Optimal Collateralized Contracts*

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August 20, 2018

Abstract

We examine the role of collateral in a dynamic model of optimal credit contracts in which a borrower values both housing and non-housing consumption. The borrower's private information about his income is the only friction. An optimal contract is collateralized when in some state, some portion of the borrower's net worth is forfeited to the lender. We show that optimal contracts are always collateralized. The total value of forfeited assets is decreasing in income, highlighting the role collateral as a deterrent to manipulation. Some assets, those that generate consumable services will necessarily be collateralized while others may not be. Endogenous default arises when the borrower's initial wealth is low, as with subprime borrowers, and/or his future earnings are highly variable.

JEL Codes: D86, D14, G21

Key Words and Phrases: Optimal contract, asymmetric information, collateral, forfeiture, collateralized contract, full recourse.

*We thank Luca Anderlini, Axel Anderson, Douglas Gale, Pedro Gete, Johannes Horner, Billy Jack, Arik Levinson, Jim Peck, Debraj Ray, John Rust, Florian Scheuer, Ricardo Serrano-Padial, Huanxing Yang, Lixin Ye, Ivan Werning, John Weymark, an anonymous referee, and numerous seminar participants for useful comments and suggestions.

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1 Introduction

Collateralized credit contracts are commonplace. They arise in durable goods purchases, consumption loans, repurchase agreements, and many other types of credit transactions. The value of residential mortgages alone amounted to over $14 trillion in the U.S. at the end of 2016.[1]

At a basic level, the rationale for using collateral is unclear. In a frictionless world, an optimal contract between a risk neutral lender and a risk averse borrower with an uncertain income stream will have the lender offer a loan and repayment schedule that fully insures the borrower. By contrast, a contract with collateral transfers an asset from borrower to lender in the uncertain event the loan is not repaid. Collateralized contracts thus indemnify the lender — the wrong party according to the baseline theory.

Explanations of collateral therefore incorporate a particular friction, or set of frictions. These include Stiglitz and Weiss (1981), Lacker (2001), Rampini (2005), Geanakoplos (2010), Cao, Lorenzoni, and Walentin (2016), Rampini and Viswanathan (2013), and Gorton and Ordonez (2014).

A standard feature of these models is that they give rise to "fixed default" contracts in which the borrower's consumption of the collateralized asset is constant across all incomes below a threshold and/or all incomes above a (possibly different) threshold. A simple example is the two-step semi-pooling contract in asset forfeiture as displayed in Figure 1. Below the threshold, the borrower is held to be in default. In some of the models, the fixed default structure is ad hoc.[2] In other cases, the fixed default contract results from institutional frictions such as limited liability or legal limits on contractual contingency. Indeed, if these institutional frictions are largely responsible for the use of collateral, then fixed default would appear to be an inherent characteristic of such contracts.

This "fixed default" narrative applies to many common *de jure* contracts. The *de jure*, or legally recognized, contract in many jurisdictions entails the full loss of a "collateralized asset" (a house) if the borrower cannot make good on repayment. The *de jure* contract aptly describes non-recourse loans where forfeiture claims are legally restricted to the

[1]Board of Governors, Federal Reserve System, Release Date: December 2016.

[2]The fixed default narrative is common in the large literature that embeds collateral exogenously as a means to explain other features of credit markets. It is assumed for instance in explanations of credit rationing (Stiglitz and Weiss (1981), Bester (1985)), bank screening (Besanko and Thakor (1987), Manove et al. (2001)), and speculation and financial crises (Geanakoplos (2010), Simsek (2013), Cao and Nie (2017), Cao (2018)).

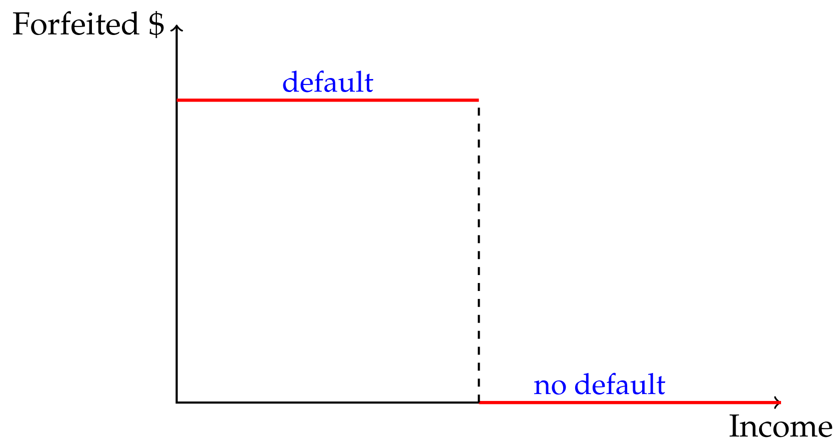

Figure 1. Semi-pooling value of forfeited assets

particular asset on which the loan was based.

Nevertheless, the fixed default contract is often an inaccurate description of the *de facto* credit arrangement. Empirical studies by Agarwal et al. (2011) and Adelino et al. (2013) show that forfeiture is a relatively smooth function of realized income due to loan modifications. Agarwal et al. describe several common options that resemble partial defaults, including short sale and deed-in-lieu.[3] The conclusions of these studies are reinforced by the widely reported incidence of side agreements that allow for varying degrees of partial default.[4]

Moreover, most states in the U.S. allow some form of recourse, i.e., allow seizure of other forms of assets and income beyond the purchased asset. Europe does as well. Ghent and Kudlyak (2011) provide evidence showing that even when recourse is not exercised by lenders, its availability significantly alters borrowers' incentives to default and therefore alters the terms of the implicit contract.

Finally, Adelino et al. (2013) make the case that asymmetric information about the borrower's ability to repay is the key friction underlying the *de facto* mortgage contract, more so than institutional factors. Using LPS data,[5] they show that there is little or no difference in loan modification rates among securitized and non-securitized mortgages

[3]See Figure 1 in their paper.

[4]See, for instance, a 2012 CNN report on home sales by delinquent borrowers: *http://money.cnn.com/2012/02/10/real_estate/short_sale_incentives/*.

[5]The Lender Processing Services (LPS) data consists of individual mortgage loans that, according to Adelino et al., "contains detailed information on the characteristics of both purchase-money mortgages and mortgages used to refinance existing debt."

(i.e. loans held by the lender) whereas institutional frictions arising from securitization would imply otherwise.

Given problems with the "fixed default" model, we take a fresh look at the foundations of collateralized contracts. We present a dynamic optimal contracting model in which collateral arises endogenously from an informational friction of the type emphasized by Adelino et al. (2013). Asset forfeiture is shown to vary continuously over the borrower's income stream, and the contract itself differs systematically across ex ante different kinds borrowers. The structure of a contract offered, say, to subprime borrowers, i.e., those with low levels of wealth or in unstable occupations, will largely differ from those offered to prime borrowers. The resulting contract also looks quite different from the fixed default model, and this suggests that an important mechanism driving the dispersion in wealth, credit availability, and consumption may have been missing.

The model consists of a dynamic contracting scenario with two agents, a risk neutral lender and a risk averse borrower. In each period, the borrower values a composite consumption good and an additional capital good that generates use value for the borrower. A household's consumption of non-housing and housing services is the prime example and is used as the concrete setting for the model.

The borrower receives a random income/return in consumption units. The housing good is non-stochastic. The realization of consumption income is determined from a continuous distribution; low realizations correspond to a negative shocks such as job loss or poor returns on investments. Consequently, the borrower seeks a loan that allows him to upgrade his housing good consumption and smooth his total consumption across time and across his uncertain income stream. The borrower, however, has private information about his income shock.

An *optimal contract* maximizes the borrower's expected payoff subject to period-by-period incentive constraints and a 0-profit constraint of the lender. We look for an optimal contract across a large range of possibilities, including those contingent on arbitrary combinations of assets, and on any dollar value of those assets. No consumption bounds or legal limitations, e.g., limited liability, limited enforcement, or limited contingency, are presumed. A contract is *collateralized* if the overall value of the borrower's consumption and savings falls short of his net worth. In that case some of the borrower's assets are forfeited to the lender in some state of nature.

Our characterization of the optimal contract yields three main findings. First, optimal contracts in our model are always collateralized. By contrast, under complete information or under some forms of market incompleteness, they are not. We show that the total value of all forfeited assets is decreasing in realized income. This is displayed in Figure 2a. The result highlights the role of collateral in reversing the natural flow of indemnification. This is necessary to deter manipulation of information or strategic default by the borrower. Some portion of the borrower's net worth will be seized by the lender in precisely those states of the world in which the borrower wants to be indemnified.

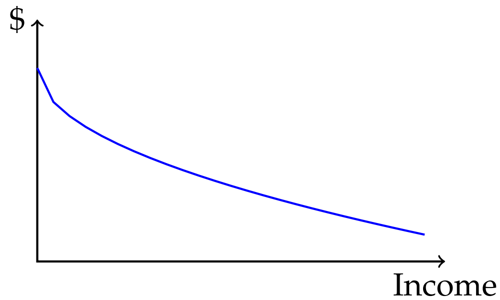

Figure 2a: Decreasing value of all forfeited assets

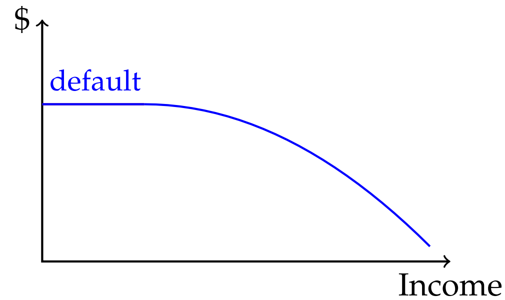

Figure 2b: Semi-pooling forfeiture in housing at low income levels

Second, assets that directly generate consumable services, independent of their inter-temporal properties, will necessarily be collateralized while assets used only for inter-temporal substitution may or may not be collateralized. Thus the model provides a rationale for why certain assets are better suited for collateral than others. The idea is that while inter-temporal substitution can provide some partial insurance, a consumable asset provides a critical *intra-temporal margin* with which to manage the insurance-incentive trade off. Forfeiture of all types of assets can thus occur in the optimal contract. Our results are roughly in line with Ghent and Kudlyak (2011) empirical study of recourse loans, and the present model can be interpreted as one micro-foundation for the types of loan modifications documented in the literature.

Third, while it may not be surprising that collateral can be a part of an optimal contract with frictions, the form it takes can differ widely across different types of borrowers. We describe natural conditions under which semi-pooling in the repayment schedule and in the consumption of the housing asset occurs at the bottom of the income distribution. Specifically, if the borrower's initial wealth is large and/or the dispersion of the income shock is low, then repayment and housing consumption both vary monotonically across the entire distribution. By contrast, semi-pooling at the low end will occur endogenously whenever the borrower's initial wealth is low enough at the time he enters the contract, and/or the variance in distribution of income shocks is large enough.

A natural interpretation is that semi-pooling occurs for subprime borrowers — those with low wealth and/or unstable employment prospects. The interpretation consistent with Adelino et al. who show that self-cure rates (delinquent borrowers that eventually find a way to repay) fell dramatically in 2008-09 recession. Since repayment in a self cure must, by definition, vary continuously with the borrower's realized income, our theory predicts that self cures should decline when borrowers' wealth levels fall sufficiently.

All in all, when observations of forfeiture are limited to the housing asset, the existence of semi-pooling in the optimal contract is consistent with the "fixed default" narrative. However, under the broader "full recourse" definition of forfeiture the fixed-default narrative fails: the total value of everything seized by the lender declines monotonically even if semi-pooling in the separate contractual components occurs. Thus, semi-pooling in an optimal contract is roughly consistent with both the *de jure* formulation and the *de facto* evidence of continuously varying forfeiture (compare Fig 2b with Fig 2a).

Our results have potentially important implications on new designs of mortgage contracts. For example, in recent influential work, Mian and Sufi (2014) argue for equity-like mortgage contracts (Share Responsible Mortgages) which prevents default and foreclosures. However, our third result suggests that, because of adverse selection, these contracts are either not optimal (in terms of risk-sharing) or not incentive compatible, especially in recessions when the dispersion of income increases significantly.[6] The optimal contracts in these cases call for a pooling interval in the bottom of the income distribution - a debt-like feature.[7]

The next section, Section 2, introduces a baseline model where collateral arises as a solution to an optimal contracting problem. In our framework, collateralization refers to the potential forfeiture of all types of assets (as in full recourse loans). However, collateralization in the narrower, non-recourse, sense is also explored. Section 3 describes the main results, and then distinguishes between those environments in which semi-pooling and fully separating contracts arise using phase diagrams. Section 4 provides a complete characterization of the optimal contracts in a class of parametric models. Section 5 examines related models, in particular Lacker (2001) and Rampini (2005), in greater detail. We compare our definition to the literature and examine how key differences in modeling

[6] Mayer et al. (2014) find direct evidence for consumers' strategic defaults when they are offered simple loan modification programs that do not take into account consumers' potential manipulative behaviors.

[7] Other implications are discussed in Section 5.

assumptions inform the various definitions and results. The proofs are in the Appendix.

2 A Model of Collateralized Contracts

2.1 Basic Elements

We lay out a simple T-period contracting problem (with $T \leq \infty$) with two agents, a risk neutral lender and a risk averse borrower (a household). In each period, the borrower values a consumption good c_t which may be a composite of many types of goods, and a capital good k_t. The capital good k_t is a service-generating durable good (e.g., housing, cars, appliances). A borrower with low initial wealth entering the period desires a loan to smooth consumption over time and/or fund durable good purchases.

For concreteness, from here on we will refer to the capital good as "housing," though it should be emphasized that it can be any durable good that produces consumable services for the borrower. For simplicity we rule out depreciation as it plays no role in the analysis.

The borrower's flow payoff is $U_t(c_t, k_t) = u_t(c_t) + v_t(k_t)$ where u_t and v_t are strictly concave, increasing and twice continuously differentiable.[8] His initial wealth is A_0 . There are no ad hoc bounds on either c or k. Lower bounds may be implied for certain payoff functions, for instance when U is the log function.

At beginning of t, the borrower realizes an random income shock θ_t to his consumption income. The shocks are assumed to be i.i.d and distributed according to F with support $[\underline{\theta}, \overline{\theta}]$ and mean $\Theta = \int \theta dF(\theta)$. F is common knowledge. Significantly, the realized income shock θ_t at each date t is privately observed by borrower.

The spot price q of housing is fixed and exogenous as we focus only on the internal credit arrangement.[9] The consumption good spot price at each date is normalized to one. For its part, the lender is risk neutral and belongs to a perfectly competitive set of intermediaries, all of whom offer loans with a market return of R.

[8]When $T = 1$ or $T = 2$, i.e., a static or a 2-period contracting problem, our results are valid for general non-separable utility functions with c, k being normal goods. Indeed, Appendix B presents the analyses for general utility functions.

[9]An earlier version of the paper allowed for time-varying housing prices. It is straightforward to allow for stochastically varying prices as well.

2.2 A Contracting Problem

At the beginning of a period the borrower obtains is offered a contract and obtains a loan and before the realization of the income shock. At the end of the period, state dependent consumption and savings occur.

The optimal contracting problem has two critical features. First, the borrower has private information on his income. Second, we assume that assets held by the borrower, including housing, are observable to the lender while the borrower's non-housing consumption is not observable. Given the unobservability of non-housing consumption, the lender cannot back out the information on the borrower's income. Therefore, the contract is incentive-constrained.

The contracting problem is described recursively as follows. The value of all assets net of incurred debt held by the borrower entering date t is given by $A_t \equiv qk_{t-1} + B_t$ where B_t is the borrower's liquid savings (or debt if $B_t < 0$) entering the period, and qk_{t-1} is the value of housing services.

A loan contract or simply a "contract" maps from the borrower's position A_t at each date to a contracted loan/repayment, housing consumption, and savings contingent on his realized income θ_t. Formally, a *contract* between the lender and borrower consists of a contingent sequence

$$(L^*, y^*, k^*, B^*) \;=\; \left\{L_t^*(A_t),\; \left(y_t^*(\theta_t, A_t), k_t^*(\theta_t, A_t), B_{t+1}^*(\theta_t, A_t)\right)\right\}$$

consisting of the following: L_t^* is the value of a new loan, excluding the rollover of existing debt, at the beginning of t.[10] After the loan, y^* is a repayment schedule in which $y_t^*(\theta_t, A_t) < 0$ is a repayment to the lender and $y_t^*(\theta_t, A_t) > 0$ represents loan forgiveness to some degree. The borrower then consumes $c_t^*(\theta_t, A_t) = \theta_t + y_t^*(\theta_t, A_t)$ units of the consumption good. $k_t^*(\theta_t, A_t)$ is the amount of housing consumed in that period, and so the borrower's payoff is $U_t = u_t(c_t^*(\theta_t, A_t)) + v_t(k_t^*(\theta_t, A_t))$. Finally, the contract specifies $B_{t+1}^*(\theta_t, A_t)$ units of savings or debt for the borrower entering next period. Hence, the borrower's wealth entering the next period is

$$A_{t+1}^*(\theta_t, A_t) = B_{t+1}^*(\theta_t, A_t) + qk_t^*(\theta_t, A_t). \tag{1}$$

[10] In a complete and competitive credit market, the borrower can borrow $L_t^*(A_t)$ up to the full value of the present value of his future expected income Θ.

All relevant features of the interaction between lender and borrower are built into the contract, including the borrower's consumption and savings, and "loan forgiveness".[11] Note that A_t may be negative in which case it represents the rollover of existing debt.

Taking the perspective of the borrower, we characterize the optimal contract taking perfect competition in the lending sector as given.

A contract (y^*, k^*, B^*) is *optimal* if it satisfies at each date t and for each wealth A_t,

$$V_t\left(A_t\right) = \max_{y_t^*, k_t^*, B_{t+1}^*} \int_{\underline{\theta}}^{\bar{\theta}} \left[U_t\left(c_t^*(\theta_t, A_t), k_t^*(\theta_t, A_t)\right) + \beta V_{t+1}\left(A_{t+1}^*(\theta_t, A_t)\right)\right] dF_t(\theta_t) \quad (2)$$

subject to

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(c_t^*(\theta_t, A_t) + q k_t^*(\theta_t, A_t) + \frac{1}{R} B_{t+1}^*(\theta_t, A_t)\right) dF_t(\theta_t) \ \leq \ A_t + L_t^*(A_t) \quad (3)$$

and

$$\begin{aligned} &U_t\left(\theta_t + y_t^*(\theta_t, A_t), k_t^*(\theta_t, A_t)\right) \ + \ \beta V_{t+1}\left(A_{t+1}^*(\theta_t, A_t)\right) \ \geq \\ &U_t\left(\theta_t + y_t^*(\tilde{\theta}_t, A_t), k^*(\tilde{\theta}_t, A_t)\right) \ + \ \beta V_{t+1}\left(A_{t+1}^*(\tilde{\theta}_t, A_t)\right) \quad \forall\, A_t,\ \theta_t, \tilde{\theta}_t, \end{aligned} \quad (4)$$

where A_{t+1}^* is given by (1).

The problem above is a dynamic Bayesian mechanism design problem. Equation (4) is the incentive constraint. Equation (3) is the resource constraint in which $R > 0$ is the risk-free interest rate. In order to guarantee that the objective function is well-defined, we require that the user-cost of capital is strictly positive: $\hat{q} = q(1 - \frac{1}{R})$. The resource constraint in equation (3) can also be interpreted as a zero profit or participation constraint for a competitive lender.[12] As in Green (1987), we can show that this recursive definition of an optimal contract is also optimal ex ante, and so there is no loss of generality in framing the problem this way.

[11] However, consumption c_t is not directly observable to the lender; only y_t, k_t, and B_{t+1} are. By the Revelation Principle, the optimal contract induces truth-telling, therefore the borrower reports her private information θ_t and receives the designed consumption $\theta_t + y_t(\theta_t)$. As in the standard Mirleesian optimal nonlinear taxation literature, the optimal contract can also be offered as a nonlinear repayment schedule $\{\hat{y}_t(k, B)\}$. Therefore, the borrower self-selects into the allocation assigned to his type and the lender does not have to "force" him to choose his allocation.

[12] In this interpretation, the borrower's net worth $A_t + L_t^*(A_t)$ is held by the lender in deposit, thus constituting the lender's revenue. The borrower's consumption and savings are the contractual outflow, hence cost, to the lender.

2.3 Collateral

A workable definition of collateral depends on whether and to what degree all the borrower's assets are subject to forfeiture. Broadly speaking, a borrower forfeits something when the *realized* value of what he spends or saves falls short of his net worth.

2.3.1 Definitions

In the model, two notions of collateralization can be defined, depending whether one has in mind the *de facto* full recourse contractual environment, or a *de jure* non-recourse environment. In the full recourse world all assets are subject to forfeiture. Under full recourse, the amount forfeited depends on the realized state. The *value of assets forfeited by the borrower* (and thus seized by the lender) with asset value A_t and realized income θ_t is

$$\Gamma_t(\theta_t, A_t) \equiv \underbrace{A_t + L_t^*(A_t)}_{\text{net worth at } t} - \underbrace{\left(qk_t^*(\theta_t, A_t) + c_t^*(\theta_t, A_t) + \frac{1}{R}B_{t+1}^*(\theta_t, A_t)\right)}_{\text{realized value of expenditures and net savings at } t} \tag{5}$$

Recall that A_t is the value of all assets net of incurred debt at the beginning of t, and $L_t^*(A_t)$ is the new loan given to the borrower at the beginning of the period. With asymmetric information as the only friction in an otherwise complete markets setting, the borrower's *ex ante* net worth is the relevant sum from which assets are seized.[13] Forfeiture represents a wedge between (i) the borrower's net worth after receiving the loan and (ii) his realized consumption and net savings (savings leftover after any loan repayment) at the end of the period.

Notice also that forfeiture can be positive or negative.[14] In states of nature where forfeiture is negative, the value of the realized consumption exceeds borrower's initial net worth. This can happen for a variety of reasons including, for instance, a higher-than-expected income realization.

[13] In an incomplete market setting, it may be that the relevant measure of net worth is the borrower's *ex post* net worth after the realization of the shock. An example is provided later on in the section.

[14] This must be so since $\int \Gamma(\theta, A)dF = 0$ when the resource constraint is binding.

Definition. A contract is *collateralized* at date t if

$$\max_{\theta_t \in [\underline{\theta}, \overline{\theta}]} \Gamma_t(\theta_t, A_t) \; - \; \min_{\theta_t \in [\underline{\theta}, \overline{\theta}]} \Gamma_t(\theta_t, A_t) \; > \; 0 \tag{6}$$

A contract can be regarded as "collateralized" if the largest possible forfeiture exceeds the smallest. In this definition all forms of assets can be potentially collateralized and forfeited. Again, since Γ_t defines forfeiture of all assets, the definition applies to the full recourse world.

In contrast with the above definition, one can restrict attention to the de jure non-recourse contract in which only the housing asset is subject to forfeiture.

Definition. A contract is *individually collateralized in asset k* at date t if (6) applies to the value of k forfeited by the borrower. This value is given by

$$\Gamma_t^k(\theta_t, A_t) \; = \; q \left(k_{t-1} \; - \; k_t^*(\theta_t, A_t) \right) \tag{7}$$

Individual collateralization restricts attention to the wedge in the housing asset k. It excludes initial savings or debt, non-housing consumption, and future savings or debt. Equation (7) is often regarded as the canonical, *de jure* form of contractual forfeiture.[15]

We emphasize again that by limiting attention to individual assets is to overlook some important features of the contract. Even when the de jure legal environment is non-recourse, "backdoor" methods exist for lenders to claim other assets. An emergent literature documents the myriad of ways that credit contracts effectively allow the lender full recourse regardless of legal formalisms (Agarwal et al. (2011) and Adelino et al. (2013) Ghent and Kudlyak (2011)). In terms of the model, this means that both goods can be purchased with debt and so the notion that the loan is only secured by the "asset purchased from the loan" is inaccurate. Hence, it is difficult to determine precisely how binding a no-recourse constraint might be, or whether say Γ_t^k really captures the effective amount actually forfeited by the borrower. For these reasons, we favor the full recourse definition (5) as a closer approximation of the *de facto* credit arrangement.

[15] One can apply a similar definition to liquid savings, although non-recourse contracts applied to liquid assets are uncommon.

2.3.2 Broader Discussion of the Definitions

Collateralization is not automatic. It does not occur in the complete information benchmark. To see why, observe that in the absence of the IC constraint, the solution to (2) corresponds to a *full information optimal contract* (y^{FI}, k^{FI}, B^{FI}). One can easily verify that under full information, the contract provides provides full insurance. That is,

$$\begin{aligned} y_t^{FI}(\theta_t, A_t) &= \bar{c}_t - \theta \\ k_t^{FI}(\theta_t, A_t) &= \bar{k}_t \\ A_t^{FI}(\theta_t, A_{t-1}) &= \sum_{\tau=t}^{\infty} (\frac{1}{R})^{\tau-t} \left[\Theta - \bar{c}_\tau + q(1 - \frac{1}{R})\bar{k}_\tau \right] + A_{t-1}^{FI} \end{aligned} \tag{8}$$

for optimal values of $\bar{c}_t$ and $\bar{k}_t$.[16] The contract fully insures the borrower in the consumption good c within each period. The market also provides certainty equivalent wealth available for consumption and savings/borrowing.

It follows that $\Gamma(\theta_t, A_t)$ is independent of realized income θ_t, and so $\Gamma(\theta_t, A_t) = 0$ for all θ_t. In other words, the value of seized collateral for every type θ_t is zero. No collateral is pledged. None is seized.

Under full information the housing good is not needed to achieve full insurance in (8). The contractual repayment function y_t^* suffices to fully insure the borrower. The function y_t^* serves much the same purpose as an Arrow security that moves income to different states of the world. Spot markets then determine that values of k_t^* and B_{t+1}^* following the realization of θ_t.[17]

We make three final observations before concluding the Section. First, if housing were absent from the model, liquid assets may be forfeited in certain states of the world. Influential models by Townsend (1982), Green (1987), and Thomas and Worrall (1990) show this in models without housing (and with finitely many types). Though these papers do not explicitly refer to the borrower's contingent balances as "collateral", we would label them as such since these assets can be pledged and seized just as easily as housing when

[16] The values $\bar{c}_t$ and $\bar{k}_t$ are independent of θ_t but may vary over time if $\beta R \neq 1$.

[17] Under full information the spot market values k_t^* and B_{t+1}^* need not vary in θ because the Arrow security does all the work to smooth consumption.

all frictions other than asymmetric information are absent.[18]

Second, while the definition suggests a relationship between collateralization and imperfect insurance, the two are not the same. To see this, notice that y_t^* can be constant (no insurance) even as $\Gamma_t = 0$ (no collateralization). The pairing of constant y_t^* with $\Gamma_t = 0$ can occur if variations in k_t^* are exactly offset by variations in B_{t+1}^*. Our results later on show that this will not happen in our specific model where asymmetric information is the only friction in an otherwise complete markets model. Still there is nothing in principle to rule it out.

The combination of imperfect insurance and no collateralization can in fact arise in an optimal contract when markets are incomplete. To illustrate the point, consider an alternative model with a severe form of market incompleteness. In this case the loan market is nonexistent, and so there is no ex ante contracting stage. All "contracting" is done ex post, after the realization of the shock. This means there is no insurance at all. The agent cannot borrow against his/her ex ante net worth. In this case, the value of forfeited assets (the analogue of Equation (5) where income θ_t is in lieu of loan $L_t^*(A_t)$) is defined relative to the ex post net worth:

$$\Gamma(\theta_t, A_t) = A_t + \theta_t - c_t^*(\theta_t, A_t) - qk_t^*(\theta_t, A_t) - B_{t+1}^*(\theta_t, A_t) \tag{9}$$

which is zero over all θ in order to balance the borrower's ex post budget. In this case, there is no collateralization yet there is imperfect insurance.

When the credit market is complete, the contract is offered before the realization of θ. It is well known that asymmetric information gives rise to imperfect insurance. We will show later on that contracts with collateral are optimal under asymmetric information. Therefore, both forfeiture and imperfect insurance are shown to occur when asymmetric information is the sole friction.

Third, the definition of collateralization is similar but not identical to notions in Lacker (2001) and Rampini (2005). Both posit mechanism design models with asymmetric information and a collateralizable capital good. Both offer definitions that come closer to our description of a *de jure* contract with fixed default. Both describe collateralization in terms of specific pooling features. In Lacker's definition, a collateralized contract is one with fixed (pooled) housing consumption above an income threshold, and zero consumption

[18]Note that both k and B are equally observable. It is also the case that the contracting authority has the same degree of control over k as it does over B.

of the nondurable good below the threshold. Rampini describes collateralized contracts in terms of housing consumption that is pooled above a threshold, and is pooled below a possibly different threshold. Our definition includes both possibilities. The next section shows, however, that neither of these types of contracts will be optimal in our setting. Key differences in assumptions and results are discussed in detail in Section 5.

3 Collateralization, Semi-Pooling, and Default

The full information contract is obviously not incentive compatible when income is private information. If full insurance were offered, higher types would always mimic lower ones to obtain the insurance payout. In this section optimal contracts are shown to be collateralized under the broader definition of forfeiture in (5), and under the narrower definition in (7) when the *de jure* contract is referenced.

We then show that the total value of forfeited collateral is decreasing in realized income. This means that collateral is not just a wedge preventing full insurance but assumes a particular form.

To simplify the notations, we omit the subscripts t from $t-$variables when the context is clear.

3.1 Optimal Contracts are Collateralized

Our first result establishes some general properties that imply collateralization.

Proposition 1. *Let (y^*, k^*, B^*) be an optimal contract, and let A be any initial value of the borrower's assets. Then:*

(i) (Monotonicity and no pooling at the top). The schedule $y^*(\theta, A)$ is weakly decreasing and $k^*(\theta, A)$ is strictly increasing in θ. There is a threshold income $\hat{\theta} < \overline{\theta}$ above which the contract is strictly separating, i.e., y^* is strictly decreasing, k^* is strictly increasing, and A^* is strictly increasing in the interval $[\hat{\theta}, \overline{\theta}]$.

(ii) (Declining forfeiture at the top and bottom) There exist thresholds of income $\underline{\theta} < \check{\theta} \leq \tilde{\theta} < \overline{\theta}$ such that

$$\frac{d\Gamma^*(\theta, A)}{d\theta} < 0$$

for all $\theta \in [\underline{\theta}, \breve{\theta}]$ *and* $\theta \in [\tilde{\theta}, \overline{\theta}]$.

The main import is that any partial insurance provided by payment schedule y^* is offset in part by regressivity in housing consumption k^*. This is strictly true at the top of the distribution. Overall, the contract tends toward regressivity. An immediate corollary is:

Corollary. *For any initial value A, every optimal contract is collateralized, and k is an individually collateralized asset.*

Recall that in the full information contract, the three goods c_t, k_t, and B_{t+1} are all constant over income. The contract's only role is to smooth consumption over time, and this does not require collateral. When the shock hits c_t, the full information contract fully insures against income losses. This, in turn, frees up k_t and B_{t+1} to be chosen by the borrower to satisfy a budget constraint, and so they can be regarded as unused for incentive purposes.

When θ is private information, however, the repayment schedule y^* cannot be adjusted over θ to fully insure the borrower. In and of itself, this is a well known consequence of private information in contracting. However, the result also implies that either k_t^* or B_{t+1}^* (or both) must vary with θ to compensate for the limited variation in y^*. The fact that k_t^* varies with θ tells us that housing k will be collateralized. This may not be the case for savings B, depending on relative prices.

The role of k_t is a key difference between this model and those of Townsend (1982), Green (1987), and Thomas and Worrall (1990) cited earlier is the presence here of an intratemporal margin between liquid balances and capital. When a capital good such as k *is* available, it is shown to be a *superior* instrument in the optimal contract. It will be used regardless of whether the borrower's balances may be collateralized. To our knowledge this is a novel result.

To better understand why this is so and what the consequences are, a more tractable statement of the planner's problem is formulated below.

$$\begin{aligned}
\mathcal{L} &= \int_{\underline{\theta}}^{\bar{\theta}} \left(U(\theta + y^*(\theta, A), k^*(\theta, A)) + \beta V\left(A^*(\theta, A)\right)\right) dF(\theta) \\
&+ \int_{\underline{\theta}}^{\bar{\theta}} \Bigg\{ \mu(\theta) \left[\, U(\theta + y^*(\theta, A), k^*(\theta, A)) + \beta V\left(A^*(\theta, A)\right) \right. \\
&- \left. \left(U\left(\underline{\theta} + y^*(\underline{\theta}, A), k^*(\underline{\theta}, A)\right) + \beta V\left(A^*(\underline{\theta}, A)\right)\right) \right] - u'\left(\theta + y^*(\theta, A)\right) \int_{\theta}^{\bar{\theta}} \mu(\tilde{\theta}) d\tilde{\theta} \Bigg\} d\theta \\
&+ \int_{\underline{\theta}}^{\bar{\theta}} \left[\eta(\theta)\left(y^*(\theta, A) - y^*(\underline{\theta}, A)\right) + j(\theta) \left(\gamma(\theta) + \int_{\theta}^{\bar{\theta}} \eta\left(\tilde{\theta}\right) d\tilde{\theta}\right) \right] d\theta \\
&+ \lambda(A) \left(A + \Theta - \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta + y^*(\theta, A) + \hat{q} k^*(\theta, A) + \frac{1}{R} A^*(\theta, A)\right) dF(\theta) \right).
\end{aligned} \tag{10}$$

The multiplier $\lambda(A)$ (shadow cost) is clearly associated with the budget constraint (3). Adapting a familiar result from mechanism theory, we show that incentive constraints in (4) can be fully characterized by two conditions, and envelope constraint and a monotonicity constraint on $y^*(.)$, the latter implying Part (i) in the Proposition.[19] The envelope constraint given by the $\{\cdot\}$ term in (10) is associated with the multiplier $\mu(\theta)$. It is an integration-by-parts reformulation of the first order condition

$$u'(\theta + y^*(\theta, A)) \frac{dy^*(\theta, A)}{d\theta} + v'(k^*(\theta, A)) \frac{dk^*(\theta, A)}{d\theta} + \beta V'\left(A^*(\theta, A)\right) \frac{dA^*(\theta)}{d\theta} = 0 \tag{11}$$

required for truthful revelation of θ to be a local optimum. The monotonicity constraint then extends local truth-telling constraint to a global solution of the borrower's type selection problem. Monotonicity is expressed as two separate constraints associated with multipliers $\eta(\theta)$ and $\gamma(\theta)$, respectively, that describe an equivalent and more tractable way of stating $j(\theta) = -dy^*/d\theta \geq 0$.

The first order conditions for schedules B^*, k^* and y^* in the Saddle problem are, and

$$u'\left(\theta + y^*(\theta, A)\right)\left(f(\theta) + \mu(\theta)\right) + \eta(\theta) = \lambda(A) f(\theta) + u''\left(\theta + y^*(\theta, A)\right) \int_{\theta}^{\bar{\theta}} \mu(\tilde{\theta}) d\tilde{\theta}, \tag{12a}$$

[19] This result is not a direct application of a single-crossing condition as in the Mirleesian optimal taxation literature because the number of screening devices, y, k, A', is strictly greater than the dimension of type plus 1. Matthews and Moore (1987) show that in this case, it is possible that the optimal contract is non-monotone even if the utility function satisfies the usual single-crossing condition.

and

$$v'\left(k^*(\theta,A)\right)\left(f(\theta)+\mu(\theta)\right) \;=\; \lambda(A)\hat{q}f(\theta), \tag{12b}$$

and

$$\beta V'(A^*(\theta,A))\left(f(\theta)+\mu(\theta)\right) \;=\; \lambda(A)\frac{1}{R}f(\theta). \tag{12c}$$

Significantly, the monotonicity of y^* also implies monotonicity of A^* and k^*, namely that both are weakly increasing in θ. To see this observe that the first order conditions (12b) and (12c) imply that there is no relative distortion between the choice of k and A. Because both v and V are differentiable and concave,[20] (k^*, A^*) solves

$$W(\tilde{A}) = \max_{\{k,A':\ \hat{q}k+A'/R=\tilde{A}\}} \left[v(k) \;+\; \beta V(A')\right],$$

where $\tilde{A} = \hat{q}k^* + A^*/R$.

Therefore, the original problem (2) can be rewritten as:

$$\max_{y^*,\tilde{A}^*} \int_{\underline{\theta}}^{\bar{\theta}} \left[u\left(\theta + y^*(\theta,A)\right) + W\left(\tilde{A}^*(\theta,A)\right)\right] dF(\theta) \tag{13}$$

subject to the budget constraint:

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(y_t^*(\theta,A) + \tilde{A}^*(\theta,A)\right) dF(\theta) \;\;\leq\;\; A$$

and the incentive constraint:

$$u(\theta + y^*(\theta,A)) \;+\; W(\tilde{A}^*(\theta,A)) \;\geq\; u(\theta + y^*(\hat{\theta},A)) \;+\; W(\tilde{A}^*(\hat{\theta},A)),$$

for all $\theta, \hat{\theta} \in [\underline{\theta}, \overline{\theta}]$.

Since the y^* is at least weakly decreasing, the deterrence of misreporting can only be achieved if $\tilde{A}^*(\theta,A)$ is increasing in θ. Then, since k and A' are normal goods (in the definition of W), both k^* and A^* are weakly increasing in θ as well. Extending this argument a bit further, it means that if y^* is strictly decreasing on some interval (i.e., some partial insurance is provided by the lender), then both A^* and k^* must be strictly increasing. Of course, A^* is a composite asset made up of the individual assets k and B,

[20] Differentiability and concavity of the continuation value function V can be shown using the standard analysis for the dual cost minimization problem as in Green (1987) and Thomas and Worrall (1990).

and our argument does not extend to the liquid asset B. Part (i) of the proposition implies, in fact, that it must be the housing asset k and not necessarily B that is collateralized.

Applying Lemma 2 in Appendix B to problem (13),[21] we obtain for all interior income realizations $\theta \in (\underline{\theta}, \bar{\theta})$,

$$\frac{v'(k^*(\theta, A))}{u'(\theta + y^*(\theta, A))} > \hat{q}, \tag{14}$$

and at the top income $\overline{\theta}$,

$$\frac{v'(k^*(\overline{\theta}, A))}{u'(\overline{\theta} + y^*(\overline{\theta}, A))} \;=\; \hat{q}. \tag{15}$$

In the full information solution, marginal rates of substitution match the relative prices and so the equality (15) holds everywhere. Hence, when compared to the full information solution, the optimal contract allocates too little housing consumption and too much disposable consumption at every wealth level A and every realized income θ.

The intuition is similar to the rationale for distortions in optimal taxation.[22] Ideally, an optimal contract provides at least partial insurance in the form of a larger transfer value $T(\theta) \equiv y^*(\theta) + qk^*(\theta)$ for lower types θ. In order to do this while deterring mimicry by higher types, the contract must introduce an intra-temporal distortion in the form of (14). In particular, holding fixed one's actual income θ, the left-hand side of (14) is increasing as the borrower manipulates his reported income downward. The fact that the distortion is in the direction of underconsumption of housing is due to the single crossing property, which implies that housing is a normal good. Higher income types prefer more housing, on the margin, relative to non-housing. The increased distortion thus penalizes him more severely the larger his "lie." Since the highest type $\overline{\theta}$ will not be mimicked, no such distortion for his type is required.

From (14) and (15), one can see why the second part of Part (i) of the Proposition —- the contract must be separating for income realizations at the top end of the distribution — holds. If the contract pooled types at the top end, then the marginal rate of substitution (the left-hand side of (14)) would be increasing in the interval due to single crossing, thus contradicting (15).

[21]Notice that, by the definition of W, $v'(k^*) = \hat{q}W'(\tilde{A}^*)$.

[22]However, most of the papers in the literature, such as Mirrlees (1971) and Seade (1982), show analogous results assuming that the monotonicity constraint never binds. This is essentially an assumption that the agent's first order condition sufficiently characterizes incentives. The discussion a few paragraphs forward discusses the limitations of this assumption.

Distortions of this type, it should be noted, are found in optimal tax models. The key difference is that these models usually restrict attention to a "relaxed" planner's problem where it is assumed $\eta(\theta) = 0$, i.e., the monotonicity constraint does not bind.[23] This common assumption may limit the scope of any study of credit contracts. With this assumption, the literature restricts attention to environments in which only separating contracts exist, taking the fixed default narrative off the table from the start. The next section shows that there are relevant situations in which the optimal contract does, in fact, resemble the fixed default *de jure* contract, and that these situations are consequential.

3.2 Separating vs Semi-Pooling

We call a contract fully separating if the monotonicity constraint on y does not bind, i.e., y differ across realizations of θ. We call a contract semi-pooling if y is constant over an interval of θ, i.e., a fixed-default region, but not over the full support of θ. Proposition 1 tells us that the optimal contract cannot be fully pooling.

This section presents results on separating versus semi-pooling optimal contracts for a class of exponential income distributions. Certain features of the income distribution and initial wealth will generate semi-pooling (fixed-default) contracts, while other features will generate strictly separating contracts. A parametric model in the next section derives a closed form contract with semi-pooling at the low end of the income distribution.

Proposition 2. *Suppose that u'' is (weakly) increasing and f comes from a class of exponential distributions, $f(\theta) = \phi \exp(\psi\theta)$ with $\phi, \psi \geq 0$. Given a shadow cost of funds, λ, there exists $\Delta > 0$ such that if $\left(\overline{\theta} - \underline{\theta}\right) > \Delta$, the optimal contract is specifies a region of fixed default, i.e., it is semi-pooling over a subinterval in $[\underline{\theta}, \overline{\theta}]$.*

Thus optimal contracts are semi-pooling in certain environments. When semi-pooling occurs, the borrower in the pooled income region is forced into a fixed quantity of housing consumption. This means the "fixed default" narrative applies when collateral is confined to forfeiture of the housing asset. Yet, in the broader full-recourse definition, forfeiture is declining continuously in the pooling region.

[23] When monotonicity does not bind, Werning (2000) provides a simple and elegant argument for distortion. Difficulties arise when the monotonicity constraint can potentially bind, i.e. $\gamma(\theta) > 0$ and $\eta(\theta) \neq 0$ for some θ, as shown in Proposition 2 below. Hence, we have to to deal with this issue explicitly. The most closely related result in the prior literature (that we are aware of) appears in Hellwig (2007) who works with general but static, non-separable preferences of the form $u(\theta, y, k)$ and allow for potentially binding monotonicity constraint. Our proof strategy is similar to Hellwig's, though the two models are not nested.

In fact, one can verify from (5) that in the pooled region $d\Gamma^*/d\theta = -1$. In other words, forfeiture declines by a dollar for every additional dollar of increased income.

To derive the result, we assume that the optimal contract is fully separating, i.e., that the monotonicity constraint on $y(\cdot)$ does not bind. Letting

$$x(\theta) \equiv u'(c(\theta))$$
$$z(\theta) \equiv v'(k(\theta)),$$

the first order conditions from the Lagrangian problem can be expressed as a boundary value problem

$$\frac{d}{d\theta}\begin{bmatrix} x \\ \\ z \end{bmatrix} = F(x, z) \tag{16}$$

with $x(\overline{\theta})/z(\overline{\theta}) = \hat{q} = x(\underline{\theta})/z(\underline{\theta})$ (see Appendix D).[24] This system, in turn, is represented in a phase diagram as displayed Figure 3.

An optimal and fully separating contract must correspond to a trajectory (curve) traced out by the ODE (16) in the phase diagram which starts on the diagonal $z = \hat{q}x$ and also ends on the diagonal. Figure 3 displays three such curves. The red solid curve has $z'(\underline{\theta}) = 0$ and $z'(\theta) < 0$ for all $\theta > \underline{\theta}$.[25] The blue dotted curve has $z'(\theta) < 0$ for all $\theta \geq \underline{\theta}$. The two curves yield contracts that satisfy the monotonicity constraint (k^* strictly increasing, which is implied by z being strictly decreasing and v being strictly concave) and are thus optimal contracts. When $\overline{\theta} - \underline{\theta}$ exceeds a threshold, the solution to the ODE (16) lies above the red curve such as the blue dashed curve. For these solutions the monotonicity constraint on $k(\cdot)$, and therefore on $y(\cdot)$, is violated at and near $\theta = \underline{\theta}$. So the corresponding contracts are not incentive compatible and thus are not optimal contracts. In this case the optimal contract features some pooling interval (the flat portion intersecting with the dashed blue curve in Figure 3).

The argument can be framed in more intuitive terms. Notice that a strictly separating contract only works if either (1) the highest types are sufficiently rewarded to deter their mimicry of lower types, or (2) the lowest types are sufficiently penalized to prevent

[24] Because of exponential distribution, given λ, changing $\underline{\theta}$ does not alter the optimal allocation c, k, A' and only changes the total budget required to support the allocation.

[25] In Appendix D, we show that, for this curve, $(x(\underline{\theta}), z(\underline{\theta})) = (x^*, z^*) = (2\lambda, 2\hat{q}\lambda)$. For reference, the full information solution corresponds to a dot $(x^{FI}, z^{FI}) = (\lambda, \hat{q}\lambda)$.

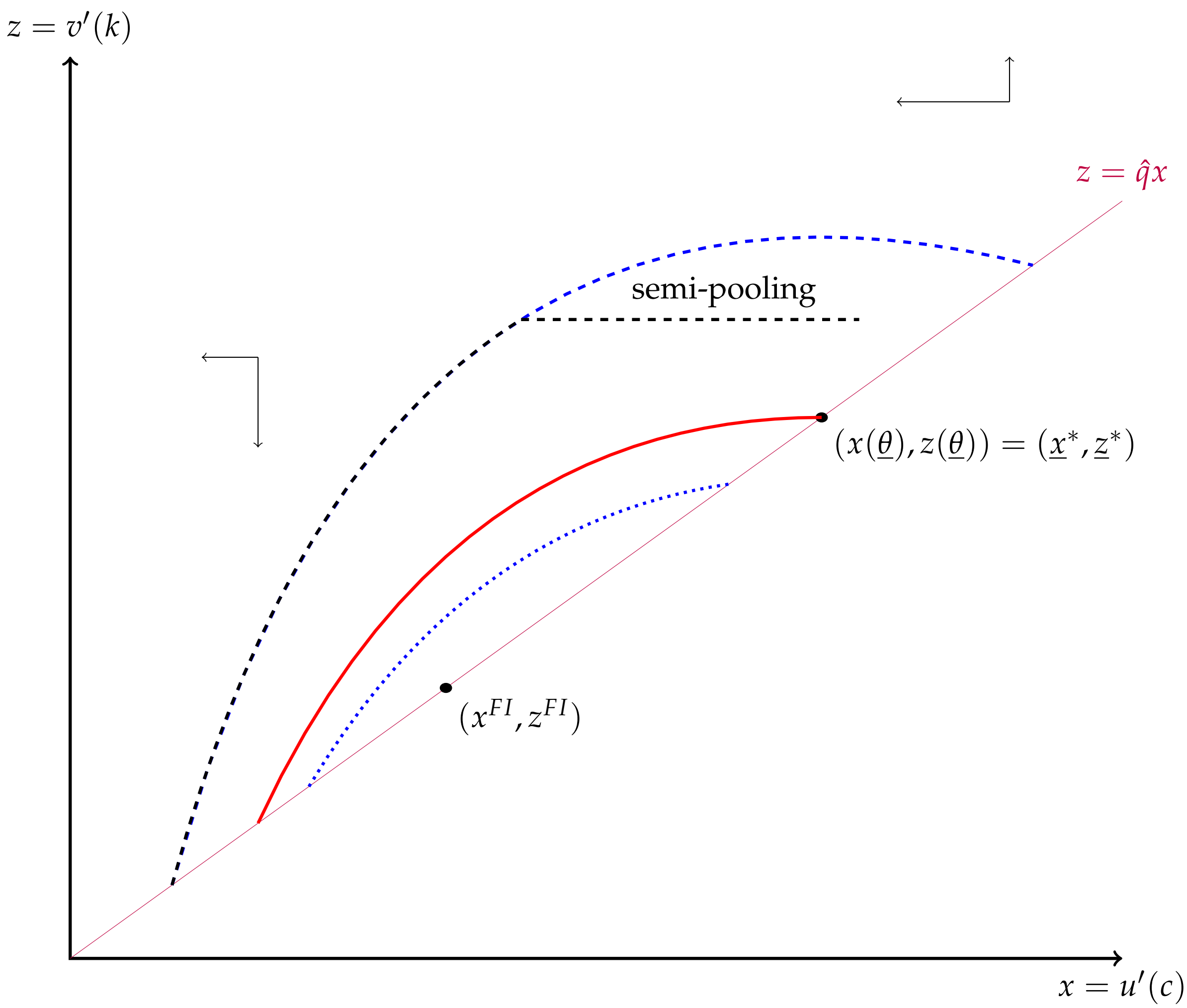

Figure 3: Phase Diagram for Optimal Contracts

mimicry by higher types. Thus when the support is stretched out, the reward-punishment scheme in the separating contract will eventually run out of room — at some point it will violate the resource constraint. The solution is to pool types at the bottom, effectively shrinking the region of rewards and punishments.

It must be noted that the Proposition does not literally show that pooling must occur only at the bottom of the distribution, although we cannot find an example where it occurs elsewhere. (By Proposition 1, it cannot occur at the top). The next section analyzes a canonical case in which the contract has a closed form solution and exhibits semi-pooling at the bottom of the distribution.

4 A Parametric Model with Endogenous Fixed Default

This section presents a parametric "log-uniform" version of the model. The optimal contract in this parametric model is solved in closed form. Housing consumption is weakly convex in income, and is strictly convex whenever it is strictly increasing. We show that if the support of the distribution is sufficiently large and/or the borrower's initial wealth is sufficiently small, the optimal contract is semi-pooling in some interval at the bottom of the distribution. Losses in housing consumption are fixed in some interval of income types below a threshold. Consequently, in this region endogenous default occurs.

By contrast, when the support is small and/or the borrower's initial wealth is large, then the contract is strictly separating. In both cases, forfeiture is globally regressive — it is strictly decreasing in income over the entire distribution.

Let $f(\theta) = \bar{f} = \frac{1}{\bar{\theta}-\underline{\theta}}$ and $U = \frac{1}{2}\log c + \frac{1}{2}\log k$ (V can be any arbitrary increasing, concave, and differentiable function). As before, we let A denote the borrower's initial wealth heading into the period. To capture the effects of variations of A we focus attention on the multiplier $\lambda(A)$ in the Langrangian Saddle problem. For simplicity we drop the notational dependence on A and consider the comparative statics on λ as proxy for variations in A.

Adapting first order conditions (12a) and (12b) to the log-uniform model, we obtain

$$\frac{1}{2c(\theta)}\left(\bar{f}+\mu(\theta)\right) + \eta(\theta) \;=\; \lambda\bar{f} \;-\frac{1}{c(\theta)^2}\int_{\theta}^{\bar{\theta}}\mu(\theta)d\theta \tag{17a}$$

and

$$\frac{1}{2k(\theta)}\left(\bar{f}+\mu(\theta)\right) \;=\; \lambda\hat{q}\bar{f}. \tag{17b}$$

We first conjecture and verify the existence of a separating equilibrium for some set of parameter values. Under full separation, the monotonicity constraint does not bind. The multipliers associated with the monotonicity constant therefore vanish: $\gamma(\theta) \equiv 0$ and $\eta(\theta) \equiv 0$. Incorporating these into the the conditions above and performing some calculations, one obtains

$$y(\theta) \;=\; \sqrt{\left(c(\underline{\theta})\right)^2+\frac{\theta-\underline{\theta}}{2\lambda}} \;-\; \theta. \tag{18}$$

and

$$k(\theta) \;=\; D\frac{\exp(4\lambda c(\theta))}{c(\theta)} \tag{19}$$

where D, a constant, is equal to $\hat{q}(c(\underline{\theta}))^2 \exp(-4\lambda c(\underline{\theta}))$. It is not difficult to show that $k(\theta)$ globally is strictly convex in θ. In other words, *the ratio of housing consumption to realized income is increasing.*

Notice that y is weakly decreasing, as required, if and only if $c(\underline{\theta}) \geq \frac{1}{4\lambda}$. In addition, one can back out the boundary points using the fact there is no distortion at either end of the distribution when monotonicity does not bind. Thus we obtain

$$c(\underline{\theta}) = \hat{q}k(\underline{\theta}) \quad \text{and} \quad c(\overline{\theta}) = \hat{q}k(\overline{\theta}). \tag{20}$$

and so one can solve for $c(\underline{\theta})$ by evaluating (19) at $\theta = \overline{\theta}$ to obtain

$$\log\left(c(\underline{\theta})^2 + \frac{\overline{\theta}-\underline{\theta}}{2\lambda}\right) \;=\; \log(c(\underline{\theta})^2) \;+4\lambda\left((c(\underline{\theta}))^2 + \frac{\overline{\theta}-\underline{\theta}}{2\lambda}\right)^{1/2} - 4\lambda c(\underline{\theta}). \tag{21}$$

Together, the equations (18), (19) (20), and (21) completely characterize a separating contract.

The conditions under which this contract is consistent with the underlying constraints is not unrestricted. For instance, it is required that $c(\underline{\theta}) \geq \frac{1}{4\lambda}$ holds. Consider then an optimal contract with semi-pooling. Above some threshold level $\hat{\theta}$, the semi-pooling contract resembles the separating contract with $\hat{\theta}$ as the lower bound. Namely,

$$y(\theta) \;=\; \sqrt{\left(c(\hat{\theta})\right)^2 + \frac{\theta-\underline{\theta}}{2\lambda}} \;-\; \theta \tag{22}$$

and

$$k(\theta) \;=\; D^{*}\frac{\exp(4\lambda c(\theta))}{c(\theta)^{1/2}} \tag{23}$$

where D^{*}, in this case, is equal to $k(\hat{\theta})c(\hat{\theta})\exp(-4\lambda c(\hat{\theta}))$. Below the pooling threshold $\hat{\theta}$, $k(\theta) = k(\hat{\theta}),\ y(\theta) = y(\hat{\theta})$, and $c(\theta) = c(\hat{\theta}) + \theta - \hat{\theta}$.

In order to calculate the threshold $\hat{\theta}$, the pooling levels of housing $k(\hat{\theta})$ and non-housing $c(\hat{\theta})$ are found by evaluating (22) and (23) at $\theta = \overline{\theta}$ and using the no-distortion condition $\hat{q}c(\overline{\theta}) = k(\overline{\theta})$. Together with the first order conditions on the multipliers $\mu(\theta), \gamma(\theta)$ over $[\underline{\theta}, \hat{\theta}]$, we obtain three equations with three unknowns $c(\hat{\theta})$, $k(\hat{\theta})$, and $\hat{\theta}$ (see Appendix E for details).

The result are summarized as follows.

Proposition 3. *In any optimal contract in the log-uniform model, the value of forfeiture $\Gamma^*(\theta)$ is strictly decreasing in θ. In addition there exists $\delta > 0$ such that*

(i) (Separating contract). If $\lambda(\overline{\theta} - \underline{\theta}) \leq \delta$, then the optimal contract is strictly separating and satisfies (18), (19) (20), and (21). Notably, k^* is strictly increasing and strictly convex in θ, y^* is strictly decreasing and strictly concave.

(ii) (Endogenous Fixed Default). If $\lambda(\overline{\theta} - \underline{\theta}) > \delta$, then there exists a cutoff type $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ such that the optimal contract is separating in the interval $[\hat{\theta}, \overline{\theta}]$ and is pooling in the interval $[\underline{\theta}, \hat{\theta})$. Above the cutoff $\hat{\theta}$, k^* is strictly convex in θ, y^* is strictly concave.

The parameter set is partitioned into those that generate fully separating contracts and those that generate semi-pooling ones.[26] Both the support $[\underline{\theta}, \overline{\theta}]$ and the multiplier λ play a critical role. The intuition is very much like that of Proposition 2. A larger support increases the incentive to misreport in a separating equilibrium since the reward-punishment scheme would have to be stretched beyond the resource constraint. Incentives can be only be brought into line by a pooling contract at the lower end. In that case there is no further gain from mimicry below the threshold $\hat{\theta}$.[27]

As for the multiplier λ, an increase in λ also diminishes the range in which the separating contract exists. In the Proposition, λ is treated as a parameter. However, λ is an implicit solution to the resource constraint evaluated at y and k, given budget A. As a shadow price of the constraint, λ is continuously decreasing in A. Hence, what the proposition is, in effect , saying is that there is a threshold $\hat{A}$ so that if the borrower's initial wealth A is above the threshold, then the optimal continuation contract (y, k) is strictly separating. Whereas, if the borrower's initial wealth is below the threshold then there exists a cutoff type $\hat{\theta}$ such that the contract is separating above $\hat{\theta}$ and is pooling below it. In all cases forfeited collateral is decreasing in income.

This result indicates that the optimal use of collateral must be, in a sense, regressive. Regardless of whether the contract is separating or semi-pooling, forfeiture is strictly de-

[26]In fact, the parameter δ can be pinned down precisely: $\delta = (\epsilon - 1)/8$ where ϵ is the unique scalar that satisfying $\log(\epsilon) = \epsilon^{1/2} - 1$ ($\epsilon \approx$ 12.34).

[27]In a phase diagram as in Figure 3, the semi-pooling contract corresponds to a flat section at the bottom of the income distribution then a separating section at the top following the phase diagram, similar to the black dashed curve in the figure.

creasing in realized income over the entire distribution. Individuals with low income realizations forfeit more.

Recall that in the pooling region $d\Gamma^*/d\theta = -1$, whereas in the separating region $d\Gamma^*/d\theta > -1$. More concretely, take two individuals with different initial wealth, one above and one below the threshold $\hat{A}$. For the poorer one, any small negative shock to income is not compensated by a reduction in forfeiture. In other words, there is no loan forgiveness following a reduction in the borrower's income.

For the wealthier borrower, a reduction in income results in partial forgiveness of the loan. Loan forgiveness is, in effect, the subsidy given to the high income types to deter them from misrepresenting income. But the subsidy can only be taken so far before resource constraints are violated. The pooling at the bottom is a penalty to low types in order to prevent manipulation by high types when subsidies are limited. Thus, unlucky agents pay the price for incentive problems faced by the luckier ones. Ex ante poorer agents — those with low initial wealth A — or those with high variation in future income pay the price by bearing the full risk of the income shock in the pooling region.

Illustration with parameter values. The optimal collateralized contract is illustrated in two parametric special cases, one is static and the other one is fully dynamic in infinite horizon.

Example 1 (Static)**.** In this example, we consider the static version of the parametric model in this section, i.e., $U(c,k) \equiv \frac{1}{2}\log c + \frac{1}{2}\log k$ and $V \equiv 0$. We set parameter values

$$A = 1,\ \hat{q} = 1,\ \lambda = 1,\ f \equiv \frac{1}{\overline{\theta} - \underline{\theta}}$$

where $[\underline{\theta}, \overline{\theta}] = [-0.5, 0.5]$ so that expected income is zero.

The allocations, distortions, and forfeitures are then displayed in Figure 4 for two different supports. Each column displays the optimal contract, a distribution of the distortionary wedge, and the distribution of the value of forfeiture/seizure of the asset across the realized incomes.

In the first column of Figure 4 the support is $[\underline{\theta}, \overline{\theta}] = [-0.5, 0.5]$. The optimal contract in this case is strictly separating, as per Part (i) of Proposition 3. In the second column $[\underline{\theta}, \overline{\theta}] = [-1.5, 1.5]$ so that the second support is three times as broad as the first. In this case, the optimal contract is semi-pooling, as per Part (ii) of the Proposition. The region of borrower's income below a cutoff of around -1.3 is pooled, while types above that

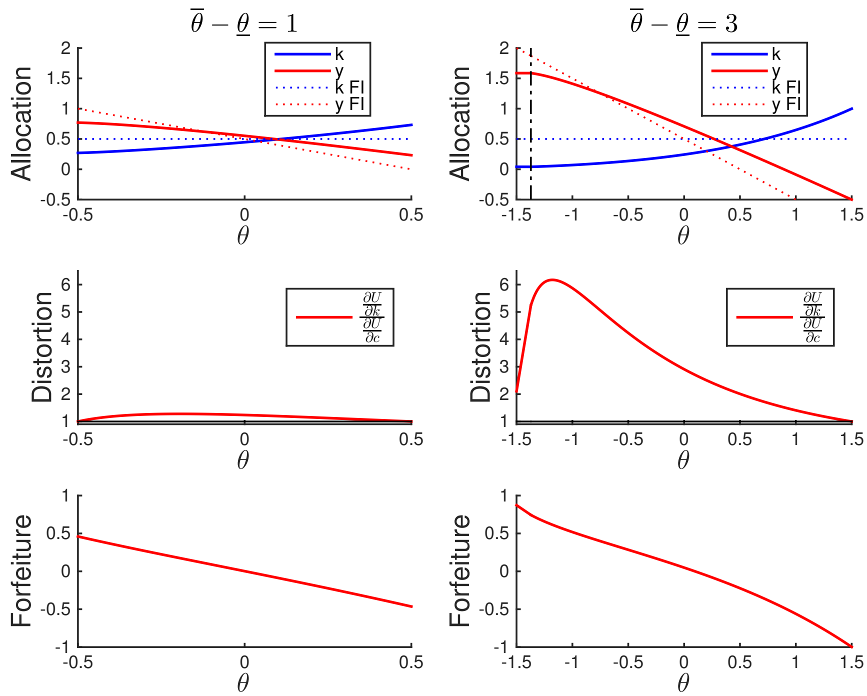

Figure 4: Static Model: Allocations, Distortions, and Forfeiture

threshold strictly separate.

In both cases, the distortion (second row in Figure 4) is a non-monotonic function of income. The wedge is largest for middle income types, and smallest for the very poor and very rich. This is largely due to the differing information incentives and the cost mitigating bad incentives across types.

Curiously, in this example the threshold value $\hat{\theta}$ below which pooling/fixed default occurs is not necessarily monotonic in initial wealth A. This can be seen in Figure 5. The Figure displays the threshold value as a function of initial wealth A when $[\underline{\theta}, \bar{\theta}] = [-0.5, 0.5]$. The threshold value first increases before decreasing to the lower bound $\underline{\theta} = -.5$ as A increases.

At the lower bound, the contract is fully separating. At first glance, one might infer from Figure 5 that in some region of the wealth distribution, default is *less* likely at lower levels of wealth. This is misleading, however, because the severity of partial default above the threshold also changes as wealth level A varies. Figure 6 displays the ratios consumption to wealth for c and k for different values of A.[28] Notice that the consumption ratios of

[28]Varying wealth levlel A introduces scale effects in consumption. To make a consistent comparison we

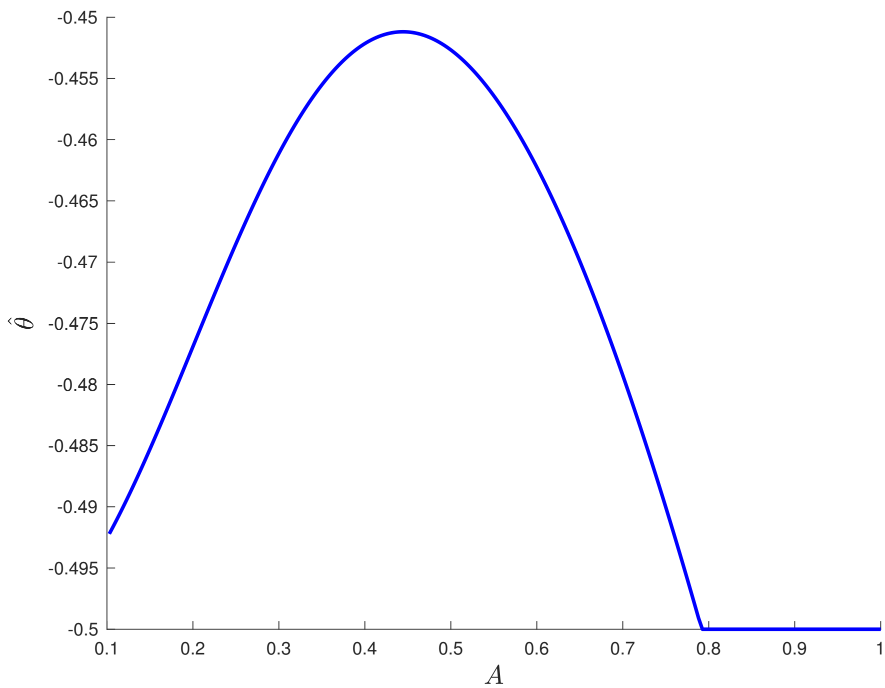

Figure 5: Non-monotonicity of the semi-pooling region as wealth varies

both types of goods appears "flatter" at higher values of A indicating superior insurance opportunities for wealthier borrower. Hence, the partial default regions are just as important as the full default regions for determining how different types of borrowers fare under collateralized contracting.

Example 2 (Dynamics)**.** In this example, we consider the dynamic, infinite horizon of the parametric model in Example 1. V is determined endogenously as a solution to a Bellman equation derived from (2).

We set the parameters value

$$\beta = 0.90 \qquad R = 1.05 \qquad \hat{q} \equiv 1 \qquad q \equiv \frac{R}{R-1},$$

and $f_t(\theta) \equiv \frac{1}{\overline{\theta}-\underline{\theta}}$ over $[\underline{\theta}, \overline{\theta}] = [-0.5, 0.5]$ so that expected income each period is zero.

The allocations in k_t and B_{t+1} and forfeitures are displayed in Figure 7 for two different values of A_t. When $A_t = 1$, the optimal contract is semi-pooling in the bottom of the

normalize consumption values by dividing by A.

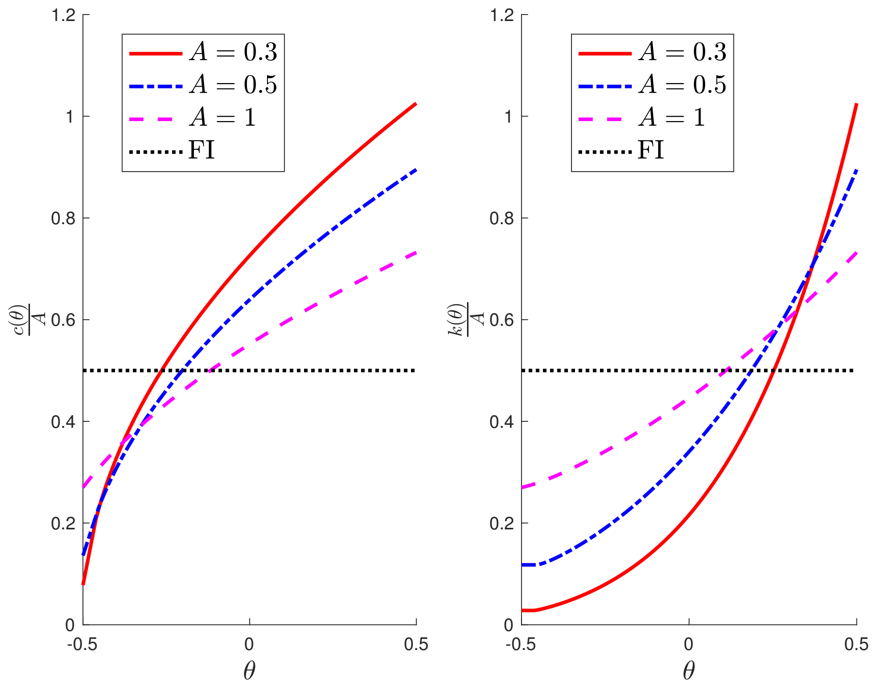

Figure 6: Variation in consumption to wealth ratio as borrower's wealth varies

distribution and B_{t+1} is non-monotone in θ.[29] When $A_t = 10$, the optimal contract is fully separating and B_{t+1} is actually decreasing in θ. In both cases total forfeiture Γ_t is strictly decreasing in θ. In addition, B_{t+1} varies little in θ relative to the variation in housing consumption qk_t and total asset Γ_t. This shows that housing and total asset are collateralized more intensively than saving as discussed earlier.

Time horizon. Example 1 illustrates the static model while Example 2 takes on the infinite horizon case. Intuitively, longer time horizon should relax incentive constraints at the beginning of the contract.

To assess this claim, one must compare models of different time horizons. To do so, we normalize so that per period expected income, $A_1 / \sum_{t=1}^{T} (1/R)^{t-1}$, is the same across different time horizon models. Under this normalization, when $\beta R = 1$ then the shadow cost λ_1 of the constraint under full information will be the same under all time horizons. This allows us to isolate the effects on incentives of asymmetric information under alternative time horizon models.

[29] $A = 1$ does not lead to pooling in Example 1 (the static model) but does in the infinite horizon since the same amount of wealth is spent over more periods, despite the benefit of longer term contract for incentives.

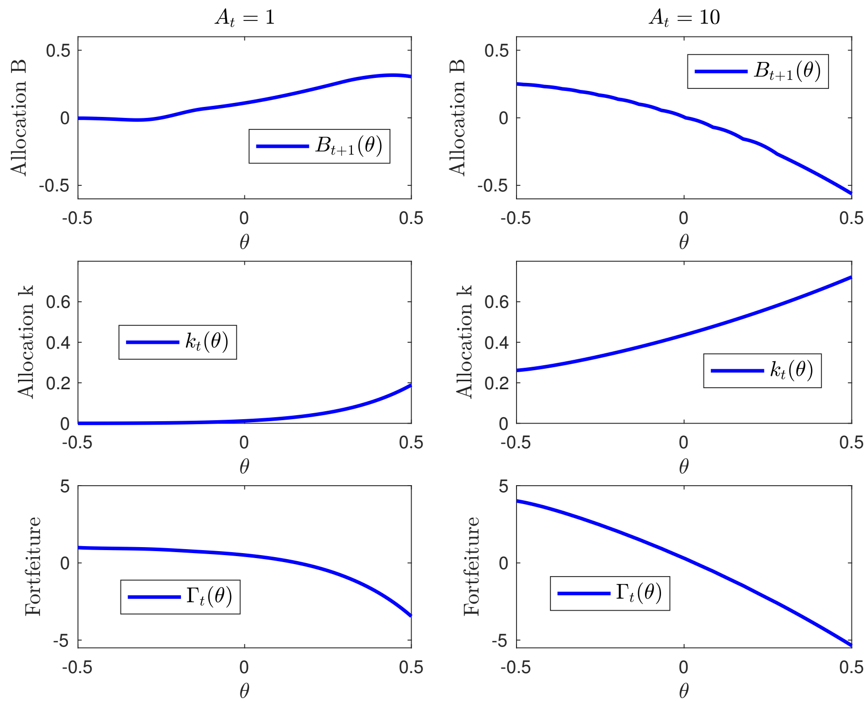

Figure 7: Dynamic model: asset allocations and forfeiture under low/high initial wealth

In a numerical example with the same parameters as Example 2, (with $\beta R = 1$), we verify that the initial multiplier λ_1, the shadow cost of funds at the beginning of the contract, decreases as time horizon T increases (Figure 8). The dotted black line in Figure 8 displays the full information graph. As expected, the shadow cost decreases as per period income increases. Figure 8 also displays the threshold as a horizontal line (given in Proposition 3 as $\frac{\delta}{\bar{\theta}-\underline{\theta}}$), for λ_1 above which there is a fixed-default region. This threshold is common across horizons. Thus, there are levels of average per-period expected income for which a fixed-default region occurs in shorter horizon economies but not in longer horizon economies. Overall, the figure suggests that there is value to longer term contracts, however we caution that the value will depend largely on life cycle attributes of the borrower.

5 Collateral and Pooling: A Comparison to the Literature

The present model is not the first to examine whether collateral of some form is a feature of an optimal contract design. However, we are unaware of design models of collateri-

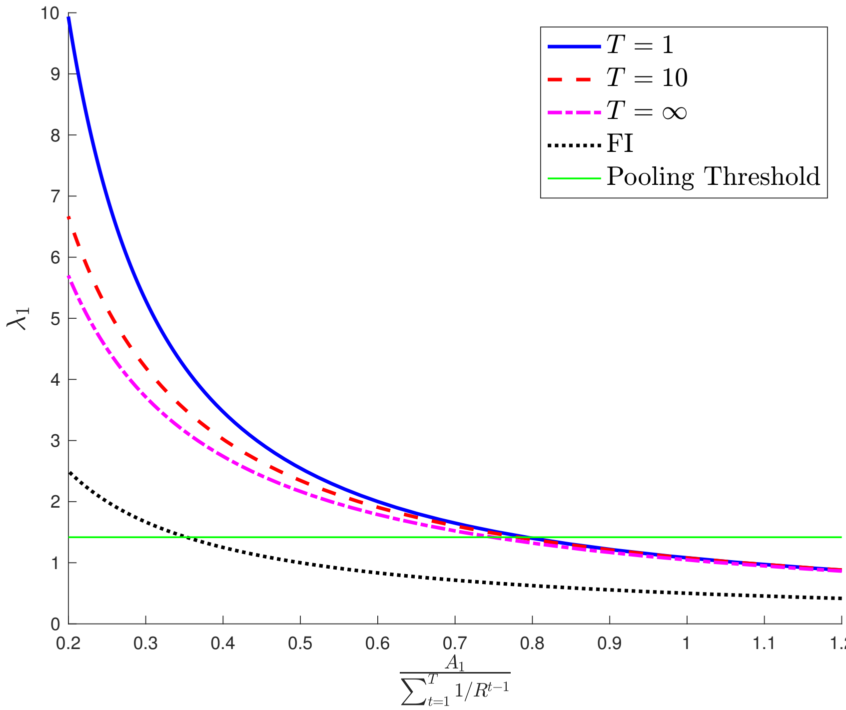

Figure 8: Shadow cost of funds under alternative time horizons

zable capital in which asymmetric information is the sole friction. We are also unaware of contract designs that distinguish between prime and subprime borrowers. It appears that both of these features are related to the way that semi-pooling or "bunching" arises and varies with parameters of the model.

Partial pooling figures into some other models of collateral. A comparison of our work with these is instructive. The ones closest to the present model are Lacker (2001), and Rampini (2005).[30] They both examine collateral in mechanism design models with asymmetric information. They posit static versions of the model that resemble ours in that (i) there are two margins — they also posit two goods c and k, (ii) the borrower has asymmetric information in the form of a shock to c, and (iii) the optimal contract maximizes the borrower's ex ante payoff subject to incentive and resource constraints.

There are also some differences between our work and theirs. The present model is

[30]The growing literature on endogenous debt-contracts arising from asymmetric information friction is also related, though more distantly. This literature includes Townsend (1979), Green (1987), Kiyotaki (2011), and more recently Dang et al. (2015). These do not explicitly incorporate collateral, although we would argue that many of these have features that resemble collateralization.

dynamic whereas these are static. The critical difference is that these models assume a fixed upper bound on housing consumption (independently of the borrower's initial net worth). In other words, $k(\theta) \leq K$ for some finite K. In our model, there is no bound in keeping with our original motivation to isolate the one key friction. It is also realistic in cases where housing consumption services can be increased (via, for instance, remodeling).

One could justify the bound as a feature of an asset with fixed consumption for the borrower in the absence of default. When the bounds are binding in the optimal contract, the designer would like to make the borrower's housing consumption larger for higher income individuals in order to deter manipulation. The designer cannot do it because the housing consumption constraint binds. Consequently, contracts with consumption bounds invariably display "pooling at the top," i.e., fixed housing and repayment above an income threshold. This top-end pooling is integral to the notions of collateralization in Lacker (2001), and Rampini (2005).

Because forfeiture will not be constant in the contracts described in these papers, the contracts will also collateralized according to our definition. However, optimal collateralized contracts under our definition are not collateralized under theirs: without consumption bounds we show that the optimal contract must be separating at high income levels.[31]

As a general matter, broad conditions under which partially-pooled contracts arise are elusive. The majority of the static Mirleesian optimal taxation literature assumes fully separating contracts, and in the dynamic Mirleesian literature the use of a first-order approach (e.g. Albanesi and Sleet (2006), Farhi and Werning (2013)) also rules out pooling of any kind.

Partial pooling can be shown to exist in specialized environment with quasi-linear utility functions. In that case pooling (equivalently "bunching") is verified using ironing argument when hazard rates are not monotonic and/or when participation constraints exist. Examples include Myerson (1981), Jullien (2000), Noldeke and Samuelson (2007), and others. With general utility functions, we show that pooling might happen even with monotone hazard rates and/or without a participation constraint. This suggests a

[31] While Lacker defines a collateralized contract as strictly separating below income thresholds, Rampini constructs an appealing numerical example of an optimal contract with pooling at both ends. His notion is thus closer to what we refer to as the *de jure* contract. Since optimal contracts in the present model may (or may not) exhibit bottom-end pooling, Rampini's results are a sort of hybrid between Lacker's model and ours.

rationale for pooling in an optimal contract very different from that of the quasi-linear paradigm.[32]

A few important issues remain unresolved. First, our understanding of the distributional attributes of collateralized contracts is incomplete. Beyond the log uniform model, we cannot say how the composition of collateralized assets changes in θ. One would like to know more about how the contract differs across ex ante heterogeneous agents.

Second, while the model might be viewed as a micro foundation for empirical work on effective credit contracts under full recourse, all attempts to document the true underlying contract face a common difficulty. Namely, an omitted variables problem arises as summarized in Guiso et al. (2013).

> "The main problem in studying strategic defaults is that such defaults are de facto unobservable events. While we do observe defaults, we cannot observe whether a default is strategic as strategic defaulters have incentives to disguise themselves as people who cannot afford to pay and hence they are difficult to identify in the data." - Guiso, Sapienza, and Zingales (2013)

Our theory suggests some qualitative properties of collateralization, distortion, and default. Still, the inferences thus far are based on indirect evidence, comparing recourse with non-recourse, and securitized with non-securitized contracts, and so on.

Finally, the baseline theory takes prices as given, and so our baseline describes a locally optimal contract between the two contracting parties. The extension to a general equilibrium model suggests that perfect competition can "partially mitigate" the distortion and thus reduce forfeiture. It remains unclear to what extent this is so. Our extension also suggests that optimal collateralized contracts have different equilibrium implications than the ad hoc fixed-default contracts. To the extent that the majority of the recent macro-finance literature with housing, such as Iacoviello (2005), Corbae and Quintin (2015) and Berger et al. (2015), assume fixed-default contracts and, as we argue in the introduction, that fixed-default contracts might be far from reality, it would be interesting to examine the robustness of the findings in the literature with respect to the optimal collateralized contracts.

[32] Both Lollivier and Rochet (1983) and Rampini (2005) also construct examples with semi-pooling in quasi-linear environments. Unlike the auction literature, they examine contract design under resource rather than participation constraints. For this reason, we think their rationale for pooling is closer to the present model than the auction literature.

These questions indicate that the theory is far from complete, but also suggest a number of viable paths forward.

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Appendix

A Deriving the Lagrangian

In this appendix, we present the derivation of the planner given in (10). First we start with the following lemma that characterizes the set of incentive compatible contracts.

Lemma 1. *A continuous and piecewise differentiable contract $y(\theta, A), k(\theta, A), B(\theta, A)$ is incentive compatible if and only if y is weakly decreasing in θ and the envelope condition is satisfied.*

Proof. To simplify the notations, we suppress the dependence on A of the contract. First we show that incentive compatibility implies that $y(\theta)$ is decreasing in θ (it is standard to show that local incentive compatibility constraint implies Envelope Condition). Let

$$\varphi(\theta', \theta) = u(\theta' + y(\theta)) + v(k(\theta)) + \beta V(A(\theta)).$$

Fixing any θ, from the incentive constraint,

$$\Delta(\theta') = \varphi(\theta', \theta') - \varphi(\theta', \theta) \geq 0$$

and by definition of φ,

$$\Delta(\theta) = 0.$$

Therefore,

$$\Delta'(\theta) = 0 \quad \text{and} \quad \Delta''(\theta) \geq 0.$$

From the definition of $\Delta(.)$, we have:

$$\Delta'(\theta') = u'(\theta' + y(\theta')) - u'(\theta' + y(\theta))$$

and

$$\Delta''(\theta') = u''(\theta' + y(\theta'))\frac{dy(\theta')}{d\theta'} + u''(\theta' + y(\theta')) - u''(\theta' + y(\theta)).$$

Since $u'' < 0$, this implies, from $\Delta''(\theta) \geq 0$,

$$\frac{dy(\theta)}{d\theta} \leq 0.$$

Now if $\frac{dy(\theta)}{d\theta} \leq 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. We show that the incentive compatibility constraint is satisfied. Indeed, for $\theta > \theta'$:

$$\begin{aligned}
& u\left(\theta + y(\theta)\right) + v(k(\theta)) + \beta V\left(A(\theta)\right) - u\left(\theta + y(\theta')\right) - v(k(\theta')) - \beta V\left(A(\theta')\right) \\
&= \int_{\theta'}^{\theta} \left(u'(\theta + y(\tilde{\theta})) \frac{dy(\tilde{\theta})}{d\tilde{\theta}} + v'(k(\tilde{\theta})) \frac{dk(\tilde{\theta})}{d\tilde{\theta}} + \beta V'\left(A(\tilde{\theta})\right) \frac{dA(\tilde{\theta})}{d\tilde{\theta}} \right) d\tilde{\theta} \\
&\geq \int_{\theta'}^{\theta} \left(u'(\tilde{\theta} + y(\tilde{\theta})) \frac{dy(\tilde{\theta})}{d\tilde{\theta}} + v'(k(\tilde{\theta})) \frac{dk(\tilde{\theta})}{d\tilde{\theta}} + \beta V'\left(A(\tilde{\theta})\right) \frac{dA(\tilde{\theta})}{d\tilde{\theta}} \right) d\tilde{\theta} \\
&= 0,
\end{aligned}$$

where the inequality comes from $0 < u'(\theta + y(\tilde{\theta})) \leq u'(\tilde{\theta} + y(\tilde{\theta}))$ and $\frac{dy(\tilde{\theta})}{d\tilde{\theta}} \leq 0$, and the last equality comes from the Envelope Condition. The proof for $\theta < \theta'$ is identical. □

Lemma 1 allows us to replace the IC constraint, (4), by the Envelope Condition, (11) and the constraint that $y(\theta)$ is decreasing. We use the integral form of the Envelope Condition:

$$\begin{aligned}
& u\left(\theta + y(\theta)\right) + v(k(\theta)) + \beta V(A(\theta)) \\
&= u\left(\underline{\theta} + y(\underline{\theta})\right) + v(k(\underline{\theta})) + \beta V(A(\underline{\theta})) \\
&+ \int_{\underline{\theta}}^{\theta} u'(\tilde{\theta} + y(\tilde{\theta})) d\tilde{\theta},
\end{aligned}$$

and let $\mu(\theta)$ denote the multiplier on this constraint.

The monotonicity of $y(.)$ can be written as:

$$y(\theta) = y(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} j(\tilde{\theta}) d\tilde{\theta},$$

where $j = -\frac{dy}{d\theta} \geq 0$. Let $\eta(\theta)$ denote the multiplier on this constraint and $\gamma(\theta)$ denote the multiplier on the positivity constraint on $j(\theta)$.

The Lagrangian of the planner problem is then:

$$
\begin{aligned}
\mathcal{L} = & \int_{\underline{\theta}}^{\bar{\theta}} \left(u\left(\theta + y(\theta)\right) + v(k(\theta)) + \beta V\left(A(\theta)\right)\right) dF(\theta) \\
& + \int_{\underline{\theta}}^{\bar{\theta}} \mu(\theta) \left(u\left(\theta + y(\theta)\right) + v(k(\theta)) + \beta V\left(A(\theta)\right)\right) d\theta \\
& - \left(\int_{\underline{\theta}}^{\bar{\theta}} \mu(\theta) d\theta\right) \left(u\left(\underline{\theta} + y(\underline{\theta})\right) + v(k(\underline{\theta})) + \beta V\left(A(\underline{\theta})\right)\right) \\
& - \int_{\underline{\theta}}^{\bar{\theta}} \mu(\theta) \int_{\underline{\theta}}^{\theta} u'(\tilde{\theta} + y(\tilde{\theta})) d\tilde{\theta} d\theta \\
& + \int_{\underline{\theta}}^{\bar{\theta}} \eta(\theta) \left(y(\theta) - y(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} j(\tilde{\theta}) d\tilde{\theta}\right) d\theta + \int_{\underline{\theta}}^{\bar{\theta}} \gamma(\theta) j(\theta) d\theta \\
& + \lambda \left(A - \int_{\underline{\theta}}^{\bar{\theta}} \left(y(\theta) + \hat{q} k(\theta) + \frac{1}{R} A(\theta)\right) dF(\theta)\right).
\end{aligned}
$$

Using Fubini Theorem to switch the order of integrals:

$$
\int_{\underline{\theta}}^{\bar{\theta}} \mu(\theta) \int_{\underline{\theta}}^{\theta} u'(\tilde{\theta} + y(\tilde{\theta})) d\tilde{\theta} d\theta = \int_{\underline{\theta}}^{\bar{\theta}} u'(\theta + y(\theta)) \left(\int_{\theta}^{\bar{\theta}} \mu(\tilde{\theta}) d\tilde{\theta}\right) d\theta
$$

and

$$
\int_{\underline{\theta}}^{\bar{\theta}} \eta(\theta) \int_{\underline{\theta}}^{\theta} j(\tilde{\theta}) d\tilde{\theta} d\theta = \int_{\underline{\theta}}^{\bar{\theta}} j(\theta) \left(\int_{\theta}^{\bar{\theta}} \eta(\tilde{\theta}) d\tilde{\theta}\right) d\theta,
$$

and plugging them into the expression for $\mathcal{L}$, we arrive at (10).

B Distortion in the General Static Problem

First, we consider a more general static optimal contracting problem with two good c and k in which the utility function $\mathcal{U}(c, k)$ is not necessarily separable in the two goods. Instead, we assume only that it is twice continuously differentiable, strictly increasing, strictly concave, and both goods are normal goods:

$$
\max_{y(.), k(.)} \int_{\underline{\theta}}^{\bar{\theta}} \mathcal{U}(\theta + y(\theta), k(\theta)) dF(\theta) \tag{24}
$$

subject to

$$\int_{\underline{\theta}}^{\overline{\theta}} (y(\theta) + qk(\theta))\, dF(\theta) \leq A$$

and

$$\mathcal{U}(\theta + y(\theta), k(\theta)) \geq \mathcal{U}(\theta + y(\hat{\theta}), k(\hat{\theta})) \forall \theta, \hat{\theta} \in [\underline{\theta}, \overline{\theta}]. \tag{25}$$

It is standard to show that that the normality of c and k is equivalent to a weak complementarity condition:

$$\frac{\partial^2 \mathcal{U}}{\partial c \partial k} > \max \left\{ \frac{\partial \mathcal{U}/\partial c}{\partial \mathcal{U}/\partial k} \frac{\partial \mathcal{U}^2}{\partial k^2}, \frac{\partial \mathcal{U}/\partial k}{\partial U/\partial c} \frac{\partial \mathcal{U}^2}{\partial c^2} \right\}, \tag{26a}$$

or equivalently,

$$\frac{\partial \mathcal{U}^2}{\partial c^2} \frac{\partial \mathcal{U}}{\partial k} - \frac{\partial^2 \mathcal{U}}{\partial c \partial k} \frac{\partial \mathcal{U}}{\partial c} < 0 \tag{26b}$$

and

$$\frac{\partial \mathcal{U}^2}{\partial k^2} \frac{\partial \mathcal{U}}{\partial c} - \frac{\partial^2 \mathcal{U}}{\partial c \partial k} \frac{\partial \mathcal{U}}{\partial k} < 0. \tag{26c}$$

The condition (26b) is equivalent to the Strict Single Crossing Condition (SSCC) for

$$\hat{\mathcal{U}}\,(y, k, \theta) \equiv \mathcal{U}(\theta + y, k).$$

Indeed,

$$\frac{\partial}{\partial \theta} \left(\frac{\frac{\partial \hat{\mathcal{U}}}{\partial k}}{\frac{\partial \hat{\mathcal{U}}}{\partial y}} \right) = \frac{\frac{\partial^2 \mathcal{U}}{\partial k \partial c} \frac{\partial \mathcal{U}}{\partial c} - \frac{\partial \mathcal{U}^2}{\partial c^2} \frac{\partial \mathcal{U}}{\partial k}}{\left(\frac{\partial \mathcal{U}}{\partial c} \right)^2} > 0. \tag{26d}$$

The following lemma characterizes the distortion in the solution to problem (24).

Lemma 2. *A continuous and piecewise differential solution to problem (24), $y(.), k(.)$ satisfy:*

(i) $y(\theta)$ is strictly decreasing and $k(\theta)$ is strictly increasing in a neighborhood of $\overline{\theta}$ and

$$\frac{\frac{\partial \mathcal{U}}{\partial k}(\overline{\theta} + y(\overline{\theta}), k(\overline{\theta})}{\frac{\partial \mathcal{U}}{\partial c}(\overline{\theta} + y(\overline{\theta}), k(\overline{\theta}))} = q$$

(ii) For all $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\frac{\frac{\partial \mathcal{U}}{\partial k}(\theta + y(\theta), k(\theta))}{\frac{\partial \mathcal{U}}{\partial c}(\theta + y(\theta), k(\theta))} > q.$$

To show this result, we first characterize the properties of incentive compatible contracts.

Lemma 3. *A contract* (y, k) *satisfies the incentive constraint* (25) *if and only if for* $\theta > \theta'$*, we have*

$$y(\theta) \leq y(\theta') \quad \text{and} \quad k(\theta) \geq k(\theta'), \tag{27}$$

$$\frac{d\mathcal{U}^-}{d\theta}\left(\theta + y(\theta), k(\theta)\right) \geq \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k(\theta)\right), \text{ and} \tag{28}$$

$$\frac{d\mathcal{U}^+}{d\theta}\left(\theta + y(\theta), k(\theta)\right) \leq \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k(\theta)\right) \tag{29}$$

where $\frac{dU^-}{d\theta}$ *and* $\frac{dU^+}{d\theta}$ *are the left and right one-sided derivatives respectively.*

Proof. From the IC constraint for type θ:

$$\mathcal{U}\left(\theta + y(\theta), k(\theta)\right) \geq \mathcal{U}\left(\theta + y(\theta'), k(\theta')\right),$$

we rule out the possibility that $y(\theta') \geq y(\theta)$ and $k(\theta') \geq k(\theta)$, with at least on strictly inequality. Similarly from the IC constraint for type θ', we rule out the possibility that $y(\theta') \leq y(\theta)$ and $k(\theta') \leq k(\theta)$, with at least one strict inequality. Therefore to obtain this lemma, we just need to eliminate the possibility that $y(\theta) \geq y(\theta')$ and $k(\theta) \leq k(\theta')$ with at least one strict inequality.

We show this by contradiction. Suppose that it is true. Let

$$\tilde{\mathcal{U}}(\tilde{y}, k, \theta) \equiv \mathcal{U}(\theta - \tilde{y}, k).$$

Then $\tilde{\mathcal{U}}$ satisfies the Strict Single Crossing Condition (SSCC) because it satisfies the Spence-Mirlees condition (see Milgrom and Shannon (1994) for the exact definition of these conditions):

$$\frac{\partial}{\partial \theta}\left(\frac{\partial \tilde{\mathcal{U}}/\partial \tilde{y}}{\partial \tilde{\mathcal{U}}/\partial k}\right) = \frac{\partial}{\partial \theta}\left(-\frac{\partial \mathcal{U}(\theta - \tilde{y}, k)/\partial c}{\partial \mathcal{U}(\theta - \tilde{y}, k)/\partial k}\right) > 0,$$

where the last inequality is equivalent to (26b). By Milgrom and Shannon (1994, Theorem 3), since

$$\left(-y(\theta), k(\theta)\right) < \left(-y(\theta'), k(\theta')\right),$$

and

$$\tilde{\mathcal{U}}\left(-y(\theta'), k(\theta'), \theta'\right) \geq \tilde{\mathcal{U}}\left(-y(\theta), k(\theta), \theta'\right),$$

we have

$$\tilde{\mathcal{U}}\left(-y(\theta'),k(\theta'),\theta\right) > \tilde{\mathcal{U}}\left(-y(\theta),k(\theta),\theta\right),$$

or equivalently

$$\mathcal{U}\left(\theta+y(\theta'),k(\theta')\right) > \mathcal{U}\left(\theta+y(\theta),k(\theta)\right),$$

which contradicts the IC constraint for θ. Therefore by contradiction, we obtain the monotonicity property.

Now we turn to the envelope conditions. For $\theta' < \theta$, we write the IC constraint for type θ as

$$\mathcal{U}\left(\theta+y(\theta),k(\theta)\right) - \mathcal{U}\left(\theta'+y(\theta'),k(\theta')\right) \geq \mathcal{U}\left(\theta+y(\theta'),k(\theta')\right) - \mathcal{U}\left(\theta'+y(\theta'),k(\theta')\right).$$

Dividing both-side by $\theta - \theta'$ and take the limit $\theta' \to \theta$, we obtain

$$\frac{d\mathcal{U}^-}{d\theta}(\theta+y(\theta),k(\theta)) \geq \frac{\partial\mathcal{U}}{\partial c}\left(\theta+y(\theta),k\left(\theta\right)\right).$$

The second half of the envelope condition is obtained similarly by considering the IC constraint for type θ'.

□

Now going back to the original problem (24), we consider a "weakly relaxed problem" in which only downward incentive compatibility and monotonicity are imposed, the multiplier $\xi(\theta)$ on the local downward IC constraint (28) is positive by assumption. We then show that the optimal solution to this "weakly relaxed problem" also satisfies the global incentive constraint, therefore it is also the optimal solution to the original problem.

Let $w(\theta) \equiv \mathcal{U}(\theta+y(\theta),k(\theta))$. The local downward IC constraint (28) can be written as (for simplicity we assume differentiability)

$$w'(\theta) - \frac{\partial\mathcal{U}}{\partial c}\left(\theta+y(\theta),k\left(\theta\right)\right) \geq 0. \tag{30}$$

We also require that $k(.)$ is non-decreasing in θ:

$$k'(\theta) = i(\theta) \geq 0. \tag{31}$$

We denote $\gamma(\theta) \geq 0$ the multiplier on this constraint.

We write the Lagrangian of the "weakly relaxed problem" as

$$\begin{aligned}\mathcal{L}_C = &\int_{\underline{\theta}}^{\bar{\theta}} \mathcal{U}\left(\theta + y(\theta), k(\theta)\right) dF(\theta) \\ &+ \lambda \left(A - \int_{\underline{\theta}}^{\bar{\theta}} \left(y(\theta) + qk(\theta) \right) dF(\theta) \right) \\ &+ \int_{\underline{\theta}}^{\bar{\theta}} \xi(\theta) \left(w'(\theta) - \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k\left(\theta\right)\right) \right) d\theta \\ &+ \int_{\underline{\theta}}^{\bar{\theta}} \eta(\theta) \left(k(\theta) - k(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} i(\tilde{\theta}) d\tilde{\theta} \right) d\theta + \int_{\underline{\theta}}^{\bar{\theta}} \gamma(\theta) i(\theta) d\theta.\end{aligned}$$

Because of the constraints (30) and (31) , $\xi \geq 0$ and $\gamma \geq 0$.

Using the derivations similar to the ones in Appendix A and let $\mu(\theta) = -\xi'(\theta)$, we rewrite $\mathcal{L}_C$ as

$$\begin{aligned}\mathcal{L}_C = &\int_{\underline{\theta}}^{\bar{\theta}} \mathcal{U}\left(\theta + y(\theta), k(\theta)\right) \left(f(\theta) + \mu(\theta) \right) d\theta \\ &+ \lambda \left(A - \int_{\underline{\theta}}^{\bar{\theta}} \left(y(\theta) + qk(\theta) \right) f(\theta) d\theta \right) \\ &- \int_{\underline{\theta}}^{\bar{\theta}} \xi(\theta) \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k\left(\theta\right)\right) d\theta + \xi(\bar{\theta}) w(\bar{\theta}) - \xi(\underline{\theta}) w(\underline{\theta}) \\ &+ \int_{\underline{\theta}}^{\bar{\theta}} \eta(\theta) \left(\, k(\theta) - k(\underline{\theta}) \right) d\theta \;-\; \int_{\underline{\theta}}^{\bar{\theta}} i(\theta) \left(\int_{\theta}^{\bar{\theta}} \eta\left(\tilde{\theta}\right) d\tilde{\theta} + \gamma(\theta) \right) d\theta.\end{aligned}$$

We use shortcuts $\frac{\partial \mathcal{U}}{\partial c}(\theta), \frac{\partial^2 \mathcal{U}}{\partial c^2}(\theta)$ to write the F.O.Cs as follow.

F.O.C. in $y(\theta)$

$$\frac{\partial \mathcal{U}}{\partial c}(\theta) \left(\mu(\theta) + f(\theta) \right) - \lambda f(\theta) = \frac{\partial^2 \mathcal{U}}{\partial c^2}\left(\theta\right) \xi(\theta). \tag{32}$$

F.O.C. in $k(\theta)$

$$\frac{\partial \mathcal{U}}{\partial k}(\theta) \left(\mu(\theta) + f(\theta) \right) + \eta(\theta) - \lambda q f(\theta) = \frac{\partial^2 \mathcal{U}}{\partial c \partial k}\left(\theta\right) \xi(\theta). \tag{33}$$

F.O.C. in $i(\theta)$

$$\gamma(\theta) - \int_{\theta}^{\bar{\theta}} \eta\left(\tilde{\theta}\right) d\tilde{\theta} = 0. \tag{34}$$

F.O.C. in $k(\bar{\theta})$

$$\mathcal{U}_k\left(\bar{\theta}\right) \xi(\bar{\theta}) = 0. \tag{35}$$

F.O.C. in $y(\underline{\theta})$:

$$\mathcal{U}_c\left(\underline{\theta}\right) \xi(\underline{\theta}) = 0. \tag{36}$$

F.O.C. in $k(\underline{\theta})$:

$$-\mathcal{U}_k(\underline{\theta})\xi(\underline{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \eta(\theta)d\theta = 0. \tag{37}$$

Because $\mathcal{U}_k(\bar{\theta}) > 0$, (35) implies that $\xi(\bar{\theta}) = 0$. Similarly, because $\mathcal{U}_c(\underline{\theta}) > 0$, (36) implies $\xi(\underline{\theta}) = 0$.

Combining this result with (37) yields

$$\int_{\underline{\theta}}^{\bar{\theta}} \eta(\theta)d\theta = 0. \tag{38}$$

Therefore, from (34),

$$\gamma(\underline{\theta}) = 0. \tag{39}$$

Also from (34),

$$\gamma(\overline{\theta}) = 0. \tag{40}$$

Armed with these properties, Lemma 6 below show that in the optimal solution of the "weakly relaxed problem," the local incentive constraint is satisfied. Lemma 6 uses the following two Lemmas.

Lemma 4. *In an optimal solution to the "weakly relaxed problem", if $k(\theta)$ is constant over some interval $[\theta_1, \theta_2] \in [\underline{\theta}, \overline{\theta}]$ then $y(\theta)$ is constant over the same interval.*

Proof. Assume that $k(\theta) = k^*$ over $[\theta_1, \theta_2] \in [\underline{\theta}, \overline{\theta}]$. By downward incentive compatibility, $y(\theta)$ is non-decreasing over $[\theta_1, \theta_2]$.

We show the result in this lemma by contradiction. Assume that $y(\theta)$ is not constant over the same interval because y is continuous, there exists a non-degenerate subinterval $[\theta', \theta''] \in [\theta_1, \theta_2]$ such that $y(.)$ is strictly increasing over this interval.

In this interval

$$\begin{aligned} w'(\theta) &= \frac{d}{d\theta}\left(\mathcal{U}\left(\theta + y(\theta), k^*\right)\right) \\ &= \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k^*\right) + \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k^*\right)\frac{dy}{d\theta} \\ &> \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k^*\right). \end{aligned}$$

Therefore (30) does not bind, i.e., $\xi(\theta) = 0$ for $\theta \in [\theta', \theta'']$. Since $\mu = -\xi'$, $\mu(\theta) = 0$ for $\theta \in [\theta', \theta'']$. (32) then implies

$$\frac{\partial \mathcal{U}}{\partial c}(\theta + y(\theta), k^*) = \lambda$$

for $\theta \in [\theta', \theta'']$. Differentiate both sides with respect to θ, we have

$$\frac{\partial^2 \mathcal{U}}{\partial c^2}(\theta + y(\theta), k^*)\left(1 + \frac{dy}{d\theta}\right) = 0.$$

This is a contradiction since $\frac{\partial^2 \mathcal{U}}{\partial c^2} < 0$ and $\frac{dy}{d\theta} > 0$. □

Lemma 5. *In an optimal solution to the "weakly relaxed problem," for each $\theta^* \in \left(\underline{\theta}, \overline{\theta}\right)$, if $\xi(\theta^*) = 0$ then $\gamma(\theta^*) > 0$.*

Proof. We show this result by contradiction. Assume that $\gamma(\theta^*) = 0$.

We rewrite (32) as

$$\xi'(\theta) + \frac{\frac{\partial^2 \mathcal{U}}{\partial c^2}(\theta)}{\frac{\partial \mathcal{U}}{\partial c}(\theta)}\xi(\theta) = f(\theta)\left(1 - \frac{\lambda}{\frac{\partial \mathcal{U}}{\partial c}(\theta)}\right)$$

where $\frac{\partial \mathcal{U}}{\partial c}(\theta)$, $\frac{\partial^2 \mathcal{U}}{\partial c^2}(\theta)\frac{\partial^2 \mathcal{U}}{\partial c \partial k}(\theta)$ are shortcuts. Using the fact that $\xi(\underline{\theta}) = 0$, we obtain

$$\xi(\theta) = \exp\left(-\int_{\underline{\theta}}^{\theta} \frac{\frac{\partial^2 \mathcal{U}}{\partial c^2}(\tilde{\theta})}{\frac{\partial \mathcal{U}}{\partial c}(\tilde{\theta})} d\tilde{\theta}\right) g_1(\theta),$$

where

$$g_1(\theta) = \int_{\underline{\theta}}^{\theta} \exp\left(\int_{\underline{\theta}}^{\theta_1} \frac{\frac{\partial^2 \mathcal{U}}{\partial c^2}(\tilde{\theta})}{\frac{\partial \mathcal{U}}{\partial c}(\tilde{\theta})} d\tilde{\theta}\right) \left(1 - \frac{\lambda}{\frac{\partial \mathcal{U}}{\partial c}(\theta_1)}\right) dF(\theta_1).$$

Because $g \geq 0$, $g_1(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. In addition $g_1(\theta^*) = 0$. Therefore θ^* is a local minimum of g_1. Therefore, $g_1'(\theta^*) = 0$ and $g_1''(\theta^*) \geq 0$. By the definition of g_1, this is equivalent to,

$$1 - \frac{\lambda}{\frac{\partial \mathcal{U}}{\partial c}(\theta^*)} = 0$$

and

$$\frac{d}{d\theta}\left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial c}(\theta)}\right) \geq 0$$

and $\theta = \theta^*$. Equivalently, at $\theta = \theta^*$

$$\mathcal{X} = \frac{d}{d\theta}\left\{\frac{\partial \mathcal{U}}{\partial c}(\theta)\right\} \geq 0. \tag{41}$$

Similarly, we rewrite (33) as

$$\xi'(\theta) + \frac{\frac{\partial^2 \mathcal{U}}{\partial c \partial k}(\theta)}{\frac{\partial \mathcal{U}}{\partial k}(\theta)} \xi(\theta) = f(\theta)\left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta)} + \frac{\eta(\theta)}{\frac{\partial \mathcal{U}}{\partial k}(\theta) f(\theta)}\right).$$

Again, since $\xi(\underline{\theta}) = 0$, we have

$$\xi(\theta) = \exp\left(-\int_{\underline{\theta}}^{\theta} \frac{\frac{\partial^2 \mathcal{U}}{\partial c \partial k}(\tilde{\theta})}{\frac{\partial \mathcal{U}}{\partial k}(\tilde{\theta})} d\tilde{\theta}\right) g_2(\theta),$$

where

$$g_2(\theta) = \int_{\underline{\theta}}^{\theta} \exp\left(\int_{\underline{\theta}}^{\theta_1} \frac{\frac{\partial^2 \mathcal{U}}{\partial c^2}(\tilde{\theta})}{\frac{\partial \mathcal{U}}{\partial c}(\tilde{\theta})} d\tilde{\theta}\right) \left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta_1)} + \frac{\eta(\theta_1)}{\frac{\partial \mathcal{U}}{\partial k}(\theta_1) f(\theta_1)}\right) dF(\theta_1).$$

Because $g \geq 0$, $g_2 \geq 0$. In addition, $g_2(\theta^*) = 0$. Therefore θ^* is a local minimum of g_2. Thus, $g_2'(\theta^*) = 0$ and $g_2''(\theta^*) \geq 0$. By the definition of g_2, this is equivalent to

$$1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta^*)} + \frac{\eta(\theta^*)}{\frac{\partial \mathcal{U}}{\partial k}(\theta^*) f(\theta^*)} = 0 \tag{42}$$

and at $\theta = \theta^*$:

$$\frac{d}{d\theta}\left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta)} + \frac{\eta(\theta)}{\frac{\partial \mathcal{U}}{\partial k}(\theta) f(\theta)}\right) \geq 0. \tag{43}$$

Notice also that $\gamma \geq 0$ and $\gamma'(\theta) = -\eta(\theta)$ and $\gamma''(\theta) = -\eta'(\theta)$. In addition $\gamma(\theta^*) = 0$. Therefore θ^* is a local minimum of γ. Thus $\gamma'(\theta^*) = -\eta(\theta^*) = 0$ and $\gamma''(\theta^*) = -\eta'(\theta^*) \geq 0$. Plugging the first equality into (42) implies

$$1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta^*)} = 0.$$

Plugging the second inequality into (43) implies

$$\begin{aligned}
\frac{d}{d\theta}\left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta)} + \frac{\eta(\theta)}{\frac{\partial \mathcal{U}}{\partial k}(\theta) f(\theta)}\right) &= \frac{d}{d\theta}\left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta)}\right) + \frac{d}{d\theta}\left(\frac{\eta(\theta)}{\frac{\partial \mathcal{U}}{\partial k}(\theta) f(\theta)}\right) \\
&= \frac{d}{d\theta}\left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta)}\right) + \frac{1}{\frac{\partial \mathcal{U}}{\partial k}(\theta) f(\theta)} \frac{d}{d\theta}(\eta(\theta)) \\
&\geq 0. \qquad (44)
\end{aligned}$$

Since $\frac{d}{d\theta}(\eta(\theta)) \leq 0$, the inequality above implies that $\frac{d}{d\theta}\left(1 - \frac{\lambda q}{\frac{\partial \mathcal{U}}{\partial k}(\theta)}\right) \geq 0$ at $\theta = \theta^*$. Therefore,

$$\mathcal{Z} = \frac{d}{d\theta}\left\{\frac{\partial \mathcal{U}}{\partial k}(\theta^*)\right\} \geq 0. \tag{45}$$

However, (41) and (45) contradict the normality of c and k.

Indeed by total differentiation

$$\mathcal{Z} = \frac{\partial^2 \mathcal{U}}{\partial k \partial c}\frac{dc}{d\theta} + \frac{\partial^2 \mathcal{U}}{\partial k^2}\frac{dk}{d\theta}$$

and

$$\mathcal{X} = \frac{\partial^2 \mathcal{U}}{\partial c^2}\frac{dc}{d\theta} + \frac{\partial^2 \mathcal{U}}{\partial c \partial k}\frac{dk}{d\theta}.$$

So

$$\begin{bmatrix} \frac{dc}{d\theta} \\ \\ \frac{dk}{d\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathcal{U}(c,k)}{\partial c^2} & \frac{\partial^2 \mathcal{U}(c,k)}{\partial c \partial k} \\ \\ \frac{\partial^2 \mathcal{U}(c,k)}{\partial c \partial k} & \frac{\partial^2 \mathcal{U}(c,k)}{\partial k^2} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{X} \\ \\ \mathcal{Z} \end{bmatrix}.$$

Besides,

$$\frac{d}{d\theta}\left\{\mathcal{U}(c(\theta),k(\theta))\right\} = \begin{bmatrix} \frac{\partial \mathcal{U}}{\partial c} & \frac{\partial \mathcal{U}}{\partial k} \end{bmatrix} \begin{bmatrix} \frac{dc}{d\theta} \\ \\ \frac{dk}{d\theta} \end{bmatrix} = \frac{\partial \mathcal{U}(c(\theta),k(\theta))}{\partial c} > 0.$$

On the other hand

$$\begin{bmatrix} \frac{\partial \mathcal{U}}{\partial c} & \frac{\partial \mathcal{U}}{\partial k} \end{bmatrix} \begin{bmatrix} \frac{dc}{d\theta} \\ \\ \frac{dk}{d\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{U}}{\partial c} & \frac{\partial \mathcal{U}}{\partial k} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \mathcal{U}(c,k)}{\partial c^2} & \frac{\partial^2 \mathcal{U}(c,k)}{\partial c \partial k} \\ \\ \frac{\partial^2 \mathcal{U}(c,k)}{\partial c \partial k} & \frac{\partial^2 \mathcal{U}(c,k)}{\partial k^2} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{X} \\ \\ \mathcal{Z} \end{bmatrix} \leq 0,$$

since, by (26a)

$$\begin{bmatrix} \frac{\partial \mathcal{U}}{\partial c} & \frac{\partial \mathcal{U}}{\partial k} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \mathcal{U}(c,k)}{\partial c^2} & \frac{\partial^2 \mathcal{U}(c,k)}{\partial c \partial k} \\ \\ \frac{\partial^2 \mathcal{U}(c,k)}{\partial c \partial k} & \frac{\partial^2 \mathcal{U}(c,k)}{\partial k^2} \end{bmatrix}^{-1} < 0$$

and $\mathcal{X}, \mathcal{Z} \geq 0$. This is the desired contradiction. □

Given Lemma 4 and Lemma 5, it is relatively straightforward to show the main result.

Lemma 6. *In the optimal solution to the "weakly relaxed problem," for all* $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$,

$$w'(\theta) - \frac{\partial \mathcal{U}}{\partial c}\left(\theta + y(\theta), k\left(\theta\right)\right) = 0.$$

Proof. We show this result by contradiction. If there exists θ^* such that this is not true:

$$w'(\theta^*) > \frac{\partial \mathcal{U}}{\partial c}\left(\theta^* + y(\theta^*), k\left(\theta^*\right)\right). \tag{46}$$

Then $\xi(\theta^*) = 0$. By Lemma 5, $\gamma(\theta^*) > 0$, therefore by continuity $\gamma(\theta) > 0$ in some neighborhood of θ^*. So $k(\theta) = k^*$ in this neighborhood. By Lemma 4, $y(\theta) = y^*$ in this neighborhood. This however contradicts (46). □

We have established that the optimal solution to the "weakly relaxed problem" satisfies the local incentive compatibility constraint (envelope condition), therefore it is also an optimal solution to the original problem, (24).[33] This implies that the optimal solution to (24) is characterized by the F.O.C.s (32)-(40) with $\xi \geq 0$. With this property, we are now ready to prove Lemma 2.

Proof of Lemma 2, Part i. Assume by contradiction that the first property does not hold, i.e. there is pooling at the top. Let $[\theta^*, \bar{\theta}]$ denote the maximum pooling interval. If $\theta^* > \underline{\theta}$. By the definition of θ^*, $\gamma(\theta^*) = 0$. If $\theta^* = \underline{\theta}$, we also have $\gamma(\theta^*) = \gamma(\underline{\theta}) = 0$ by (39).

Evaluating (32) and (33) at $\bar{\theta}$, we have

$$\frac{\partial \mathcal{U}}{\partial c}(\bar{\theta}) \left(\mu(\bar{\theta}) + f(\bar{\theta})\right) - \lambda f(\bar{\theta}) = 0.$$

This implies $\left(\mu(\bar{\theta}) + f(\bar{\theta})\right) > 0$. And

$$\frac{\partial \mathcal{U}}{\partial k}(\theta) \left(\mu(\bar{\theta}) + f(\bar{\theta})\right) + \eta\left(\bar{\theta}\right) - \lambda q f(\theta) = 0.$$

Since $\gamma(\bar{\theta}) = 0$ and $\gamma(\theta) \geq 0$ for all θ, $\eta(\bar{\theta}) = -\gamma'(\bar{\theta}) \geq 0$. Therefore

$$\frac{\frac{\partial \mathcal{U}}{\partial k}(\bar{\theta})}{\frac{\partial \mathcal{U}}{\partial c}(\bar{\theta})} \leq q. \tag{47}$$

From (32) and (33) at θ^*

$$\xi(\theta^*)\left(\frac{\partial \mathcal{U}}{\partial c}(\theta^*)\frac{\partial^2 \mathcal{U}}{\partial c \partial k}(\theta^*) - \frac{\partial \mathcal{U}}{\partial k}(\theta^*)\frac{\partial^2 \mathcal{U}}{\partial c^2}(\theta^*)\right) = \lambda f(\theta^*)\left(\frac{\partial \mathcal{U}}{\partial k}(\theta^*) - q\frac{\partial \mathcal{U}}{\partial c}(\theta^*)\right) + \eta(\theta^*)\frac{\partial \mathcal{U}}{\partial c}(\theta^*).$$

Because $\gamma(\theta^*) = 0$ and $\gamma(\theta) \geq 0$ for all $\theta \geq \theta^*$, $\eta(\theta^*) = -\gamma'(\theta^*) \leq 0$. In addition we have $\xi(\theta^*) \geq 0$ and (26b) at θ^*, so

$$\frac{\frac{\partial \mathcal{U}}{\partial k}(\theta^*)}{\frac{\partial \mathcal{U}}{\partial c}(\theta^*)} \geq q. \tag{48}$$

Combining (47) and (48), we obtain:

$$\frac{\frac{\partial \mathcal{U}}{\partial k}(\theta^*)}{\frac{\partial \mathcal{U}}{\partial c}(\theta^*)} \geq \frac{\frac{\partial \mathcal{U}}{\partial k}(\bar{\theta})}{\frac{\partial \mathcal{U}}{\partial c}(\bar{\theta})}. \tag{49}$$

[33] It is standard to show that local incentive constraint and monotonicity implies global incentive constraint, e.g., Tirole (1988).

This is a contradiction since there is pooling over $\left[\theta^*, \bar{\theta}\right]$ and therefore

$$\frac{d}{d\theta}\left(\frac{\frac{\partial \mathcal{U}}{\partial k}(\theta)}{\frac{\partial \mathcal{U}}{\partial c}(\theta)}\right) = \frac{\partial}{\partial \theta}\left(\frac{\frac{\partial \mathcal{U}}{\partial k}(\theta)}{\frac{\partial \mathcal{U}}{\partial c}(\theta)}\right) > 0,$$

for $\theta \in \left[\theta^*, \overline{\theta}\right]$, which implies

$$\frac{\frac{\partial \mathcal{U}}{\partial k}(\theta^*)}{\frac{\partial \mathcal{U}}{\partial c}(\theta^*)} < \frac{\frac{\partial \mathcal{U}}{\partial k}(\bar{\theta})}{\frac{\partial \mathcal{U}}{\partial c}(\bar{\theta})},$$

contradicting (49). Therefore, the optimal contract is separating in a neighborhood of $\overline{\theta}$. So $\eta(\overline{\theta}) = -\gamma'(\overline{\theta}) = 0$. In addition $\xi(\overline{\theta}) = 0$. Equations (32) and (33) at $\overline{\theta}$ then imply that

$$\frac{\frac{\partial \mathcal{U}}{\partial k}\left(\overline{\theta} + y(\overline{\theta}), k(\overline{\theta})\right)}{\frac{\partial \mathcal{U}}{\partial c}\left(\overline{\theta} + y(\overline{\theta}), k(\overline{\theta})\right)} = q.$$

□

Proof of Lemma 2, Part ii. Consider $\theta^* \in \left(\underline{\theta}, \overline{\theta}\right)$. There are two cases:

Case 1: $\gamma(\theta^*) > 0$ then there is pooling at θ^*. Let θ^{**} denote the left most point, such that there is pooling from θ^{**} to θ^*. Formally

$$\theta^{**} = \inf\left\{\theta \in [\underline{\theta}, \theta^*] : \gamma(\theta_1) > 0 \text{ for all } \theta_1 \in (\theta, \theta^*)\right\}.$$

Then $\gamma(\theta^{**}) = 0$ and $\gamma(\theta) > 0$ for all $\theta \in (\theta^{**}, \theta^*)$ (this comes from definition of θ^{**} if $\theta^{**} > \underline{\theta}$. If $\theta^{**} = \underline{\theta}$, then this is also true since $\gamma(\underline{\theta}) = 0$). Consequently,

$$\gamma'(\theta^{**}) = -\eta(\theta^{**}) \geq 0.$$

First, we show that

$$\frac{\frac{\partial \mathcal{U}}{\partial k}\left(\theta^{**} + y(\theta^{**}), k(\theta^{**})\right)}{\frac{\partial \mathcal{U}}{\partial c}\left(\theta^{**} + y(\theta^{**}), k(\theta^{**})\right)} \geq q. \tag{50}$$

Indeed, from (32) and (33) at θ^{**}

$$\xi(\theta^{**})\left(\frac{\partial \mathcal{U}}{\partial c}(\theta^{**})\frac{\partial^2 \mathcal{U}}{\partial c \partial k}(\theta^{**}) - \frac{\partial \mathcal{U}}{\partial k}(\theta^{**})\frac{\partial^2 \mathcal{U}}{\partial c^2}(\theta^{**})\right)$$
$$= \lambda f(\theta^{**})\left(\frac{\partial \mathcal{U}}{\partial k}(\theta^{**}) - q\frac{\partial \mathcal{U}}{\partial c}(\theta^{**})\right) + \eta(\theta^{**})\frac{\partial \mathcal{U}}{\partial c}(\theta^{**}).$$

Together with $\xi(\theta^{**}) \geq 0$, $\eta(\theta^{**}) \leq 0$ and (26b) at θ^{**}, we obtain (50).

Now, since there is pooling over (θ^{**}, θ^*), and because of (26d) we have

$$\frac{\frac{\partial \mathcal{U}}{\partial k}\left(\theta^* + y(\theta^*), k(\theta^*)\right)}{\frac{\partial \mathcal{U}}{\partial c}\left(\theta^* + y(\theta^*), k(\theta^*)\right)} > \frac{\frac{\partial \mathcal{U}}{\partial k}\left(\theta^{**} + y(\theta^{**}), k(\theta^{**})\right)}{\frac{\partial \mathcal{U}}{\partial c}\left(\theta^{**} + y(\theta^{**}), k(\theta^{**})\right)} \geq q.$$

Case 2: $\gamma(\theta^*) = 0$. By Lemma 5 $\xi(\theta^*) > 0$.

Since $\gamma(\theta^*) = 0$ and $\gamma(\theta) \geq 0$ for all θ,

$$\gamma'(\theta^*) = -\eta(\theta^*) \geq 0.$$

From (32) and (33) at θ^*

$$\xi(\theta^*) \left(\frac{\partial \mathcal{U}}{\partial c}(\theta^*) \frac{\partial^2 \mathcal{U}}{\partial c \partial k}(\theta^*) - \frac{\partial \mathcal{U}}{\partial k}(\theta^*) \frac{\partial^2 \mathcal{U}}{\partial c^2}(\theta^*) \right)$$
$$= \lambda f(\theta^*) \left(\frac{\partial \mathcal{U}}{\partial k}(\theta^*) - q \frac{\partial \mathcal{U}}{\partial c}(\theta^*) \right) + \eta(\theta^*) \frac{\partial \mathcal{U}}{\partial c}(\theta^*).$$

Together with $\xi(\theta^*) > 0$, $\eta(\theta^*) \leq 0$ and (26b) at θ^*, we obtain

$$\frac{\frac{\partial \mathcal{U}}{\partial k}\left(\theta^* + y(\theta^*), k(\theta^*)\right)}{\frac{\partial \mathcal{U}}{\partial c}\left(\theta^* + y(\theta^*), k(\theta^*)\right)} > q.$$

□

C Proof of Proposition 1

Part i is a direct application of the general result in Appendix B to $\mathcal{U}(c, \tilde{A}) = u(c) + W(\tilde{A})$ as explained in the body of the paper.[34] In this appendix, we present the proof for Part ii. We observe that

$$\frac{d\Gamma^*(\theta)}{d\theta} = -1 - \frac{dy^*}{d\theta} - \hat{q}\frac{dk^*}{d\theta} - \frac{1}{R}\frac{dA^*}{d\theta}.$$

From the incentive constraint,

$$u'(\theta + y^*)\frac{dy^*}{d\theta} + v'(k^*)\frac{dk^*}{d\theta} + \beta V'(A^*)\frac{dA^*}{d\theta} = 0.$$

[34]Conditions (26) are satisfied because $\mathcal{U}$ is separable.

From the F.O.C. in k and A', we have

$$v'(k^*) = \hat{q}R\beta V'(A^*).$$

Plugging this back into the incentive constraint, we obtain

$$\hat{q}\frac{dk^*}{d\theta} + \frac{1}{R}\frac{dA^*}{d\theta} = -\hat{q}\frac{u'(\theta + y^*)}{v'(k^*)}\frac{dy^*}{d\theta}.$$

Therefore

$$\frac{d\Gamma^*(\theta)}{d\theta} = -1 - \frac{dy^*}{d\theta}\left(1 - \frac{\hat{q}u'(\theta + y^*)}{v'(k^*)}\right).$$

In Part i, we show that $1 - \frac{\hat{q}u'(\theta+y^*)}{v'(k^*)} = 0$ at $\theta = \bar{\theta}$, therefore, by continuity, $1 - \frac{\hat{q}u'(\theta+y^*)}{v'(k^*)}$ is close to 0 in a neighborhood of $\bar{\theta}$. Consequently, $\frac{d\Gamma^*(\theta)}{d\theta} < 0$ in that neighborhood.

Similarly, if the monotonicity constraint on y^* does not bind at $\underline{\theta}$, $1 - \frac{\hat{q}u'(\theta+y^*)}{v'(k^*)} = 0$ at $\theta = \underline{\theta}$, and $\frac{d\Gamma^*(\theta)}{d\theta} < 0$ in a neighborhood of $\underline{\theta}$. If the monotonicity constraint binds, then y^*, k^*, A^* are constraint in a neighborhood of $\underline{\theta}$. In this case $\frac{d\Gamma^*(\theta)}{d\theta} = -1 < 0$ in that neighborhood.

D Proof of Proposition 2

Let $H(x)$ and $G(z)$ denote the inverses of u' and v', respectively. First we derive the dynamics of x and z, captured in the phase diagram, Figure 3, from the original first order conditions (12). From the definition of x and z, we have:

$$c(\theta) = H(x(\theta))$$
$$k(\theta) = G(x(\theta)).$$

Notice that

$$u'(H(x)) = x$$

so

$$u''(H(x)) = \frac{1}{H'(x)}.$$

Because u is strictly concave $H' < 0$.

For later derivations, we also use the fact that

$$H''(x) = -\frac{u'''(H(x))}{u''(H(x))\left(u''(H(x))\right)^2} \geq 0, \tag{51}$$

since $u''' \geq 0$ and $u'' < 0$.

In addition, we use $\xi(\theta) = \int_\theta^{\bar{\theta}} \mu(\tilde{\theta}) d\tilde{\theta}$ and, thus, $\xi'(\theta) = -\mu(\theta)$.

With these results, we rewrite the original system (12) as

$$x(\theta)\left(f(\theta) - \xi'(\theta)\right) - \lambda f(\theta) = \frac{1}{H'(x(\theta))}\xi(\theta) \tag{52a}$$

and

$$z(\theta)\left(f(\theta) - \xi'(\theta)\right) - \lambda \hat{q} f(\theta) = 0. \tag{52b}$$

Plugging $(f(\theta) - \xi'(\theta))$ from (52b) into (52a), we obtain:

$$H'(x(\theta))\left(\frac{x(\theta)}{z(\theta)}\hat{q} - 1\right)\lambda f(\theta) = \xi(\theta).$$

Differentiate both sides with respect to θ, we obtain:

$$\begin{aligned} \xi'(\theta) = H'(x(\theta))\left(\frac{x(\theta)}{z(\theta)}\hat{q} - 1\right)\lambda f'(\theta) + H''(x(\theta))x'(\theta)\left(\frac{x(\theta)}{z(\theta)}\hat{q} - 1\right)\lambda f(\theta) \\ + H'(x(\theta))\left(\frac{x'(\theta)}{z(\theta)}\hat{q} - \frac{x(\theta)}{z(\theta)}\frac{z'(\theta)}{z(\theta)}\hat{q}\right)\lambda f(\theta). \end{aligned}$$

From (52b), we also have

$$\xi'(\theta) = f(\theta) - \frac{\lambda \hat{q} f(\theta)}{z(\theta)}.$$

Therefore

$$\begin{aligned} & H'(x(\theta))\left(\frac{x(\theta)}{z(\theta)}\hat{q} - 1\right)\lambda f'(\theta) + H''(x(\theta))x'(\theta)\left(\frac{x(\theta)}{z(\theta)}\hat{q} - 1\right)\lambda f(\theta) \\ & + H'(x(\theta))\left(\frac{x'(\theta)}{z(\theta)}\hat{q} - \frac{x(\theta)}{z(\theta)}\frac{z'(\theta)}{z(\theta)}\hat{q}\right)\lambda f(\theta) \\ & = f(\theta) - \frac{\lambda \hat{q} f(\theta)}{z(\theta)}. \end{aligned}$$

Dividing both sides by $f(\theta)$ and noting that $f'(\theta) = \psi f(\theta)$ and regroups the terms on

$x'(\theta)$ and $z'(\theta)$, we obtain

$$\left(H''(x(\theta))\left(\frac{x\left(\theta\right)}{z(\theta)}\hat{q}-1\right)+H'(x(\theta))\frac{\hat{q}}{z(\theta)}\right)\lambda x'(\theta)$$
$$-H'(x(\theta))\frac{x(\theta)}{z(\theta)}\frac{z'(\theta)}{z(\theta)}\lambda\hat{q}$$
$$=1-\frac{\lambda\hat{q}}{z(\theta)}-\psi H'(x(\theta))\left(\frac{x\left(\theta\right)}{z(\theta)}\hat{q}-1\right)\lambda.$$

Now the local incentive compatibility reads

$$u'(c(\theta))\left(c'(\theta)-1\right)+v'(k(\theta))k'(\theta)=0.$$

In terms of x and z, we have

$$x(\theta)\left(H'(x(\theta)x'(\theta)-1\right)+z(\theta)G'(z(\theta))z'(\theta)=0. \tag{53}$$

Therefore

$$x'(\theta)=\frac{x(\theta)-z(\theta)G'(z(\theta))z'(\theta)}{x(\theta)H'(x(\theta))}.$$

Plugging this expression for $x'(\theta)$ into the previous equation, we obtain:

$$\left(H''(x(\theta))\left(\frac{x\left(\theta\right)}{z(\theta)}\hat{q}-1\right)+H'(x(\theta))\frac{\hat{q}}{z(\theta)}\right)\lambda\frac{x(\theta)-z(\theta)G'(z(\theta))z'(\theta)}{x(\theta)H'(x(\theta))}$$
$$-H'(x(\theta))\frac{x(\theta)}{z(\theta)}\frac{z'(\theta)}{z(\theta)}\lambda\hat{q}$$
$$=1-\frac{\lambda\hat{q}}{z(\theta)}-\psi H'(x(\theta))\left(\frac{x\left(\theta\right)}{z(\theta)}\hat{q}-1\right)\lambda.$$

This can be simplified as:

$$\mathcal{A}\left(x(\theta),z(\theta)\right)z'(\theta)=\mathcal{B}\left(x(\theta),z(\theta)\right) \tag{54}$$

where

$$\mathcal{A}\left(x,z\right)=-\left(H''(x)\left(\frac{\hat{q}x}{z}-1\right)+H'(x)\frac{\hat{q}}{z}\right)\lambda\frac{zG'(z)}{xH'(x)}-H'(x)\frac{x}{z^2}\lambda\hat{q}$$

and

$$\mathcal{B}(x,z) = 1 - \frac{2\lambda\hat{q}}{z} + \left(1 - \frac{\hat{q}x}{z}\right)\left(\lambda\frac{H''(x)}{H'(x)} + \psi H'(x)\right).$$

Notice that over

$$z > \hat{q}x$$

and because $H'' \geq 0$, by (51), and $H' < 0$, we have

$$\mathcal{A}(x,z) > 0.$$

Using this property, we can rewrite (54) as

$$z'(\theta) = F^z(x,z) \equiv \frac{\mathcal{B}(x,z)}{\mathcal{A}(x,z)}.$$

Combining (53) and (54), we obtain

$$x' = F^x(x,z) \equiv \frac{1}{H'(x)} - \frac{zG'(z)}{xH'(x)}\frac{\mathcal{B}(x,z)}{\mathcal{A}(x,z)}.$$

We can rewrite the necessary conditions for x and z as

$$\left(x', z'\right) = F(x,z) = \left(F^x(x,z), F^z(x,z)\right),$$

where the expressions for F^x and F^z are given above.

Lemma 7. *Consider a fully separating optimal contracts, $(y(\theta), k(\theta))_{\theta\in[\underline{\theta},\bar{\theta}]}$. We must have*

$$v'(k(\underline{\theta})) \geq \lambda \geq v'\left(k\left(\bar{\theta}\right)\right).$$

Proof. Let $(x(\theta), z(\theta)) = (u'(c(\theta)), v'(k(\theta)))$. Because we assume that the contract is fully separating, we have $z(\theta) = \hat{q}x(\theta)$ when $\theta \in \{\underline{\theta}, \bar{\theta}\}$ and by Lemma 2, $z(\theta) > \hat{q}x(\theta)$ for all $\theta \in (\underline{\theta}, \bar{\theta})$. Therefore, $z'(\underline{\theta}) \geq \hat{q}x'(\underline{\theta})$ and $z'(\bar{\theta}) \leq \hat{q}x'(\bar{\theta})$. By the Intermediate Value Theorem, the exists $x^* \in [x(\bar{\theta}), x(\underline{\theta})]$ such that at $(x^*, z^*) = (x^*, \hat{q}x^*)$, we have

$$F^z(x^*, z^*) = \hat{q}F^x(x^*, z^*).$$

Now

$$F^z(x^*, z^*) = \frac{\mathcal{B}(x^*, z^*)}{\mathcal{A}(x^*, z^*)}$$
$$= \frac{1 - \frac{2\lambda\hat{q}}{z^*}}{-\hat{q}\lambda\frac{G'(z^*)}{x^*} - \lambda\hat{q}H'(x)\frac{1}{z^*}}$$

and

$$F^x(x^*, z^*) = \frac{1}{H'(x^*)} - \frac{z^* G'(z^*)}{x^* H'(x^*)} \frac{1 - \frac{2\lambda\hat{q}}{z^*}}{-\hat{q}\lambda\frac{G'(z^*)}{x^*} - \lambda\hat{q}H'(x)\frac{1}{z^*}}$$

After simplifications, we arrive at:

$$\frac{z^* - \lambda\hat{q}}{z^*} = \hat{q}(\hat{q}\lambda - z^*)\frac{G'(z^*)}{x^* H'(x^*)}$$

which implies $z^* = \lambda\hat{q}$ and $x^* = \lambda$.

Therefore,

$$z(\underline{\theta}) = \hat{q}x(\underline{\theta}) \geq \hat{q}\lambda \geq \hat{q}x(\overline{\theta}) = z(\overline{\theta}),$$

which corresponds to the desired inequality. □

Armed with these derivations, now we are ready to prove the result stated in Proposition 2.

Proof of Proposition 2. First we show that there exist $\underline{\theta}^*, \bar{\theta}^*$ such that the optimal contract is fully separating and

$$v'(k^*(\underline{\theta}^*)) = 2\lambda\hat{q}.$$

Indeed, for $z < 2\lambda\hat{q}$ (and $z > \hat{q}x$), from the expression for $\mathcal{B}(x, z)$ above, we have $\mathcal{B}(x, z) < 0$ if

$$\lambda\frac{H''(x)}{H'(x)} + \psi H'(x) \leq 0.$$

This inequality holds because of (51) and $H' < 0$ and $\psi \geq 0$. Therefore,

$$z' = F^z(x, z) = \frac{\mathcal{B}(x, z)}{\mathcal{A}(x, z)} < 0.$$

Starting from an arbitrary $\underline{\theta}^*$ and $(x(\underline{\theta}^*), z(\underline{\theta}^*)) = (2\lambda, 2\lambda\hat{q})$, and following the phase diagram - the red solid curve in Figure 3 - z is decreasing. Lengthy algebras shows that the points $(x,z) = (x,0), x \geq 0$ are singular and repulsive, therefore, the trajectory must intersect the diagonal $z = \hat{q}x$ at some $\overline{\theta}^* > \underline{\theta}^*$ and $x, z > 0$, i.e.,

$$(x(\overline{\theta}^*), z(\overline{\theta}^*)) = (x(\overline{\theta}^*), \hat{q}x(\overline{\theta}^*)) > 0.$$

The trajectory from $\underline{\theta}^*$ to $\overline{\theta}^*$ forms an optimal, fully separating contract.

In any optimal fully separating contract defined over an support $[\underline{\theta}, \bar{\theta}]$, we must have $z'(\underline{\theta}) \leq 0$ and $z(\underline{\theta}) = \hat{q}x(\underline{\theta})$. From the expression for $\mathcal{B}(x,z)$ above, this implies:

$$z(\underline{\theta}) = v'(k(\underline{\theta})) \leq 2\lambda\hat{q}.$$

In Lemma 7, we show that

$$v'\left(k\left(\underline{\theta}\right)\right) \geq \lambda\hat{q}.$$

Now, for each $\underline{z} \in [\lambda\hat{q}, 2\lambda\hat{q}]$, let $\Delta(\underline{z})$ denote the time (in θ) it takes to move along the phase diagram from $\left(\frac{\underline{z}}{\hat{q}}, \underline{z}\right)$ to reach the diagonal $z = \hat{q}x$ again (at some $\left(\frac{\overline{z}}{\hat{q}}, \overline{z}\right)$ with $\overline{z} \leq \lambda\hat{q}$). In addition, let

$$\bar{\Delta} = \max_{\underline{z} \in [\lambda\hat{q}, 2\lambda\hat{q}]} \Delta\left(\underline{z}\right).$$

Then for all separating optimal contracts, $\overline{\theta} - \underline{\theta} \leq \bar{\Delta}$. □

E Proof of Proposition 3

Proof of Proposition 3 Part i. Let $s = \frac{\overline{\theta} - \underline{\theta}}{2\lambda(c(\underline{\theta}))^2}$, we rewrite (21) as:

$$\frac{\log\left(1+s\right)\left(1+\sqrt{1+s}\right)}{\sqrt{s}} = 2\sqrt{2\lambda\left(\overline{\theta} - \underline{\theta}\right)}.$$

After lengthy algebras, we show that the left hand side is strictly increasing in s and is equal to 0 at $s = 0$ and to ∞ at $s = \infty$. So there exists a unique solution s^* to this equation.

Now, from the IC constraint, we have

$$\frac{1}{k(\theta)}k'(\theta) = \frac{1}{c(\theta)}\left(1 - \frac{1}{4\lambda c(\theta)}\right).$$

Therefore, in order for k to be increasing in θ, we require $4\lambda c(\theta) \geq 1$. Because c is increasing in θ, this is equivalent to $4\lambda c(\underline{\theta}) = 4\lambda q k(\underline{\theta}) \geq 1$. From the definition of s^*, $s^* \leq 8\lambda\left(\overline{\theta}-\underline{\theta}\right)$. Equivalently

$$\frac{\log\left(1+8\lambda\left(\overline{\theta}-\underline{\theta}\right)\right)\left(1+\sqrt{1+8\lambda\left(\overline{\theta}-\underline{\theta}\right)}\right)}{\sqrt{8\lambda\left(\overline{\theta}-\underline{\theta}\right)}} > 2\sqrt{2\lambda\left(\overline{\theta}-\underline{\theta}\right)}.$$

After lengthy algebras, we show that

$$\frac{\log(1+t)\left(1+\sqrt{1+t}\right)}{t}$$

is strictly decreasing in t, and it is equal to 2 at $t = 0$ and to 0 at $t = \infty$. Therefore, the above inequality is equivalent to

$$8\lambda(\overline{\theta}-\underline{\theta}) < t^*,$$

where t^* is uniquely determined by

$$\frac{\left(1+\sqrt{1+t}\right)\log\left(1+t\right)}{t} = 1. \tag{55}$$

After some algebra manipulation, it is easy to see that $t^* = \delta - 1$, where δ is defined in the Proposition. The numerical value of t^* is 11.3402.

The convexity of k^* and the concavity of y^* can be obtained easily by twice differentiating the closed form expressions for (18) and (19). Γ^* is strictly decreasing in θ since k^*, c^* are both strictly increasing in θ. □

Proof of Proposition 3 Part ii. As stated in the Proposition, we look for continuous allocations $k(.), y(.)$ pooling over $[\underline{\theta}, \theta^*]$ and separating over $[\theta^*, \bar{\theta}]$, together with the multipliers $\mu(.), \gamma(.), \eta(.)$ that satisfy the F.O.Cs (17) given in Section 4.

It is actually easier to look for the solution to an alternative system which corresponds to the F.O.Cs on the Lagrangian in which the monotonicity constraint is impose on $k(\theta)$ as in the static problem presented in Appendix B. Let $\tilde{\mu}, \tilde{\gamma}, \tilde{\eta}$ respectively denote the multipliers on the envelope condition, the condition relating k to k', and the condition that

$k' \geq 0$. We also use $\tilde{\xi} = \int_{\theta}^{\overline{\theta}} \tilde{\mu}(\tilde{\theta}) d\tilde{\theta}$ and $\tilde{\gamma}(\theta) = \int_{\theta}^{\bar{\theta}} \tilde{\eta}(\tilde{\theta}) d\theta$.

Using this alternative Lagrangian, the F.O.C. in $y(\theta)$ is:

$$\frac{1}{2\left(\theta + \underline{y}\right)}\left(\tilde{\mu}(\theta) + \bar{f}\right) - \lambda \bar{f} = -\frac{1}{2\left(\theta + \underline{y}\right)^2} \tilde{\xi}(\theta) \tag{56a}$$

and the F.O.C in k is:

$$\frac{1}{2\underline{k}}\left(\tilde{\mu}(\theta) + \bar{f}\right) + \tilde{\eta}(\theta) - \bar{f}\lambda\hat{q} = 0. \tag{56b}$$

Given $\tilde{\mu}, \tilde{\gamma}, \tilde{\eta}$, using direct calculations, we can verify that

$$\mu(\theta) \equiv \tilde{\mu}(\theta) + \tilde{\eta}(\theta) 2\underline{k},$$

and

$$\eta(\theta) \equiv -\tilde{\eta}(\theta)\frac{\underline{k}}{\theta + \underline{y}} - \frac{\underline{k}}{(\theta + \underline{y})^2}\int_{\theta}^{\bar{\theta}} \tilde{\eta}(\tilde{\theta}) d\tilde{\theta} = \frac{d}{d\theta}\left(\frac{\underline{k}}{\theta + \underline{y}}\int_{\theta}^{\bar{\theta}} \tilde{\eta}(\tilde{\theta})\right), \tag{57}$$

solve (17).

Now, going back to (56). First, this system implies that $\tilde{\mu}, \tilde{\eta}$ are continuous in θ. The first equation, (56a) is equivalent to:

$$\frac{1}{2(\theta + \underline{y})}\bar{f} - \lambda\bar{f} = \frac{d}{d\theta}\left(\frac{1}{2(\theta + y)}\tilde{\xi}(\theta)\right)$$

So

$$\frac{1}{2(\theta + \underline{y})}\tilde{\xi}(\theta) = \frac{\bar{f}}{2}\left(\log\left(\theta + \underline{y}\right) - \log\left(\underline{\theta} + \underline{y}\right)\right) - \lambda\bar{f}\left(\theta - \underline{\theta}\right)$$

or

$$\tilde{\xi}(\theta) = \bar{f}\left(\theta + \underline{y}\right)\left(\log\left(\theta + \underline{y}\right) - \log\left(\underline{\theta} + \underline{y}\right)\right) - 2\lambda\bar{f}\left(\theta - \underline{\theta}\right)\left(\theta + \underline{y}\right) \tag{58}$$

and

$$\tilde{\mu}(\theta) = -\tilde{\xi}'(\theta) = -\bar{f}\left(\log\left(\theta + \underline{y}\right) - \log\left(\underline{\theta} + \underline{y}\right)\right) - \bar{f} + 2\lambda\bar{f}\left(\theta + \underline{y}\right) + 2\lambda\bar{f}\left(\theta - \underline{\theta}\right)$$

At $\theta = \theta^*$, $\tilde{\eta}(\theta^*) = 0$, so the second equation (56b) yields:

$$\tilde{\mu}(\theta^*) + \bar{f} - 2\bar{f}\lambda\hat{q}\underline{k} = 0.$$

Therefore, from the earlier expression for $\tilde{\mu}(\theta)$:

$$-\log\left(c^*\right)+\log\left(c^*-\left(\theta^*-\underline{\theta}\right)\right)+2\lambda c^*+2\lambda\left(\theta^*-\underline{\theta}\right)-2\lambda\hat{q}\underline{k}=0, \tag{59}$$

where $c^*=\underline{y}+\theta^*$.

Integrating (56b) from $\underline{\theta}$ to θ^*, we obtain

$$\frac{1}{2\underline{k}}\left(\tilde{\xi}(\underline{\theta})-\tilde{\xi}(\theta^*)+\bar{f}\left(\theta^*-\underline{\theta}\right)\right)+\tilde{\gamma}(\underline{\theta})-\tilde{\gamma}(\theta^*)-\bar{f}\lambda\hat{q}\left(\theta^*-\underline{\theta}\right)=0.$$

Given that $\tilde{\gamma}(\theta^*)=\tilde{\gamma}(\underline{\theta})=0$ and $\tilde{\xi}(\underline{\theta})=0$ and from the earlier expression for $\tilde{\xi}$, this is equivalent to:

$$-c^*\left(\log c^*-\log\left(c^*-\left(\theta^*-\underline{\theta}\right)\right)\right)+2\lambda\left(\theta^*-\underline{\theta}\right)c^*+\left(\theta^*-\underline{\theta}\right)-2\lambda\hat{q}\underline{k}\left(\theta^*-\underline{\theta}\right)=0 \tag{60}$$

Lastly, using the assumption that the contract is fully separating over $[\theta^*,\bar{\theta}]$ and, thus there is no distortion at the top, we obtain:

$$\begin{aligned}&\log\left(\left(c^*\right)^2+\frac{\overline{\theta}-\theta^*}{2\lambda}\right)\\&=\log(\hat{q}\underline{k})+4\lambda\sqrt{\left(c^*\right)^2+\frac{\overline{\theta}-\theta^*}{2\lambda}}-4\lambda c^*+\log c^*.\end{aligned} \tag{61}$$

We show that there is a solution $\left(\underline{y},\underline{k},\theta^*\right)$ to (59)-(61).

From the first two equations, (59) and (60), we have

$$c^*\frac{-\log c^*+\log\left(c^*-\left(\theta^*-\underline{\theta}\right)\right)}{\theta^*-\underline{\theta}}+2\lambda c^*+1=-\log\left(c^*\right)+\log\left(c^*-\left(\theta^*-\underline{\theta}\right)\right)+2\lambda c^*+2\lambda\left(\theta^*-\underline{\theta}\right).$$

Let $\zeta=\frac{\theta^*-\underline{\theta}}{c^*}$, this expression simplifies to:

$$\frac{\log(1-\zeta)}{\zeta}+1=\log(1-\zeta)+2\lambda c^*\zeta$$

So c^* is a function of ζ

$$c^*=\hat{c}(\zeta)\equiv\frac{\frac{\log(1-\zeta)}{\zeta}-\log(1-\zeta)+1}{2\lambda\zeta}.$$

Using Taylor expansion

$$\hat{c}(\zeta) = \frac{1}{2\lambda} \sum_{n=0}^{\infty} \frac{\zeta^n}{(n+1)(n+2)}, \tag{62}$$

which is strictly increasing in ζ.

Notice that $\lim_{\zeta \to 0} \hat{c}(\zeta) = \frac{1}{4\lambda}$ and $\lim_{\zeta \to 1} \hat{c}(\zeta) = \frac{1}{2\lambda}$. Because $8\lambda(\overline{\theta} - \underline{\theta}) > t^* > 4$,

$$1 < 2\lambda(\overline{\theta} - \underline{\theta}).$$

So

$$\lim_{\zeta \to 1} \zeta \hat{c}(\zeta) = \frac{1}{2\lambda} < (\overline{\theta} - \underline{\theta}),$$

which implies at $\zeta = 1$, $\theta^* = \underline{\theta} + \lim_{\zeta \to 1} \zeta \hat{c}(\zeta) < \overline{\theta}$.

Solving for $q\underline{k}$ from (59) and (61), we obtain another equation:

$$\left((c^*)^2 + \frac{\overline{\theta} - \underline{\theta}}{2\lambda} - \frac{c^* \zeta}{2\lambda} \right) \exp\left(-4\lambda \sqrt{(c^*)^2 + \frac{\overline{\theta} - \underline{\theta}}{2\lambda} - \frac{c^* \zeta}{2\lambda}} \right) \frac{\exp(4\lambda c^*)}{c^*} = \frac{\log(1-\zeta)}{2\lambda} + c^* + c^* \zeta. \tag{63}$$

From the earlier expression for c^*, (62), this corresponds to one equation, and one unknown in ζ. Let $\Delta(\zeta)$ denote the difference between the RHS and LHS of this equation.

We will show that $\Delta(0) < 0$ and $\Delta(1^-) > 0$.

It is easy to show that $\lim_{\zeta \uparrow 1} \Delta(\zeta) = +\infty > 0$, since $\lim_{\zeta \uparrow 1} \hat{c}(\zeta) = \frac{1}{2\lambda} < \infty$ and $\lim_{\zeta \uparrow 1} \log(1-\zeta) = -\infty$.

Now at $\zeta = 0$, $\hat{c}(0) = \frac{1}{4\lambda}$, so

$$\Delta(0) = \left((\hat{c}(0))^2 + \frac{\overline{\theta} - \underline{\theta}}{2\lambda} \right) \exp\left(-4\lambda \sqrt{(\hat{c}(0))^2 + \frac{\overline{\theta} - \underline{\theta}}{2\lambda}} \right) \frac{\exp(4\lambda \hat{c}(0))}{\hat{c}(0)} - \hat{c}(0).$$

Let $\tilde{s} = \frac{\overline{\theta} - \underline{\theta}}{2\lambda (\hat{c}(0))^2}$, after lengthy algebra, we desired inequality $\Delta(0) < 0$ as:

$$\frac{\log(1+\tilde{s})\left(1 + \sqrt{1+\tilde{s}}\right)}{\sqrt{\tilde{s}}} < 2\sqrt{2\lambda\left(\overline{\theta} - \underline{\theta}\right)}. \tag{64}$$

Consider the solution to the relaxed problem in Part 1 of this Proposition, $c_R(\theta)$ and $k_R(\theta)$

(k_R might not be increasing). At $\underline{\theta}$,

$$s^* = \frac{\overline{\theta} - \underline{\theta}}{2\lambda \left(c_R(\underline{\theta})\right)^2}$$

and

$$\frac{\log\left(1+s^*\right)\left(1+\sqrt{1+s^*}\right)}{\sqrt{s^*}} = 2\sqrt{2\lambda\left(\overline{\theta} - \underline{\theta}\right)}.$$

As shown in the first part of this Proposition, the RHS of (64) is strictly increasing in $\tilde{s}$. Therefore (64) is equivalent to

$$\tilde{s} < s^*$$

or, by definition of $\tilde{s}$ and s^*,

$$\hat{c}(0) = \frac{1}{4\lambda} > c_R(\underline{\theta}).$$

Indeed this is the case since $8\lambda(\overline{\theta} - \underline{\theta}) > t^*$.

So there exists $\zeta^* \in (0,1)$ such that $\Delta(\zeta^*) = 0$. We determine $\theta^*, \underline{y}$ and $\underline{k}$ as $\theta^* = \hat{c}(\zeta^*)\zeta^* + \underline{\theta}$, $\underline{y} = \hat{c}(\zeta^*) - \theta^*$ and $\underline{k}$ is a function of c^* and ζ^* as in either (59) or (61). It is easy to verify that $\left(\theta^*, \underline{y}, \underline{k}\right)$ solves (59)-(61).

We can verify that for all $\theta \in (\underline{\theta}, \theta^*)$:

$$\tilde{\gamma}(\theta) = \frac{1}{2\underline{k}}\left(-\tilde{\xi}(\theta) + \bar{f}\left(\theta - \underline{\theta}\right)\right) - \bar{f}\lambda\hat{q}\left(\theta - \underline{\theta}\right) > 0.$$

We first show by contradiction that there is no local minimum of $\tilde{\gamma}$ in $(\underline{\theta}, \theta^*)$. Assume by contradiction that there exists a local minimum $\tilde{\theta} \in (\underline{\theta}, \theta^*)$. Then $\tilde{\gamma}'(\tilde{\theta}) = 0$ and $\tilde{\gamma}''(\tilde{\theta}) \geq 0$.

From the expression for $\tilde{\gamma}$:

$$\tilde{\gamma}''(\theta) = -\frac{1}{2\underline{k}}\tilde{\xi}''(\theta) = -\frac{1}{2\underline{k}}\bar{f}\left(\frac{1}{\theta + \underline{y}} - 4\lambda\right)$$

is increasing in θ. Therefore $\tilde{\gamma}''(\theta) > \tilde{\gamma}''(\tilde{\theta}) \geq 0$ for $\theta \in \left(\tilde{\theta}, \theta^*\right)$. From the definition of θ^*, $\tilde{\gamma}'(\theta^*) = -\tilde{\eta}(\theta^*) = 0$. So $\tilde{\gamma}'(\tilde{\theta}) < 0$ which contradicts the property that $\tilde{\gamma}'(\tilde{\theta}) = 0$. So $\tilde{\gamma}$ does not have a local minimum in $(\underline{\theta}, \theta^*)$.

At $\underline{\theta}$, $\tilde{\gamma}(\underline{\theta}) = 0$ and by construction, at θ^*, $\tilde{\gamma}(\theta^*) = 0$. So $\tilde{\gamma} \geq 0$ for all $\theta \in [\underline{\theta}, \theta^*]$. In addition, $\tilde{\gamma}(\theta) > 0$ for all $\theta \in (\underline{\theta}, \theta^*)$ (otherwise, $\tilde{\gamma}$ would have a local minimum in $(\underline{\theta}, \theta^*)$).

From the construction of $\eta(\theta)$ in (57), we also have

$$\gamma(\theta) = -\int_{\theta}^{\bar{\theta}} \eta(\tilde{\theta}) d\tilde{\theta} = \frac{\underline{k}}{\theta + \underline{y}} \tilde{\gamma}(\theta) > 0.$$

Given $\theta^*, \underline{y}, \underline{k}$, we determine the allocation over $\left[\theta^*, \overline{\theta}\right]$ using the derivation in Section 4:

$$c(\theta) = \sqrt{(c^*)^2 + \frac{\theta - \theta^*}{2\lambda}}$$

and

$$\begin{aligned} \log(k(\theta)) = \log(\underline{k}) \\ &+ 4\lambda\sqrt{(c^*)^2 + \frac{\theta - \theta^*}{2\lambda}} - 4\lambda c^* \\ &- \frac{1}{2}\log\left((c^*)^2 + \frac{\theta - \theta^*}{2\lambda}\right) + \log\left(c^*\right). \end{aligned} \tag{65}$$

Since $c^* = \hat{c}(\zeta^*) > \hat{c}(0) = \frac{1}{4\lambda}$, $k(.)$ is strictly increasing over $\left(\theta^*, \overline{\theta}\right)$.

All the first order conditions are satisfied, therefore $\{y^*, k^*\}$ as constructed above is an optimal contract. The results on the shape of the allocation and distortion are derived similarly to the fully separating case in Part i.

□