n, τ_k) on consumption, labor income and net capital income each period to fund a lump-sum transfer T each period. If there is a steady-state equilibrium with constant values $(\bar{c}, \bar{n}, \bar{k}, \bar{w}, \bar{r})$ achieved with tax rates (τ_c, τ_n, τ_k) and a constant lump-sum transfer $\bar{T}$, then are the same steady-state values $(\bar{c}, \bar{n}, \bar{k}, \bar{w}, \bar{r})$ also achieved under some other constant tax rates and constant transfers?

State your answer in theorem-proof format.

2. Consider the Diamond growth model without valued leisure. This is the overlapping generations growth model with (i) two-period lived agents, (ii) neoclassical production function $y = F(k, l)$, (iii) a constant rate of depreciation of physical capital δ, (iv) constant population growth rate n and (v) only young agents can work and they each have one unit of available time.

Assume that the preferences for consumption are of the form $U(c_y, c_o) = u(c_y) + \beta u(c_o)$ regardless of when an agent is born. Recall that the aggregate feasibility condition in steady state can be written as $c_y + c_o/(1+n) + k(n+\delta) = F(k, 1)$.

If the economy has a steady-state (c_y, c_o, k, w, r) with positive capital, then what restriction(s) must hold on the steady-state value k? Explain the key steps in your answer.

3. Consider the infinitely-lived agent growth model without valued leisure. Assume that there is one firm that owns its capital and that each period makes a labor decision and an investment decision. The agent makes a decision about consumption, share holdings in the firm and its holdings of one period risk-free assets. It is understood that the agent supplies 1 unit of labor time each period and receives a competitively priced reward for doing so.

(i) Define a recursive competitive equilibrium for this economy.

(ii) In a steady state of such an equilibrium, what is the value of all the shares of the firm? Explain your answer in steps. Use only the objects in your definition of equilibrium in your answer.

4. Consider problem 3 again but with two important differences. The first difference is that the government directly charges the agent a proportional tax τ on any dividends received each period. Thus, the owners of the firm get the after-tax dividend payment. It is understood that the government uses tax revenue to make a lump-sum transfer each model period. The second difference is that you are to employ a sequential definition of equilibrium in your analysis.

(i) Define a sequential competitive equilibrium for this economy.

(ii) In a steady state of such an equilibrium, what is the value of the shares of the firm as a function of the dividend tax rate τ? Explain your answer in clear steps.

Answer Q1:

The definition of equilibrium requires (i) optimization, (ii) competitive pricing, (iii) resource feasibility and (iv) government budget hold. Write down necessary implications for interior steady-state equilibria. The first two restrictions below come from optimization, whereas the last two come from feasibility and govt budget:

$$u_1(c, 1-n) = \beta u_1(c, 1-n)(1 + r(1-\tau_k))$$

$$\frac{u_2(c, 1-n)}{u_1(c, 1-n)} = w\frac{(1-\tau_n)}{(1+\tau_c)}$$

$$c + \delta k = F(k, n)$$

$$T = \tau_c c + \tau_k rk + \tau_n wn$$

We can conclude from the first two restrictions that (1) the capital income tax rate must be unaltered and (2) the ratio $\frac{(1-\tau_n)}{(1+\tau_c)}$ must be unaltered. Thus, we arrive at the following Conjecture (or Theorem if we can prove it):

Conjecture: If (c, n, k, w, r, T) is a steady state equilibrium under taxes (τ_c, τ_n, τ_k), then $(c, n, k, w, r, \hat{T})$ is a steady state equilibrium under $(\hat{\tau_c}, \hat{\tau_n}, \hat{\tau_k})$ provided:

1. $(1-\hat{\tau_n})/(1+\hat{\tau_c}) = (1-\tau_n)/(1+\tau_c)$
2. $\hat{\tau_k} = \tau_k$ and
3. $\hat{T} = \hat{\tau_c}c + \hat{\tau_k}rk + \hat{\tau_n}wn$

Proof: Sketch

To argue that $(c, n, k, w, r, \hat{T})$ is a steady state under tax rates $(\hat{\tau_c}, \hat{\tau_n}, \hat{\tau_k})$, one needs to show that optimization, competitive pricing, feasibility and the government budget holds. It is clear that the last three hold as allocations and prices are unchanged and the government budget holds by construction.

It remains to show that optimization holds. It would be sufficient to show that the budget set $\hat{\Gamma}$ under the new tax system and conjectured equilibrium prices is the same as the budget set Γ under the old tax system.

$$\Gamma = \{\{c_t, n_t\}_{t=0}^{\infty} : \exists\{k_t\},\ k_0 = k, n_t \in [0,1], \forall t \geq 0 \ and \ (1) \ holds \ \forall t \geq 0\}$$

$$(1) \ c_t(1+\tau_c) + k_{t+1} \leq wn_t(1-\tau_n) + k_t(1 + r(1-\tau_k)) + T$$

$$\hat{\Gamma} = \{\{c_t, n_t\}_{t=0}^{\infty} : \exists \{k_t\},\ k_0 = k, n_t \in [0,1], \forall t \geq 0 \ and \ (1) \ holds \ \forall t \geq 0\}$$

$$(1)\ c_t(1+\hat{\tau}_c) + k_{t+1} \leq wn_t(1-\hat{\tau}_n) + k_t(1+r(1-\hat{\tau}_k)) + \hat{T}$$

Alternatively, it would be sufficient to show that the proposed consumption-labor plan is in the budget set under the new tax plan and that any budget-feasible deviation from the proposed plan under new taxes would decrease utility.

Showing that either of these sufficient conditions hold is above the level of these notes. Nevertheless, one possible strategy for showing that the budget sets are the same could be based on showing that Claim 1-5 hold. To prove such claims would involve placing more structure on the asset sequences $\{k_t\}_{t=0}^{\infty}$ than we have pursued so far in this course.

Claim 1: $\Gamma = \{\{c_t, n_t\}_{t=0}^{\infty} : \sum_{t=0}^{\infty}(\frac{1}{1+r(1-\tau_k)})^t[w_t n_t(\frac{1-\tau_n}{1+\tau_c}) - c_t + \frac{T}{1+\tau_c}] + k(\frac{1+r(1-\tau_k)}{1+\tau_c}) \geq 0\}$.

Claim 2: $\hat{\Gamma} = \{\{c_t, n_t\}_{t=0}^{\infty} : \sum_{t=0}^{\infty}(\frac{1}{1+r(1-\tau_k)})^t[w_t n_t(\frac{1-\hat{\tau}_n}{1+\hat{\tau}_c}) - c_t + \frac{\hat{T}}{1+\hat{\tau}_c}] + k(\frac{1+r(1-\tau_k)}{1+\hat{\tau}_c}) \geq 0\}$.

Claim 3: $\sum_{t=0}^{\infty}(\frac{1}{1+r(1-\tau_k)})^t \frac{T}{1+\tau_c} + k(\frac{1+r(1-\tau_k)}{1+\tau_c}) = \sum_{t=0}^{\infty}(\frac{1}{1+r(1-\tau_k)})^t \frac{\hat{T}}{1+\tau_c} + k(\frac{1+r(1-\tau_k)}{1+\hat{\tau}_c})$

Claim 4: $\frac{T(\frac{1+r(1-\tau_k)}{r(1-\tau_k)})+k(1+r(1-\tau_k))}{1+\tau_c} = \frac{\hat{T}(\frac{1+r(1-\tau_k)}{r(1-\tau_k)})+k(1+r(1-\tau_k))}{1+\hat{\tau}_c}$

Claim 5: $(1+\hat{\tau}_c)T + \hat{\tau}_c kr(1-\tau_k)) = (1+\tau_c)\hat{T} + \tau_c kr(1-\tau_k))$.

Answer Q2:

If the economy is in steady state, then the four restrictions below hold. The first restriction holds if one assumes that $u'(0) = \infty$.

$$u'(c_y) = \beta u'(c_o)(1 + F_1(k,1) - \delta)$$

$$c_y + c_o/(1+n) + k(n+\delta) = F(k,1)$$

$$c_y = F_2(k,1) - k(1+n) \textit{ and } c_o = k(1+n)(1 + F_1(k,1) - \delta)$$

These restrictions reduce to a restriction on the capital labor ratio k:

$$(*)\ u'(F_2(k,1) - k(1+n)) = \beta u'(k(1+n)(1 + F_1(k,1) - \delta))(1 + F_1(k,1) - \delta)$$

Thus, we have the following Claim:

Claim: Assume $u'(0) = \infty$. *In a steady state with* $k > 0$ *restriction (*) holds.*

One could try to make more progress on characterizing solutions to this last restriction by making some additional restrictions on technology. One possibility is to assume that $kF_1(k,1)$ increases in k. This assumption implies that the right-hand-side of the last restricition decreases as k increases. However, this assumption does not seem to be strong enough to produce unique steady states, absent further restrictions on the technology, as the left-hand side of $(*)$ is not guaranteed to be increasing in k.

Answer Q3:

Let $x = (K, a, s)$ denote an agent's state variable, consisting of aggregate capital K, assets a and stock holding s. Let (k, K) be state variables for the firm and let $p(K)$ denote the ex-dividend price of the firm. Q3 asks what is $p(K)$ at the steady state value K^* (i.e. at $K^* = G(K^*)$).

Definition: A recursive equilibrium is $(c(x), a(x), s(x), v(x))$, $(n(k, K), g(k, K), z(k, K))$ and $(w(K), r(K), p(K), G(K))$ such that

1. (c, a, s, v) solve BE-1.
2. (n, g, z) solve BE-2.
3. (i) $c(K, 0, 1) + g(K, K) - K(1 - \delta) = F(K, 1)$
   (ii) $s(K, 0, 1) = 1$
   (iii) $a(K, 0, 1) = 0$,
   (iv) $n(K, K) = 1$
4. $G(K) = g(K, K)$

Below we state Bellman's equation for both problems:

(BE-1)
$v(K, a, s) = \max u(c) + \beta v(K', a', s')$ *s.t.*
$c + a' + p(K)s' \leq w(K) + a(1 + r(K)) + s[p(K) + F(K, 1) - w(K) - (G(K) - K(1 - \delta))]$ and $K' = G(K)$

(BE-2)
$z(k, K) = \max_{k',n} F(k, n) - w(K)n - (k' - k(1 - \delta)) + \frac{1}{1+r(K')} z(k', K')$
given $K' = G(K)$

BE-1 implies that the first equation below holds. The second equation below then holds in steady-state equilibrium, where steady state capital is the capital value K^* such that $K^* = G(K^*)$. The third equation puts a restriction on $r(K)$ at steady state K^*. BE-1 also resticts the share price $p(K)$ at the steady state value of K^* as indicated by the fifth equation below.

$$u'(c(K, a, s)) = \beta v_2(K', a', s') = \beta u'(c(K', a', s'))(1 + r(K'))$$

$$u'(c(K^*, 0, 1)) = \beta u'(c(K^*, 0, 1))(1 + r(K^*))$$

$$1 = \beta(1 + r(K^*)) \Rightarrow r(K^*) = 1/\beta - 1$$

$$p(K) = \frac{\beta u'(c(K', a', s'))}{u'(c(K, a, s))}(p(G(K)) + F(G(K), 1) - W(G(K)) - [G(G(K)) - G(K)(1-\delta)])$$

$$p(K^*) = \beta(\frac{F(K^*, 1) - W(K^*) - K^*\delta}{1 - \beta})$$

BE-2 has the two first-order necessary conditions below. The first necessary condition implies a restriction on the interest rate function. The second condition restricts the wage function $w(K)$.

$$1 = \frac{1}{1 + r(K')} z_1(k', K') \ and \ F_2(K, 1) = W(K)$$

$$1 = \frac{1}{1 + r(K)}(1 + F_1(K, 1) - \delta) \Rightarrow r(K) = F_1(K, 1) - \delta$$

Combining the results above gives the answer. The value of the firm is its steady-state capital level K^* and K^* is pinned down by the second equation:

$$p(K^*) = \beta(\frac{F(K^*, 1) - W(K^*) - K^*\delta}{1 - \beta}) = \beta\frac{(F_1(K^*, 1) - \delta)K^*}{1 - \beta} = \beta\frac{r(K^*)K^*}{1 - \beta} = \beta\frac{(1/\beta - 1)K^*}{1 - \beta} = K^*$$

$$1 = \beta(1 + F_1(K^*, 1) - \delta)$$

Answer Q4:

Definition: An equilibrium is $\{c_t, s_t, b_t, k_t, l_t, p_t, w_t, r_t, T_t\}_{t=0}^{\infty}$ st

1. $\{c_t, l_t s_t, b_t\}$ solves P1.
2. $\{k_t, l_t\}$ solves P2.
3. $c_t + k_{t+1} = F(k_t, l_t) + k_t(1-\delta)$ and $b_t = 0$ and $s_t = 1$
4. $T_t = \tau[F(k_t, l_t) - w_t l_t - [k_{t+1} - k_t(1-\delta)]]$

P1 $\max \sum_{t=0}^{\infty} \beta^t u(c_t)$ *st*
$c_t + p_t s_{t+1} + b_{t+1} \leq w_t + s_t(p_t + d_t(1-\tau)) + b_t(1+r_t) + T_t$ and $s_t, b_{t+1} \geq 0$
$d_t = F(k_t, l_t) - w_t l_t - [k_{t+1} - k_t(1-\delta)]$ and $l_t \in [0,1]$

P2 $\max[d_0 + \sum_{t=1}^{\infty} \prod_{j=1}^{t} (\frac{1}{1+r_j})(d_j)]$ st
$d_t = (F(k_t, l_t) - w_t l_t - [k_{t+1} - k_t(1-\delta)]), l_t \geq 0, k_t \geq 0, \forall t \geq 1$ given k_0.

Analysis:

An interior solution to P1 implies 1-2 and steady state implies 3:

1. $u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1})$
2. $u'(c_t)p_t = \beta u'(c_{t+1})(p_{t+1} + d_{t+1}(1-\tau))$
3. $1 = \beta(1+r)$ *and* $p = \beta(p + d(1-\tau))$

The following three equalities then hold. The first restates the restriction above. The second substitutes for dividends. The third uses the necessary condition for a firm's best labor choice ($w = F_2(k,l)$) and constant returns.

$$p = \frac{\beta d(1-\tau)}{1-\beta} = \frac{\beta(F(k,1) - wl - k\delta)(1-\tau)}{1-\beta} = \frac{\beta k(F_1(k,1) - \delta)(1-\tau)}{1-\beta}$$

The first equation below uses the necessary condition for firm maximization (i.e. $1 + r = 1 + F_1(k,1) - \delta$ and consumer maximization $1 = \beta(1+r)$.

$$p = \frac{\beta k(F_1(k,1) - \delta)(1-\tau)}{1-\beta} = \frac{\beta k r(1-\tau)}{1-\beta} = k(1-\tau)$$

$$1 = \beta(1+r) = \beta(1 + F_1(k,1) - \delta)$$

$$k = F_1(\cdot, 1)^{-1}[1/\beta - 1 + \delta]$$

We conclude that (1) the steady state value of the firm is proportional to $(1-\tau)$ and to the steady state capital level k and that (2) the steady state capital level k is invariant to the tax rate τ. Note that in this problem the tax rate directly taxes dividends but not the returns to the alternative risk-free asset.