The Folk Theorem in Dynastic Repeated Games: Technical Addendum

T.1. Notation

Point of Notation T.1.1: Abusing the notation we established for the standard repeated game, we adopt the following notation for continuation payoffs in the dynastic repeated game. Let an assessment (g, µ, Φ) be given.

Recall that we denote by \( v_1^i(g, µ|m_i, x^t, \Phi_i^{IB}) \) the continuation payoff to player ⟨i, t⟩ given the profile (g, µ), after he has received message \( m_i \), has observed the realization \( x^t \), and given that his beliefs over the \( n−1 \)-tuple \( m_{i−1}^t \) are \( \Phi_i^{IB} \) (see footnote 47). In view of our discussion at the beginning of Section 4, it is clear that the only component of the system of beliefs \( Φ \) that is relevant to define this continuation payoff is in fact \( Φ_i^{IB} \). Our discussion there also implies that the argument \( m_i \) is redundant once \( Φ_i^{IB} \) has been specified. We keep it in our notation since it helps streamline some of the arguments below.

We let \( v_i^j(g, µ|m_i, x^t, a^t, y^j, \Phi_i^{JE}) \) denote the continuation payoff (viewed from the beginning of period \( t + 1 \)) to player ⟨j, t⟩ given the profile \( g, µ \), after he has received message \( m_i^t \), has observed the triple \( (x^t, a^t, y^j) \), and given that his beliefs over the \( n−1 \)-tuple \( m_{i−1}^t \) are given by \( \Phi_i^{JE} \) . In view of our discussion at the beginning of Section 4, it is clear that once \( \Phi_i^{JE} \) has been specified, the arguments \( (m_i^t, x^t, a^t, y^j) \) are redundant in determining the end-of-period continuation payoff to player ⟨j, t⟩. Whenever this does not cause any ambiguity (about \( Φ_i^{JE} \)) we will write \( v_i^j(g, µ|\Phi_i^{JE}) \) instead of \( v_i^j(g, µ|m_i^t, x^t, a^t, y^j, \Phi_i^{JE}) \).

As we noted in the text all continuation payoffs clearly depend on \( δ \) as well. To keep notation down this dependence will be omitted whenever possible.

Point of Notation T.1.2: We will abuse our notation for \( Φ_i^{IB}(\cdot), Φ_i^{JE}(\cdot) \) and \( Φ_i^{IR}(\cdot) \) slightly in the following way. We will allow events of interest and conditioning events to appear as arguments of \( Φ_i^{IB}, Φ_i^{JE} \) and \( Φ_i^{IR} \), to indicate their probabilities under these distributions.

So, for instance when we write \( Φ_i^{IB}(m_{i−1}^t = (z, \ldots, z)|m_i^t) = c \) we mean that according to the beginning-of-period beliefs of player ⟨i, t⟩, after observing \( m_i^t \), the probability that \( m_{i−1}^t \) is equal to the \( n−1 \)-tuple \( (z, \ldots, z) \) is equal to \( c \).

Point of Notation T.1.3: Whenever the profile \( (g, µ) \) is a profile of completely mixed strategies, the beliefs \( Φ_i^{IB}(\cdot), Φ_i^{JE}(\cdot) \) and \( Φ_i^{IR}(\cdot) \) are of course entirely determined by what player ⟨i, t⟩ observes and by \( (g, µ) \) using Bayes’ rule. In this case we will allow the pair \( (g, µ) \) to appear as a “conditioning event.”

So, for instance, \( Φ_i^{IB}(m_{i−1}^t|g, µ) \) is the probability of the \( n−1 \)-tuple \( m_{i−1}^t \), after \( m_i^t \) has been received, obtained from the completely mixed profile \( g, µ \) via Bayes’ rule. Events may appear as arguments in this case as well, consistently with our Point of Notation T.1.2 above.

Moreover, since the completely mixed pair \((g, µ)\) determines the probabilities of all events, concerning for instance histories, messages of previous cohorts and the like, we will use the notation \( \Pr \) to indicate such probabilities, using the pair \((g, µ)\) as a conditioning event.

So, given any two events \( L \) and \( J \), the notation \( \Pr(L|J, g, µ) \) will indicate the probability of event \( L \), conditional on event \( J \), as determined by the completely mixed pair \((g, µ)\) via Bayes’ rule.

T.2. A Preliminary Result

Definition T.2.1: Consider the dynastic repeated game described in full in Section 3. Now consider the dynastic repeated game obtained from this when we restrict the message space of player ⟨i, t⟩ to be \( M_{i++}^t \subseteq H_i^{t+1} \), with all other details unchanged.

We call this the restricted dynastic repeated game with message spaces \( \{M_{i}^t\}_{i,t \geq 1} \). For any given \( δ \in (0,1) \), \( \vec{x} \) and \( \vec{y} \), we denote by \( \mathcal{G}^D(δ, \vec{x}, \vec{y}, \{M_i^t\}_{i,t \geq 1}) \) the set of SE strategy profiles, while we write \( \mathcal{E}^D(δ, \vec{x}, \vec{y}, \{M_i^t\}_{i,t \geq 1}) \) for the set of SE payoff profiles of this restricted dyfastic repeated game with restricted message spaces.
Lemma T.2.1: Let any $\delta \in (0,1)$, $\hat{x}$ and $\hat{y}$ be given. Consider now any restricted dyadic repeated game with message spaces $(M_i^t)_{t \geq 1}$. Then $\mathcal{E}^D(\delta, \hat{x}, \hat{y}, \{M_i^t\}_{t \geq 1}) \subseteq \mathcal{E}^D(\delta, \hat{x}, \hat{y})$.

Proof: Let a profile $(g^*, \mu^*) \in \mathcal{G}^D(\delta, \hat{x}, \hat{y}, \{M_i^t\}_{t \geq 1})$ with associated beliefs $\Phi^*$ be given. To prove the statement, we proceed to construct a new profile $(g^{**}, \mu^{**}) \in \mathcal{G}^D(\delta, \hat{x}, \hat{y})$ and associated beliefs $\Phi^{**}$ that are consistent with $(g^{**}, \mu^{**})$, and which gives every player the same payoff as $(g^*, \mu^*)$.

Denote a generic element of $M_i^t$ by $z_i^t$. Since $M_i^t \subseteq H^i$, we can partition $H^i$ into $\{M_i^t\}$ non-empty mutually exclusive subsets, and make each of these subsets correspond to an element $z_i^t$ of $M_i^t$. In other words, we can find a map $\rho_i^t : M_i^t \rightarrow 2^{H^i}$ such that $\rho_i^t(z_i^t) \neq \emptyset$ for all $z_i^t \in M_i^t$, and $\rho_i^t(z_i^t) \cap \rho_i^t(z_i'^t) = \emptyset$ whenever $z_i^t \neq z_i'^t$, and $\bigcup_{z_i^t \in M_i^t} \rho_i^t(z_i^t) = H^i$.

We can now describe how the profile $(g^{**}, \mu^{**})$ is obtained from the given $(g^*, \mu^*)$. We deal first with the action stage. For any player $(i,t)$, and any $z_i^t \in M_i^t$, set

$$g_i^{t**}(m_i^t, x) = g_i^{t*}(z_i^t), \quad \forall m_i^t \in \rho_i^t(z_i^t) \tag{T.2.1}$$

At the message stage, for any player $(i,t)$, any $(z_i^t, x^t, a^t, y^t)$, any $m_i^t \in \rho_i^t(z_i^t)$, and any $z_i^{t+1} \in \text{Supp}(\mu_i^{t*}(z_i^t, x^t, a^t, y^t))$, set

$$\mu_i^{t**}(m_i^{t+1}, m_i^t, x^t, a^t, y^t) = \frac{1}{\|\rho_i^{t+1}(z_i^{t+1})\|} \mu_i^{t*}(z_i^{t+1}, m_i^t, x^t, a^t, y^t), \quad \forall m_i^{t+1} \in \rho_i^{t+1}(z_i^{t+1}) \tag{T.2.2}$$

Next, we describe $\Phi^{**}$, starting with the beginning-of-period beliefs. For any player $(i,t)$, any $z_i^t \in M_i^t$ and any $z_{i-1}^t \in M_{i-1}^t$, set

$$\Phi_i^{t**}(m_{i-1}^t | m_i^t) = \frac{\Phi_i^{t*}(z_{i-1}^t | z_i^t)}{\|\rho_j^{t+1}(z_j^{t+1})\|}, \quad \forall m_j^t \in \rho_j^t(z_j^t), \quad \forall m_{i-1}^t \in \Pi_{j \neq i} \rho_j^t(z_j^t) \tag{T.2.3}$$

Similarly, concerning the end-of-period beliefs, for any player $(i,t)$, any $(z_i^t, x^t, a^t, y^t)$ and any $z_{i-1}^t \in M_{i-1}^t$, set

$$\Phi_i^{t**}(m_{i-1}^{t+1} | m_i^t, x^t, a^t, y^t) = \frac{\Phi_i^{t*}(z_{i-1}^{t+1} | z_i^t, x^t, a^t, y^t)}{\|\rho_j^{t+1}(z_j^{t+1})\|}, \quad \forall m_j^t \in \rho_j^t(z_j^t), \quad \forall m_{i-1}^{t+1} \in \Pi_{j \neq i} \rho_j^{t+1}(z_j^{t+1}) \tag{T.2.4}$$

Since the profile $(g^*, \mu^*)$ is sequentially rational given $\Phi^*$, it is immediate from (T.2.1), (T.2.2), (T.2.3) and (T.2.4) that the profile $(g^{**}, \mu^{**})$ is sequentially rational given $\Phi^{**}$, and we omit further details of the proof of this claim.

Of course, it remains to show that $(g^{**}, \mu^{**}, \Phi^{**})$ is a consistent assessment.

Let $(g_e^*, \mu_e^*)$ be parameterized completely mixed strategies which converge to $(g^*, \mu^*)$ and give rise, in the limit as $\varepsilon \rightarrow 0$, to beliefs $\Phi^*$ via Bayes’ rule.

Given any $\varepsilon > 0$, let $(g_e^*\varepsilon, \mu_e^*\varepsilon)$ be a profile of completely mixed strategies obtained from $(g_e^*, \mu_e^*)$ exactly as in (T.2.1) and (T.2.2).

We start by verifying the consistency of the beginning-of-period beliefs. Observe that for any given $z^t = (z_i^t, z_{i-1}^t)$, from (T.2.2) we know that whenever $m^t = (m_i^t, m_{i-1}^t) \in \Pi_{j \in T} \rho_j^t(z_j^t)$,

$$\Pr(m_i^t, m_{i-1}^t | g_e^*\varepsilon, \mu_e^*\varepsilon) = \frac{\Pr(z_i^t, z_{i-1}^t | g_e^*, \mu_e^*)}{\|\rho_j^t(z_j^t)\|^T} \tag{T.2.5}$$
Similarly, using (T.2.2) again we know that whenever \( m_i^t \in \rho_i^t(z_i^t) \)

\[
\Pr(m_i^t | g_{i}^{**}, \mu_{i}^{**}) = \frac{\Pr(z_i^t | g_{i}^{**}, \mu_{i}^{**})}{\| \rho_i^t(z_i^t) \|}.
\] (T.2.6)

Taking the ratio of (T.2.5) and (T.2.6) and taking the limit as \( \varepsilon \to 0 \) now yields that for any any \( z_i^t \in M_i^t \) and any \( z_i^t \in M_i^t \)

\[
\lim_{\varepsilon \rightarrow 0} \Phi_i^{tB^{**}}(m_i^t | m_i^t, g_{i}^{**}, \mu_{i}^{**}) = \frac{\Phi_i^{tB^{**}}(z_i^t | z_i^t)}{\Pi_j \| \rho_j^t(z_j^t) \|} \quad \forall m_i^t \in \rho_i^t(z_i^t), \quad \forall \mu_i^t \in \Pi_j \rho_j^t(z_j^t).
\] (T.2.7)

Hence we have shown that the beginning-of-period beliefs as in (T.2.3) are consistent with \((g^{**}, \mu^{**})\).

The proof that the end-of-period beliefs as in (T.2.4) are consistent with \((g^{**}, \mu^{**})\) runs along exactly the same lines, and we omit the details. □

### T.3. Proof of Theorem 1: Preliminaries

**Lemma T.3.1:** Let \((g, \mu)\) be a strategy profile in the dynastic repeated game and assume \(\mu\) is truthful according to Definition 2.

Then there exists a system of beliefs \(\Phi\) that is consistent with \((g, \mu)\) and such that for every \(i \in I, t \geq 0, m_i^t \in H^t, x^t \in X, a^t \in A\) and \(y^t \in Y\) we have that

\[
\Phi_i^{tB}[m_{i-1}^t, \ldots, m_i^t | m_i^t] = 1
\] (T.3.1)

and

\[
\Phi_i^{tE}[m_{i-1}^{t+1}, \ldots, m_i^t, x^t, a^t, y^t] = 1
\] (T.3.2)

In other words, \(\Phi\) is such that every player \((i, t)\) at the beginning of the period believes with probability one that all other players in his cohort have received the same message as he has, and at the end of the period believes that all other players in his cohort are sending the same (truthful and hence pure) message as he is.

**Proof:** We construct a sequence of completely mixed strategies in which deviations at the action stage are much more likely than deviations at the message stage. We parameterize the sequence of perturbations by a small positive number \(\varepsilon\), which will eventually be shrunk to zero.

Given \(\varepsilon\), the completely mixed strategy for player \((i, t)\) at the action stage is denoted by \(g_i^t\). Recall that \(g_i^t(m_i^t, x^t)\) is itself a mixed strategy in \(\Delta(A_i)\). Then \(g_i^t\) is given by

\[
g_i^t(m_i^t, x^t) = (1 - \varepsilon/(n+1)^{0.5}) g_i^0(m_i^t, x^t) + \varepsilon/(n+1)^{0.5} \nu(A_i)
\] (T.3.3)

Given \(\varepsilon\), the completely mixed strategy for player \((i, t)\) at the message stage is denoted by \(\mu_i^t\). Recall that \(\mu_i^t(m_i^t, x^t, a^t, y^t)\) is itself a mixed strategy in \(\Delta(H^{t+1})\). Then \(\mu_i^t\) is given by

\[
\mu_i^t(m_i^t, x^t, a^t, y^t) = (1 - \varepsilon) \mu_i^0(m_i^t, x^t, a^t, y^t) + \varepsilon \nu(H^{t+1})
\] (T.3.4)

In words, at the action stage, player \((i, t)\) deviates from \(g_i^t\) with probability \(\varepsilon/(n+1)^{0.5}\) and all deviations are equally likely. At the message stage, player \((i, t)\) deviates from \(\mu_i^t\) with probability \(\varepsilon\), again with all deviations equally likely. Denote by \((g_\varepsilon, \mu_\varepsilon)\) the profile of completely mixed strategies we have just described. Clearly, as \(\varepsilon \to 0\) the profile \((g_\varepsilon, \mu_\varepsilon)\) converges pointwise to \((g, \mu)\) as required.

\(^{T.1}\nu(\cdot)\) stands for the the uniform distribution. See also footnote 44.
Of course, to prove (T.3.1) it is enough to show that
\[ \lim_{\varepsilon \to 0} \Phi_{t_i}^{B}[m_{t_i}^\varepsilon] = \{m_i^1, \ldots, m_i^\varepsilon\} \mid m_i^t, g_i, \mu_i \} = 1 \quad (T.3.5) \]

Notice now that (T.3.5) follows almost immediately from the way we have defined the completely mixed profile \((g_i, \mu_i)\) in (T.3.3) and (T.3.4) above.

To see this, notice if \(m_j^t \neq m_i^t\) for some \(j \neq i\) then it must be that at least one player \(k \in I, \tau\) with \(\tau = 0, 1, \ldots, t - 1\) has “lied” his successor in the same dynasty. Given (T.3.4) this happens with a probability that is an infinitesimal in \(\varepsilon\) of order 1 or higher.\(^T.2\) This needs to be compared with the overall probability of observing \(m_i^t\). Clearly many paths of play could have generated this outcome. However, one way in which \(m_i^t\) can arise is certainly that the true history \(h^t\) is equal to \(m_i^t\) and that no player has ever deviated from truth-telling. In the worst case the true history being equal to \(m_i^t\) will involve all players \((j \in I, \tau)\) deviating from \(g_j^\tau\) in every \(\tau \leq t - 1\). Using (T.3.3) this is an infinitesimal in \(\varepsilon\) of order at most \(n \sum_{\tau=0}^{t-1} 1/\varepsilon^{n+1} < n/(n + 1)\). Hence, the overall probability of observing \(m_i^t\) cannot be lower than an infinitesimal of order \(n/(n + 1)\). Since \(1 > n/(n + 1)\), equation (T.3.5) now follows.\(^T.3\)

The proof of (T.3.2) is the exact analogue of the proof of (T.3.1) we have just given, and hence we omit the details. ■

### T.4. Proof of Theorem 1

Fix a \(\delta \in (0, 1)\), a \(\hat{x}\) and any profile \(g^* \in G^S(\delta, \hat{x})\). Then there exists a profile \(\mu^*\) of message strategies which are truthful in the sense of Definition 2 and such that \((g^*, \mu^*) \in G^D(\delta, \hat{x}, \tilde{g})\) for every finite random variable \(\tilde{g}\).

Since the strategies \(\mu^*\) are truthful, we know from Lemma T.3.1 that there is a system of beliefs \(\Phi^*\) that is consistent with \((g^*, \mu^*)\) for which (T.3.1) and (T.3.2) hold. We will show that beliefs \(\Phi^*\) support the strategies \((g^*, \mu^*)\) as an SE of the dynastic repeated game regardless of \(\hat{g}\). Indeed, notice that since \(\mu_i^*\) is truthful for every player \((i,t)\), we know from Definition 2 that all players in fact ignore the realization of the message-stage correlation device. Hence our argument below is trivially valid for any \(\hat{g}\).

We simply check that no player \((i,t)\) has an incentive to deviate either at the action or at the communication stage.

Consider player \((i,t)\) at the communication stage, after observing \((m_i^t, x^t, a^t, g^t)\). If he sends message \(m_i^{t+1} = (m_i^1, x^t, a^t)\) as prescribed by \(\mu_i^{g^*}\), given that his beliefs are \(\Phi_i^{E_{g^*}}\) his expected continuation payoff is
\[ E_{2t+1} v_i(g_i^*, m_i^1, x^t, a^t, \tilde{x}^{t+1}). \]
Notice that by construction we know that this continuation payoff is equal to \(E_{2t+1} v_i(g_i^*|h^{t+1}, \tilde{x}^{t+1})\) when we set \(h^{t+1} = (m_i^1, x^t, a^t)\), namely the expected continuation payoff to player \(i\) given \(g^*\) in the standard repeated game after history \(h^{t+1} = (m_i^t, x^t, a^t)\) has taken place, and before the realization of \(\tilde{x}^{t+1}\) has been observed.

We now need to check that player \((i,t)\) cannot gain by deviating and sending any other (mixed) message \(\phi_i^t \in \Delta(H^{t+1})\). Given that the strategies \(\mu^*\) are truthful and that his beliefs are \(\Phi_i^{E_{g^*}}\), his expected continuation payoff following such deviation clearly cannot be above \(\max_{g_i} E_{2t+1} v_i(g_i, g_i^*|h^{t+1}, \tilde{x}^{t+1})\) when we set \(h^{t+1} = (m_i^1, x^t, a^t)\). In other words it cannot exceed the maximum (by choice of \(g_i^*\)) expected continuation payoff that player \(i\) can achieve in the standard repeated game after history \(h^{t+1} = (m_i^t, x^t, a^t)\) given that all other players are playing according to \(g_j^*\). However, since \(g^* \in G^S(\delta, \hat{x})\) we know that \(E_{2t+1} v_i(g_i^*|h^{t+1}, \tilde{x}^{t+1}) = \max_{g_i} E_{2t+1} v_i(g_i, g_i^*|h^{t+1}, \tilde{x}^{t+1})\). Hence, we can conclude that no player \((i,t)\) cannot gain by deviating in this way.

\(^T.2\) Throughout, we use the words “infinitesimal in \(\varepsilon\) of order \(z^\varepsilon\)” to indicate any quantity that can be written as a constant times \(z^\varepsilon\). Similarly, we use the words “infinitesimal of order higher than \(z^\varepsilon\)” to mean any quantity that can be written as a constant times \(z^\varepsilon\), with \(z > \varepsilon\).

\(^T.3\) It is instructive to notice essentially the same argument we are following here is enough to show that in fact, upon receiving \(m_i^t\) player \((i,t)\) will assign probability one to the event that the true history of play is equal to \(m_i^t\). Formally this would be expressed as \(\lim_{\varepsilon \to 0} \Pr \{h^t = m_i^t \mid m_i^t, g_i, \mu_i\} = 1\).
Now consider player \( (i,t) \) at the action stage, after observing \( (m_t^i, x^i) \). If player \( (i,t) \) follows the prescription of \( g_t^* \) given that his beliefs are \( \Phi_t \), his expected continuation payoff is given by \( v_i(g_t^*|m_t^i, x^i) \). If he deviates to playing any other \( \sigma_i \in \Delta(A_i) \), given his beliefs, his expected continuation payoff is \( v_i(\sigma_i, g_t^*-i, g_t^*-i|m_t^i, x^i) \). Since \( g^* \in G^n(\delta, \tilde{x}) \), by (4) of Remark 1 we can then conclude that he cannot gain by deviating in this way.

### T.5. Proof of Theorem 2: Beliefs

**Definition T.5.1:** Using the notation of Definition A.1, let

\[
\hat{v} = \frac{1}{1-q} \sum_{\ell=2}^{\left| A_i \right|} p(a(\ell))u(a(\ell))
\]

and

\[
v' = qu(a') + (1-q)\hat{v}
\]

**Remark T.5.1:** Let \( \hat{v} \) and \( v' \) be as in Lemma T.5.1, then

\[
v^* = qu(a^*) + (1-q)\hat{v}
\]

and

\[
v^* - v' = q[u(a^*) - u(a')]
\]

so that using our assumptions about \( a^* \) and \( a' \)

\[
v^*_i - v'_i > 0 \quad \forall i \in I
\]

**Remark T.5.2:** Given that \( \alpha \in (0,1) \) is such that such that (A.3) of Definition A.3 holds, then simple algebra shows that the interval

\[
\left( \frac{\pi_i - u_i}{(1-\alpha)(v^*_i - v'_i)}, \frac{u_i(a') - u_i(a_i^*, a^*_{i-1})}{\alpha(v^*_i - v'_i)} \right)
\]

is not empty for every \( i \in I \).

**Definition T.5.2:** Using Remark T.5.2, for each \( i \in I \), define \( r_i \) to be a number in the interval in (T.5.6). Moreover, for each \( i \in I \) define \( \beta_i(\delta) = (1-\delta)r_i \). Notice that, given \( r_i \), as \( \delta \) grows towards one, clearly \( \beta_i(\delta) \in (0,1) \).

**Definition T.5.3.** Beginning-of-Period Beliefs: The beginning-of-period beliefs of all players \( (i \in I, 0) \) are trivial. Of course, all players believe that all other players have received the null message \( m_0^i \).

The beginning-of-period beliefs \( \Phi^B_t(m_t^i) \) of any other player \( (i,t) \), depending on the message he receives from player \( (i, t-1) \) are as follows

- If \( m_t^i = m^* \) then \( m_{t-1}^i = (m^*, \ldots, m^*) \) with probability 1
- If \( m_t^i = m^A \) then \( m_{t-1}^i = (m^A, \ldots, m^A) \) with probability 1
- If \( m_t^i = m^B \) then \( m_{t-1}^i = \begin{cases} (m^*, \ldots, m^*) \text{ with probability } \beta_i(\delta) \\ (m^B, \ldots, m^B) \text{ with probability } 1 - \beta_i(\delta) \end{cases} \)
Definition T.5.4. End-of-Period Beliefs: For ease of notation, we divide our description of the end-of-period beliefs of player \((i,t)\) into two cases: \(x^t = x(1)\), and \(x^t = x(t)\) with \(t \geq 2\).

For any player \((i,t)\), whenever \(x^t = x(1)\), let \(\Phi^E_i(m^t, x(1), a^t, y^t)\) be as follows:

\[
\begin{align*}
\text{if } a^t &= (a^t_j, a^t_{-j}) \text{ and } y^t = y(0) & &\text{then } m^t_{t-1} = (m^A, \ldots, m^A) \text{ with probability 1} \\
\text{if } m^t_i &= m^A, \text{ } a^t = (a^t_j, a^t_{-j}) \text{ and } y^t = y(0) & &\text{then } m^t_{t-1} = (m^A, \ldots, m^A) \text{ with probability 1} \\
\text{if } m^t_i &= m^A, \text{ } a^t = a^t' \text{ and } y^t = y(0) & &\text{then } m^t_{t-1} = (m^A, \ldots, m^A) \text{ with probability 1} \\
\text{if } m^t_i &= m^B \text{ and } a^t = a^t' & &\text{then } m^t_{t-1} = (m^B, \ldots, m^B) \text{ with probability 1} \tag{T.5.8} \\
\text{if } m^t_i &= m^B \text{ and } a^t = (a^t_j, a^t_{-j}) & &\text{then } m^t_{t-1} = (m^B, \ldots, m^B) \text{ with probability 1} \\
\text{if } a^t_j \not\in \{a^t_j, a^t_{-j}\} \text{ for some } j \in I & &\text{then } m^t_{t-1} = (m^B, \ldots, m^B) \text{ with probability 1} \\
\text{in all other cases} & & m^t_{t-1} = (m^*, \ldots, m^*) \text{ with probability 1} 
\end{align*}
\]

For any player \((i,t)\), whenever \(x^t = x(t)\) with \(t \geq 2\) let \(\Phi^E_i(m^t, x(t), a^t, y^t)\) be as follows:

\[
\begin{align*}
\text{if } a^t &= x(t), \text{ } m^t_i = m^A \text{ and } y^t = y(0) & &\text{then } m^t_{t-1} = (m^A, \ldots, m^A) \text{ with probability 1} \\
\text{if } a^t \neq x(t) & &\text{then } m^t_{t-1} = (m^B, \ldots, m^B) \text{ with probability 1} \\
\text{if } a^t &= x(t) \text{ and } m^t_i = m^* & &\text{then } m^t_{t-1} = (m^*, \ldots, m^*) \text{ with probability 1} \tag{T.5.9} \\
\text{if } a^t &= x(t), \text{ } m^t_i = m^A \text{ and } y^t = y(1) & &\text{then } m^t_{t-1} = (m^*, \ldots, m^*) \text{ with probability 1} \\
\text{if } a^t &= x(t) \text{ and } m^t_{t-1} = m^B & &\text{then } \begin{cases} m^t_{t-1} = (m^*, \ldots, m^*) \text{ prob. } \beta_1(\delta) \\
 m^t_{t-1} = (m^B, \ldots, m^B) \text{ prob. } 1 - \beta_1(\delta) \end{cases}
\end{align*}
\]

T.6. Proof of Theorem 2: Sequential Rationality

We begin by checking the sequential rationality of the message strategies we have defined.

Definition T.6.1: Let \(T_i^{1E}\) denote the end-of-period-\(t\) collection of information sets that belong to player \((i,t)\), with typical element \(T_i^{1E}\).

It is convenient to partition \(T_i^{1E}\) into four mutually disjoint exhaustive subsets on the basis of the associated beliefs of player \((i,t)\).

Let \(T_i^{1E}(\ast) \subset T_i^{1E}\) be the collection of information sets in which player \((i,t)\) believes that \(m^t_{t+1}\) is equal to \((m^*, \ldots, m^*)\) with probability one. These beliefs will be denoted by \(\Phi_i^{1E}(\ast)\). Notice that these information sets are those in the last case of (T.5.8) and the third and fourth case of (T.5.9).

\[\text{T.4} \text{In the interest of brevity, we avoid writing down the end-of-period beliefs for players } (i \in I, 0) \text{ separately. Equations (T.5.8) and (T.5.9) that follow can be interpreted as defining the end-of-period beliefs of the time 0 players by re-defining } m^t_0 \text{ to be equal to } m^* \text{ for all players } (i \in I, 0).\]

\[\text{T.5} \text{Using a short-hand version of notation established in Definition A.6, when, for instance, we write } a^t = (a^t_j, a^t_{-j}) \text{, we mean that this is the case for some } j \in I.\]
Let $I_i^{tE}(A) \subset I_i^{tE}$ be the collection of information sets in which player $i$ believes that $m_{t-1}^{t+1}$ is equal to $(m^A, \ldots, m^A)$ with probability one. These beliefs will be denoted by $\Phi_i^{tE}(A)$. Notice that these information sets are those in the first three cases of (T.5.8) and the first case of (T.5.9).

Let $I_i^{tE}(B) \subset I_i^{tE}$ be the collection of information sets in which player $i$ believes that $m_{t-1}^{t+1}$ is equal to $(m^B, \ldots, m^B)$ with probability one. These beliefs will be denoted by $\Phi_i^{tE}(B)$. Notice that these information sets are those in fourth, fifth and sixth cases of (T.5.8), and the second case of (T.5.9).

Finally, let $I_i^{tE}(\ast B) \subset I_i^{tE}$ be the collection of information sets in which player $i$ believes that $m_{t-1}^{t+1}$ is equal to $(m^e, \ldots, m^e)$ with probability $\beta_i(\delta)$ and to $(m^B, \ldots, m^B)$ with probability $1 - \beta_i(\delta)$. These beliefs will be denoted by $\Phi_i^{tE}(\ast B)$. Notice that these information sets are those in the last case of (T.5.9).

**Definition T.6.2:** Given the strategy profile $(g, \mu)$ that we defined in Section A.2 and given Definition T.6.1, we can appeal to the stationarity of the game and of $(g, \mu)$ to define the following.

With a slight abuse of notation, for any pair of messages $m$ and $\hat{m}$ both in $\{m^e, m^A, m^B\}$, we denote by $v_i(m, \hat{m}, \delta)$ the end-of-period-$t$ (discounted as of the beginning of period $t + 1$) payoff to player $i$, if he sends message $m$, and all other players send message $\hat{m}$.

**Lemma T.6.1:** Let the assessment $(g, \mu, \Phi)$ described in Section A.2 be given. Then the end-of-period continuation payoffs for any player $i$ at information sets $I_i^t \in \{I_i^{tE}(\ast), I_i^{tE}(A), I_i^{tE}(B)\}$ are as follows.\(^{T.6}\)

\[
v_i^t(g, \mu | \Phi_i^{tE}(\ast)) = v_i^t(m^e, m^e, \delta) = v_i^* \tag{T.6.1}
\]

\[
v_i^t(g, \mu | \Phi_i^{tE}(A)) = v_i^t(m^a, m^a, \delta) = \alpha v_i^t + (1 - \alpha) v_i^* \tag{T.6.2}
\]

\[
v_i^t(g, \mu | \Phi_i^{tE}(B)) = v_i^t(m^b, m^b, \delta) = v_i^t \tag{T.6.3}
\]

**Proof:** The first equalities in equations (T.6.1), (T.6.2) and (T.6.3) are obvious from Definitions T.6.1 and T.6.2.

Equations (T.6.1), (T.6.3) are a direct consequence of the way we have defined strategies and beliefs in Section A.2, and we omit the details. To see that (T.6.2) holds notice that we can write this continuation payoff recursively as

\[
v_i^t(m^A, m^A, \delta) = (1 - \delta)v_i^t + \delta \left[\gamma(\delta)v_i(m^e, m^e, \delta) + (1 - \gamma(\delta))v_i(m^A, m^A, \delta)\right] \tag{T.6.4}
\]

Substituting the definition of $\gamma(\delta)$ given in (A.4), substituting (T.6.1), and solving for $v_i(m^A, m^A, \delta)$ yields (T.6.2).\(\blacksquare\)

**Lemma T.6.2:** Given the beliefs described in Definition T.5.4, for $\delta$ sufficiently close to one, no player $(i, t)$ has an incentive to deviate from the message strategy described in Definition A.6 at any information set $I_i^{tE} \in I_i^{tE}(\ast)$.\(^{T.6}\)
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Proof: From Lemma T.6.1, if player \( \langle i, t \rangle \) follows the equilibrium message strategy \( \mu_i^t \), then his continuation payoff is as in (T.6.1). If he deviates and sends \( m^A \) instead of \( m^* \), we can write his payoff recursively as follows

\[
v_i(m^A, m^*, \delta) = q \{ (1 - \delta)u_i(a_i', a^*_{-i}) + \delta [\gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^A, m^A, \delta)] \} + (1 - q) \{ (1 - \delta)\hat{v}_i + \delta [\gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^A, m^A, \delta)] \}
\]

Substituting (T.6.1) and (T.6.2) and solving for \( v_i(m^A, m^*, \delta) \) yields

\[
v_i(m^A, m^*, \delta) = \frac{1}{1 - (1 - q)\delta(1 - \frac{(1 - \delta)(1 - \alpha)}{\alpha \delta})} \left\{ (1 - \delta)[qu_i(a_i', a^*_{-i}) + (1 - q)\hat{v}_i] + (1 - \delta)(1 - \alpha)\frac{v_i^*}{\alpha} + q\delta(1 - \frac{(1 - \delta)(1 - \alpha)}{\alpha \delta})(\alpha v_i' + (1 - \alpha)v_i^*) \right\}
\]

From (T.6.6) we get directly that

\[
\lim_{\delta \to 1} v_i(m^A, m^*, \delta) = \alpha v_i' + (1 - \alpha)v_i^*
\]

and since the right-hand side of (T.6.7) is obviously less than \( v_i(m^*, m^*, \delta) = v_i^* \), we can conclude that player \( \langle i, t \rangle \) cannot gain by deviating to sending message \( m^A \) instead of \( m^* \) when \( \delta \) is large enough.

If player \( \langle i, t \rangle \) deviates and sends message \( m^B \) instead of \( m^* \) we can write his payoff recursively as follows

\[
v_i(m^B, m^*, \delta) = q \{ (1 - \delta)u_i(a_i', a^*_{-i}) + \delta [\gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^B, m^A, \delta)] \} + (1 - q) \{ (1 - \delta)\hat{v}_i + \delta v_i(m^B, m^*, \delta) \}
\]

Substituting (T.6.1) and (T.6.2) and solving for \( v_i(m^B, m^*, \delta) \) yields

\[
v_i(m^B, m^*, \delta) = \frac{1}{1 - (1 - q)\delta} \left\{ (1 - \delta)[qu_i(a_i', a^*_{-i}) + (1 - q)\hat{v}_i] + (1 - \delta)(1 - \alpha)\frac{v_i^*}{\alpha} + q\delta(1 - \frac{(1 - \delta)(1 - \alpha)}{\alpha \delta})(\alpha v_i' + (1 - \alpha)v_i^*) \right\}
\]

As for the previous case, if \( \delta \) is sufficiently large, the deviation does not pay since from (T.6.9) we get directly that

\[
\lim_{\delta \to 1} v_i(m^B, m^*, \delta) = \alpha v_i' + (1 - \alpha)v_i^*
\]
Hence the lemma is proved. ■

**Lemma T.6.3:** Given the beliefs described in Definition T.5.4, for $\delta$ sufficiently close to one, no player $(i,t)$ has an incentive to deviate from the message strategy described in Definition A.6 at any information set $I^E_i \in I^E_i(A)$.

**Proof:** From Lemma T.6.1, if player $(i,t)$ follows the equilibrium message strategy $\mu_i^t$, then his continuation payoff is as in (T.6.2). If he deviates and sends $m^*$ instead of $m^A$, we can write his payoff recursively as follows

$$v_i(m^*, m^A, \delta) =$$

$$q \left\{ (1 - \delta)u_i(a_i^*, a_{-i}^t) + \delta \left[ \gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^*, m^A, \delta) \right] \right\} + \frac{(1 - q) \left\{ (1 - \delta)v_i + \delta \left[ \gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^*, m^A, \delta) \right] \right\}}{1 - \frac{\gamma(\delta)}{1 - \gamma(\delta)}}$$

Substituting (T.6.1) and solving for $v_i(m^*, m^A, \delta)$ yields

$$v_i(m^*, m^A, \delta) = \alpha \left[ qu_i(a_i^*, a_{-i}^t) + (1 - q)v_i^* \right] + (1 - \alpha)v_i^*$$

and since the right-hand side of (T.6.12) is obviously less than $v_i(m^A, m^A, \delta) = \alpha v_i^* + (1 - \alpha)v_i^*$, we can conclude that player $(i,t)$ cannot gain by deviating to sending message $m^*$ instead of $m^A$ when $\delta$ is large enough.

If player $(i,t)$ deviates and sends message $m^B$ instead of $m^A$ we can write his payoff recursively as follows

$$v_i(m^B, m^A, \delta) = (1 - \delta)v_i^* + \delta \left[ \gamma(\delta)v_i(m^B, m^*, \delta) + (1 - \gamma(\delta))v_i(m^B, m^A, \delta) \right]$$

Solving for $v_i(m^B, m^A, \delta)$ using the definition of $\gamma(\delta)$ given in (A.4) yields

$$v_i(m^B, m^A, \delta) = \alpha v_i^* + (1 - \alpha)v_i(m^B, m^*, \delta)$$

Using (T.6.10) this is clearly enough to show that for $\delta$ high enough player $(i,t)$ cannot gain by deviating in this way. Hence the lemma is proved. ■

**Lemma T.6.4:** Given the beliefs described in Definition T.5.4 no player $(i,t)$ has an incentive to deviate from the message strategy described in Definition A.6 at any information set $I^E_i \in I^E_i(B)$.

**Proof:** From Lemma T.6.1, if player $(i,t)$ follows the equilibrium message strategy $\mu_i^t$, then his continuation payoff is as in (T.6.3). If he deviates and sends $m^*$ instead of $m^B$, we can write his payoff recursively as follows

$$v_i(m^*, m^B, \delta) = (1 - \delta) \left[ qu_i(a_i^*, a_{-i}^t) + (1 - q)v_i^* + \delta v_i(m^*, m^B, \delta) \right]$$

Solving for $v_i(m^*, m^B, \delta)$ yields

$$v_i(m^*, m^B, \delta) = qu_i(a_i^*, a_{-i}^t) + (1 - q)v_i^*$$

Since the right-hand side of (T.6.16) is strictly less than $v_i(m^B, m^B, \delta)$, this is clearly enough to show that for $\delta$ high enough player $(i,t)$ cannot gain by deviating in this way.
Lemma T.6.5: Given the beliefs described in Definition T.5.4, for \( \delta \) sufficiently close to one, no player \( \langle i, i \rangle \) has an incentive to deviate from the message strategy described in Definition A.6 at any information set \( \mathcal{I}_i \in \mathcal{T}_i^{IE} \).
Remark T.6.1: From Lemmas T.6.2, T.6.3, T.6.4 and T.6.5 it is clear that there exists a \( \delta \in (0,1) \) such that whenever \( \delta > \delta \) the message strategies of Definition A.6 are sequentially rational given the beliefs of Definition T.5.4.

We now turn to the sequential rationality of the action strategies we have defined in Section A.2.

Definition T.6.3: Recall that at the action stage, player \((i,t)\) chooses an action after having received a message \(m_i^t\) and having observed a realization \(x^t\) of the correlation device \(\tilde{x}^t\).

Let \(I^B_i\) denote period-\(t\) action-stage collection of information sets that belong to player \((i,t)\), with typical element \(I^B_{i,t}\). Clearly, each element of \(I^B_i\) is identified by a pair \((m_i^t,x^t)\).

It is convenient to partition \(I^B_i\) into three mutually disjoint exhaustive subsets on the basis of the message \(m_i^t\) received by player \((i,t)\).

Let \(I^B_i(\ast) \subset I^B_i\) be the collection of information sets in which player \((i,t)\) receives message \(m^\ast\). Notice that using Definition T.5.3 we know that in this case player \((i,t)\) believes that \(m_{i,t}^\ast\) is equal to \((m^\ast ..., m^\ast)\) with probability one. These beliefs will be denoted by \(\Phi^B_{i}(\ast)\).

Let \(I^B_i(A) \subset I^B_i\) be the collection of information sets in which player \((i,t)\) receives message \(m^A\). Notice that using Definition T.5.3 we know that in this case player \((i,t)\) believes that \(m_{i,t}^\ast\) is equal to \((m^A ..., m^A)\) with probability one. These beliefs will be denoted by \(\Phi^B_{i}(A)\).

Finally, let \(I^B_i(B) \subset I^B_i\) be the collection of information sets in which player \((i,t)\) receives message \(m^B\). Notice that using Definition T.5.3 we know that in this case player \((i,t)\) believes that \(m_{i,t}^\ast\) is equal to \((m^A ..., m^B)\) with probability \(1 - \beta_i(\delta)\). These beliefs will be denoted by \(\Phi^B_{i}(B)\).

Lemma T.6.6: Given the beliefs described in Definition T.5.3, for \(\delta\) sufficiently close to one, no player \((i,t)\) has an incentive to deviate from the action strategy described in Definition A.5 at any information set \(I^B_i \in I^B_i(\ast)\).

Proof: Given any \(x^t \in X\), if player \((i,t)\) follows the equilibrium action strategy he achieves a payoff that bounded below by \(v_{i,t}^\ast(1 - \delta)u_i^t + \delta v_{i,t}^\ast\) (T.6.25)

Given any \(x^t \in X\), (using the notation of Definition T.6.2) following any possible deviation player \((i,t)\) achieves a payoff that is bounded above by \((1 - \delta)\pi_i + \delta [\gamma(\delta)v_i(m^\ast,m^\ast,\delta) + (1 - \gamma(\delta))v_i(m^A,m^A,\delta)]\) (T.6.26)

Using (T.6.1), (T.6.2) and the definition of \(\gamma(\delta)\) given in (A.4), we can re-write (T.6.26) as \((1 - \delta)\pi_i + \delta \alpha v_i' \left[ 1 - \frac{(1 - \delta)(1 - \alpha)}{\delta \alpha} \right] + \delta v_i^\ast \left[ 1 - \alpha \left( 1 - \frac{(1 - \delta)(1 - \alpha)}{\delta \alpha} \right) \right]\) (T.6.27)

Taking the limit of (T.6.25) as \(\delta \rightarrow 1\) gives \(v_i^\ast\). Taking the limit of (T.6.27) as \(\delta \rightarrow 1\) gives \(\alpha v_i' + (1 - \alpha)v_i^\ast\).

Since \(v_i^\ast > \alpha v_i' + (1 - \alpha)v_i^\ast\) this is clearly enough to prove the claim. ■

Lemma T.6.7: Given the beliefs described in Definition T.5.3, for \(\delta\) sufficiently close to one, no player \((i,t)\) has an incentive to deviate from the action strategy described in Definition A.5 at any information set \(I^B_i \in I^B_i(A)\).

\(T.7\) In the interest of brevity, we avoid an explicit distinction between the \(t = 0\) players and all others. What follows can be interpreted as applying to all players re-defining \(m_{i,0}^t\) to be equal to \(m^\ast\) for players \((i \in I,0)\).
Now consider a deviation to action $a^*_t$. In this case the continuation payoff is

$$
(1 - \delta)u_i(a^*) + \delta \left[ \gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^A, m^A, \delta) \right] \tag{T.6.28}
$$

Clearly this deviation is not profitable since $u_i(a^*_t, a^*_{-t}) < u_i(a')$.

A deviation to an action $a_i \not\in \{a'_t, a^*_t\}$ yields a continuation payoff that is bounded above by

$$
(1 - \delta)\pi_i + \delta v'_i \tag{T.6.30}
$$

Taking the limit of (T.6.28) as $\delta \to 1$ yields $\alpha v'_i + (1 - \alpha)v^*_i$. Hence, for $\delta$ large enough, the quantity in (T.6.28) is greater than the quantity in (T.6.30). Therefore, for $\delta$ close enough to one, this deviation is not profitable either.

Now consider any information set in $\mathcal{I}^B_i(x)$ with $x^t \neq x(1)$. The continuation payoff to player $(i, t)$ from following the equilibrium strategy is bounded below by

$$
(1 - \delta)\omega_i + \delta \left[ \gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^A, m^A, \delta) \right] \tag{T.6.31}
$$

Taking the limit of (T.6.31) as $\delta \to 1$ yields $\alpha v'_i + (1 - \alpha)v^*_i$. Hence, for $\delta$ large enough, the quantity in (T.6.31) is greater than the quantity in (T.6.32). Therefore, for $\delta$ close enough to one, no deviation is profitable at any information set in $\mathcal{I}^B_i(x)$ with $x^t \neq x(1)$.

**Lemma T.6.8:** Given the beliefs described in Definition T.5.3, for $\delta$ sufficiently close to one, no player $(i, t)$ has an incentive to deviate from the action strategy described in Definition A.5 at any information set $\mathcal{I}^B_i \in \mathcal{I}^B_i(x)$.

**Proof:** We distinguish between the information set in $\mathcal{I}^B_i(B)$ that has $x^t = x(1)$ and those that have $x^t \neq x(1)$. We begin with the information set in which $x^t = x(1)$.

After having observed the pair $(m^B, x(1))$, if he follows the equilibrium strategy, player $(i, t)$ achieves a continuation payoff of

$$
\beta(\delta) \left[ (1 - \delta)u_i(a'_t, a^*_{-t}) + \delta(\gamma(\delta)v_i(m^*, m^*, \delta) + (1 - \gamma(\delta))v_i(m^A, m^A, \delta)) \right] + (1 - \beta(\delta)) \left[ (1 - \delta)u_i(a'_t) + \delta v_i(m^B, m^B, \delta) \right] \tag{T.6.33}
$$

If on the other hand he deviates to action $a^*_t$ his continuation payoff is

$$
\beta(\delta) \left[ (1 - \delta)u_i(a^*_t) + \delta v_i(m^*, m^*, \delta) + (1 - \beta(\delta)) \right] \left[ (1 - \delta)u_i(a^*_t, a^*_{-t}) + \delta v_i(m^B, m^B, \delta) \right] \tag{T.6.34}
$$

Using (T.6.1) (T.6.2) (T.6.3) and Definition T.5.2, we can now write the equilibrium continuation payoff in (T.6.33) minus the deviation continuation payoff in (T.6.34) as

$$
(1 - \delta) \left\{ r_i(1 - \delta) \left[ u_i(a'_t, a^*_{-t}) - u_i(a^*_t) \right] + r_i(1 - \gamma(\delta))a(v_i - v^*_i) + (1 - (1 - \delta)r_i) \left[ u_i(a'_t) - u_i(a^*_t, a^*_{-t}) \right] \right\} \tag{T.6.35}
$$
Dividing (T.6.35) by $1 - \delta$, taking the limit as $\delta \to 1$ and using (A.4), yields that (up to a factor $1 - \delta$) this difference in payoffs in the limit is equal to

$$r_i (v_i^* - v_i') + u_i(a') - u_i(a_i^*, a_i')$$  \hspace{1cm} (T.6.36)

Notice now that using the (upper) bound on $r_i$ given in (T.5.6) we can verify that the quantity in (T.6.36) is positive. Hence we can conclude that deviating to $a^*_i$ is in fact not profitable for player $(i, t)$.

Next, consider a deviation to an action $a_i \not\in \{a_i', a_i^*\}$. Following this deviation, the continuation payoff to player $(i, t)$ is bounded above by

$$(1 - \delta)\mu_i + \delta v_i(m^B, m^B, \delta)$$  \hspace{1cm} (T.6.37)

Using (T.6.1), (T.6.2), (T.6.3) and Definition T.5.2, we can now write the equilibrium continuation payoff in (T.6.33) minus the deviation continuation payoff in (T.6.37) as

$$(1 - \delta) \{\delta r_i (1 - \alpha(1 - \gamma(\delta)))(v_i^* - v_i') + (1 - \delta)ur_i(a_i', a_i^*) + (1 - (1 - \delta)r_i)u_i(a') - \mu_i\}$$  \hspace{1cm} (T.6.38)

Dividing (T.6.38) by $1 - \delta$, taking the limit as $\delta \to 1$ and using (A.4) and the fact that $\beta_i(\delta) = (1 - \delta)r_i$, yields that (up to a factor $1 - \delta$) this difference in payoffs in the limit is equal to

$$r_i (1 - \alpha)(v_i^* - v_i') + u_i(a') - \mu_i$$  \hspace{1cm} (T.6.39)

Notice now that using the (lower) bound on $r_i$ given in (T.5.6) we can verify that the quantity in (T.6.39) is positive. Hence we can conclude that deviating to $a_i^*$ is in fact not profitable for player $(i, t)$.

Now consider an information set in $I_i^B(B)$ that has $x^i \neq x(1)$. If he follows his equilibrium strategy, player $(i, t)$ achieves a continuation payoff that is bounded below by

$$(1 - \delta)\mu_i + \delta[\beta_i(\delta)v_i(m^B, m^*, \delta) + (1 - \beta_i(\delta))v_i(m^B, m^B, \delta)]$$  \hspace{1cm} (T.6.40)

His continuation payoff following any deviation is bounded above by

$$(1 - \delta)\mu_i + \delta v_i(m^B, m^B, \delta)$$  \hspace{1cm} (T.6.41)

Using (T.6.3) and Definition T.5.2, we can now write the equilibrium continuation payoff in (T.6.40) minus the deviation continuation payoff in (T.6.41) as

$$(1 - \delta) \{\mu_i - \mu_i + \delta r_i[v_i(m^B, m^*, \delta) - v_i']\}$$  \hspace{1cm} (T.6.42)

Dividing (T.6.42) by $1 - \delta$, taking the limit as $\delta \to 1$ and using (T.6.10), yields that (up to a factor $1 - \delta$) this difference in payoffs in the limit is equal to

$$\mu_i - \mu_i + r_i(1 - \alpha)(v_i^* - v_i')$$  \hspace{1cm} (T.6.43)

Notice now that using the (lower) bound on $r_i$ given in (T.5.6) we can verify that the quantity in (T.6.43) is positive. Hence we can conclude that deviating from the equilibrium strategy at any information set in $I_i^B(B)$ that has $x^i \neq x(1)$ is in fact not profitable for player $(i, t)$. Therefore, the proof is now complete.

**Remark T.6.2:** From Lemmas T.6.6, T.6.7 and T.6.8 it is clear that there exists a $\delta \in (0, 1)$ such that whenever $\delta > \delta$ the action strategies of Definition A.5 are sequentially rational given the beliefs of Definition T.5.3.
T.7. Proof of Theorem 2: Consistency of Beliefs

Definition T.7.1: For every \( i \in I \), let
\[
\psi_i(\delta) = \frac{(1 - \delta)r_i \sum_{j \in I}(|A_j| - 1 - q_j)}{1 - (1 - \delta)r_i}
\]
(T.7.1)

where \( q \) is as in Definition A.1, and \( r_i \) is as in Definition T.5.2.

Remark T.7.1: For \( \delta \) sufficiently close to 1, it is clear that the \( \psi_i(\delta) \) of equation (T.7.1) is in \((0, 1)\).

Remark T.7.2: Let \((g_t, \mu_t)\) be a completely mixed strategy profile of Definitions A.8 and A.9. It is then straightforward to check that as \( \varepsilon \to 0 \) the profile \((g_t, \mu_t)\) converges pointwise to the equilibrium strategy profile described in Definitions A.5 and A.6, as required.

Lemma T.7.1: Let \((g_t, \mu_t)\) be the completely mixed strategy profile of Definitions A.8 and A.9. Let any \( t \geq 2 \) and any quadruple of the type \((m_t^i, x^i, a^i, y^i)\) be given.\(^8\)

Then\(^9\)
\[
\lim_{\varepsilon \to 0} \Pr [m_t^{i-1} = (m^*, \ldots, m^*) | m_t^i, x^i, a^i, y^i, g_t, \mu_t] = 1
\]
(T.7.2)

Proof: In order for \( m_t^{i-1} \neq (m^*, \ldots, m^*) \) to occur it is necessary that at least one player has deviated from the equilibrium strategy in some period \( t \leq t - 2 \), either at the action or at the message stage (or both). Given the completely mixed strategy profile of Definitions A.8 and A.9 and given that from (A.10) we know that trembles become more likely as \( t \) increases, we then know that the probability of event \( m_t^{i-1} \neq (m^*, \ldots, m^*) \) is an infinitesimal in \( \varepsilon_{t-2} \) of order no lower than \( 1/2.\)^1^10 Hence, using (A.10) the probability of \( m_t^{i-1} \neq (m^*, \ldots, m^*) \) is an infinitesimal in \( \varepsilon \) of order no lower than \( 1/2n^{2(t-2)} \).

The probability of \( m_t^{i-1} \neq (m^*, \ldots, m^*) \) needs to be compared with the probability of the quadruple \((m_t^i, x^i, a^i, y^i)\). Depending on the particular quadruple \((m_t^i, x^i, a^i, y^i)\), it is possible that many paths of play could have generated this outcome. However, a lower bound on this probability can be computed as follows. Assume no deviations up to and including period \( t - 2 \). From Definition A.9 the probability that message \((m_t^i)\) is sent by player \((i, t)\) is at least (if a deviation is required) an infinitesimal in \( \varepsilon_{t-1} \) of order 2. From Definition A.8, the probability of any profile \( a^i \) (depending on the number of deviations required; clearly no more than \( n \)) is at least an infinitesimal in \( \varepsilon \) of order \( n \). Hence, using (A.10), the probability of the quadruple \((m_t^i, x^i, a^i, y^i)\) is no smaller than an infinitesimal in \( \varepsilon \) of order \( 2/n^{2(t-2)} + 1/n^{2(t-1)} \).

Since \( n \geq 3 \) it is straightforward to check that \( 1/2n^{2(t-2)} > 2/n^{2(t-1)} + 1/n^{2(t-1)} \). Hence equation (T.7.2) now follows and the proof is complete.\(^1^1\)

Lemma T.7.2: Let \((g_t, \mu_t)\) be the completely mixed strategy profile of Definitions A.8 and A.9. Let any \( t \geq 2 \) and any quadruple of the type \((m_t^i, x^i, a^i, y^i)\) be given.\(^1^2\) Fix also any array \( \hat{m}_{-i} = (\hat{m}_1, \ldots, \hat{m}_{t-1}, \hat{m}_{t+1}, \ldots, \hat{m}_n) \).

\(^8\)The reason we require that \( t \geq 2 \) in (T.7.2) below is that of course all players \((i \in I, 0)\) receive message \( m_0^i \) for sure.
\(^9\)See our point of notation T.1.3 above.
\(^1^0\)See footnote T.2 for an explicit statement of our (standard) terminology concerning infinitesimals.
\(^1^1\)It is worth pointing out that the bounds on probabilities that we have used in this argument are not “tight.” We have used the ones above simply because they facilitate the exposition. Any tight bounds would necessarily involve a case-by-case treatment according to what the equilibrium strategies prescribe, the particular message vector \( m_t^{i-1} \neq (m^*, \ldots, m^*) \) and the particular quadruple \((m_t^i, x^i, a^i, y^i)\).
\(^1^2\)The reason we require that \( t \geq 2 \) in (T.7.2) below is that of course all players \((i \in I, 0)\) receive message \( m_0^i \) for sure.
Then
\[
\lim_{\varepsilon \to 0} \Pr[m_{i-1}^t = m_{i-1}] = \frac{m_i^*, \ldots, m_i^*, g_\varepsilon, \mu_\varepsilon] =
\lim_{\varepsilon \to 0} \Pr[m_{i-1}^t = m_{i-1}] = m_i^*, x^t, a^t, y^t, m_{i-1} = (m^*, \ldots, m^*), g_\varepsilon, \mu_\varepsilon] \tag{T.7.3}
\]

**Proof:** A routine application of Bayes’ rule yields
\[
\Pr[m_{i-1}^t = m_{i-1}] = \frac{m_{i-1}^t = m_i^*, x^t, a^t, y^t, m_{i-1} = (m^*, \ldots, m^*), g_\varepsilon, \mu_\varepsilon] = \Pr[m_{i-1}^t = m_i^*, x^t, a^t, y^t, m_{i-1} = (m^*, \ldots, m^*), g_\varepsilon, \mu_\varepsilon] + \sum_{m \neq (m^*, \ldots, m^*)} \Pr[m_{i-1}^t = m_i^*, x^t, a^t, y^t, m_{i-1} = m, g_\varepsilon, \mu_\varepsilon] \Pr(m_{i-1}^t = m) \tag{T.7.4}
\]

Now take the limit as \(\varepsilon \to 0\) on both sides of (T.7.4). Next, observe that by Lemma T.7.1 all terms in the summation sign must converge to zero and the second term on the right-hand-side of (T.7.4) must converge to one. Hence (T.7.3) follows and the proof is now complete. ■

**Lemma T.7.3:** The strategy profile \((g, \mu)\) described in Definitions A.5 and A.6 and the beginning-of-period beliefs described in Definition T.5.3 are consistent.

**Proof:** For the players \(i \in I, 0\) there is nothing to prove. When \(t = 1\), clearly all players \((i \in I, 1)\) believe that all preceding players received their respective \(m_i^B\). When \(t \geq 2\), given Lemma T.7.2 we can reason taking it as given that all players \((i \in I, t)\) believe that all players \((i \in I, t-1)\) received message \(m^*\). Given this observation the claim for period \(t\) follows easily by a case-by-case examination, comparing the likelihood of deviations in period \(t-1\) (orders of infinitesimals in \(\varepsilon_{t-1}\)). We omit the details entirely for the first two cases of (T.5.7) (in which messages \(m^*\) and \(m^A\) are received).

To see the consistency of the beliefs postulated in the third case of (T.5.7) (when message \(m_i^B\) is received) observe the following. According to Definition A.8 deviations at the action stage of period \(t-1\) after receiving message \(m^*\) have probability \(\varepsilon_{t-1}\). Moreover, according to Definition A.9, after receiving message \(m^*\) player \((i, t-1)\) sends message \(m_i^B\) with probability \(\psi_i(\delta) \varepsilon_{t-1}\) when the equilibrium strategy prescribes to send message \(m^*\). Using Lemma T.7.2, (T.7.1) and Definition T.5.2 it is then immediate to see that Definitions A.8 and A.9 yield
\[
\lim_{\varepsilon \to 0} \Phi_i^B[m_{i-1} = (m_i^*, \ldots, m_i^*) | m_i = m_i^B, g_\varepsilon, \mu_\varepsilon] = \frac{\psi_i(\delta)}{\psi_i(\delta) + \sum_{j \in l}(|A_j| - 1 - q)} = (1 - \delta)r_i = \beta_i(\delta) \tag{T.7.5}
\]

and
\[
\lim_{\varepsilon \to 0} \Phi_i^B[m_{i-1} = (m_i^B, \ldots, m_i^B) | m_i = m_i^B, g_\varepsilon, \mu_\varepsilon] = \frac{\sum_{j \in l}(|A_j| - 1 - q)}{\psi_i(\delta) + \sum_{j \in l}(|A_j| - 1 - q)} = 1 - (1 - \delta)r_i = 1 - \beta_i(\delta) \tag{T.7.6}
\]
as required. ■

**Lemma T.7.4:** The strategy profile \((g, \mu)\) described in Definitions A.5 and A.6 and the end-of-period beliefs described in Definition T.5.4 are consistent, as required for an SE.
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\subsection*{Proof:} When \( t = 0 \) all players receive message \( m^0_{i} \). When \( t = 1 \) the beliefs of the players about the messages received by the previous cohort are trivial. When \( t \geq 2 \), given Lemma T.7.2 we can reason taking it as given that all players \((i \in I, t)\) believe that all players \((i \in I, t - 1)\) received message \( m^* \). Given this observation the claim for period \( t \) follows easily by a case-by-case examination, comparing the likelihood of possible deviations in periods \( t - 1 \) and \( t \) (orders of infinitesimals in \( \varepsilon_{t-1} \) and \( \varepsilon_t \)). For the sake of brevity we omit most of the details, and we simply draw attention to the following facts.

The beliefs of player \((i, t)\) after any realization \( x^t = x(t) \) with \( \ell \geq 2 \) can be seen to be consistent in the following way. In period \( t \), either no deviation from the prescription of the action-stage equilibrium strategies is observed or some deviation is observed. When there are no deviations, the revised end-of-period beliefs \( \Phi^R_{t,i} \) of player \((i, t)\) are of course the same as the beginning of period beliefs \( \Phi^B_{t,i} \). Therefore, using Lemma T.7.2 the claim in this case can be verified simply checking that the beliefs described in (T.5.9) correspond to the prescriptions of the strategies described in (A.9) in the appropriate way. In the case in which some deviations occur, observe that the strategies in (A.9) prescribe that all players should send message \( m^B \), regardless of the message they received, which corresponds to the beliefs described in (T.5.9) as required.

Now consider the case in which the realization of the action-stage correlation device is \( x^t = x(1) \). Next distinguish further between two cases. First, the action profile \( a^i \) is neither equal to \( a' \) nor is of the type \((a^i, a^j_{-j})\) for some \( j \in I \). In this case, from the strategies described in (A.8) it is immediate to check that the message sent by any player \((i, t)\) does not depend on the message \( m^t_i \) he received. Therefore, the claim in this case follows immediately from the message-stage strategies described in (A.8).

On the other hand, from the strategies described in (A.8) it is immediate to check that when \( a^i \) is either equal to \( a' \) or is of the type \((a^i, a^j_{-j})\) for some \( j \in I \) the message sent by player \((i, t)\) does depend on the message \( m^t_i \) he received.

Using the completely mixed strategies described in Definition A.8 it is easy to check that for all \( m \in \{m^*, m^A, m^B\} \) and for all \( y^t \in Y \), whenever \( a^i \) is either equal to \( a' \) or is of the type \((a^i, a^j_{-j})\) for some \( j \in I \) it must be that

\[
\lim_{\varepsilon \to 0} \Phi^R_{t,i} | m^t_{-i} = (m, \ldots, m) | m^t_i = m, x(1), a', y^t, g^t, \varepsilon, \mu^t] = 1 \quad \text{(T.7.7)}
\]

Given (T.7.7), the claim in this case can now be verified simply checking that the beliefs described in (T.5.8) correspond to the prescriptions of the strategies described in (A.8) in the appropriate way.

\subsection*{T.8. Proof of Theorem 2}

Given any \( v^* \in \text{int}(V) \) and any \( \delta \in (0, 1) \), the strategies and correlation devices described in Definitions A.2, A.4, A.5 and A.6 clearly implement the payoff vector \( v^* \).

From Remarks T.6.1 and T.6.2, we know that there exists a \( \hat{\delta} \) such that whenever \( \delta > \hat{\delta} \) each strategy in the profile described in Definitions A.5 and A.6 is sequentially rational given the beliefs described in Definitions T.5.3 and T.5.4.

From Lemmas T.7.3 and T.7.4 we know that the strategy profile described in Definitions A.5 and A.6 and the beliefs described in Definitions T.5.3 and T.5.4 are consistent, as required by Definition 1 of SE.

Hence, using Lemma T.2.1, the proof of Theorem 2 is now complete.

\subsection*{T.9. Proof of Theorem 3: Beliefs}

\begin{definition}
Beginning-of-Period Beliefs: Let \( k \) be any element of \( I \), and \( j \) be any element of \( I \) not equal to \( i \).

The beginning-of-period beliefs of all players \((i \in I, 0)\) are trivial. Of course, all players believe that all other players have received the null message \( m^0_i = \emptyset \).

The beginning-of-period beliefs \( \Phi^B_{t,i}(m^t_i) \) of any other player \((i, t)\), depending on the message he receives
from player \((i, t - 1)\) are as follows\(^{T.13}\):

- If \(m_i^t = m^*\) then \(m_{i-1}^t = (m^*, \ldots, m^*)\) with probability 1.
- If \(m_i^t = \bar{m}^j\) then 
  \[
  m_{i-1}^t = (\bar{m}^j, \ldots, \bar{m}^j) \quad \text{with probability } 1.
  \]
- If \(m_i^t = m^{j,T}\) then \(m_{i-1}^t = (m^{j_1}, \ldots, m^{j,T})\) with probability 1.
- If \(m_i^t = \bar{m}^{k,T}\) then \(m_{i-1}^t = (\bar{m}^{k_1}, \ldots, \bar{m}^{k,T})\) with probability 1.

\(^{T.13}\) Notice that the second line of (T.9.1) does not fully specify the probability distribution over the component \(m_i^t\) of the beliefs of player \((i, t)\). For the rest of the argument, what matters is only that all elements of \(M(j, t)\) have positive probability, and that no message outside this set has positive probability. The distribution can be computed using Bayes’ rule from the equilibrium strategies described in Definitions B.8 and B.9 above. We omit the details for the sake of brevity.

- If \(a^0 = g^0(m^0, x^0)\) and \(y^0 = y(j)\) then \(m_{i-1,j}^t = (\bar{m}^j, \ldots, \bar{m}^j)\) with probability 1.
- If \(a^i = g^0(m^0, x^0)\) and \(y^i = y(i)\) then \(m_{i,j}^t = (\bar{m}^i, \ldots, \bar{m}^i)\) with probability 1.
- If \(a^0_k = g^0_k(m^0, x^0)\) and \(a^0_k \neq g^0_k(m^0, x^0)\) then \(m_{i,j}^t = (\bar{m}^{k_1}, \ldots, \bar{m}^{k,T})\) with probability 1.
- Otherwise \(m_{i,j}^t = (m^*, \ldots, m^*)\) with probability 1.

\(^{T.14}\) Similarly to (T.9.1), the first line of (T.9.3) does not fully specify the probability distribution over the component \(m_{i,j}^t\) of the beliefs of player \((i, t)\). For the rest of the argument, what matters is only that all elements of \(M(j, t)\) have positive probability, and that no message outside this set has positive probability. The distribution can be computed using Bayes’ rule from the equilibrium strategies described in Definitions B.8 and B.9 above. We omit the details for the sake of brevity.

**Definition T.9.2. End-of-Period Beliefs:** Let \(k\) be any element of \(I\), and \(j\) be any element of \(I\) not equal to \(i\).

We begin with period \(t = 0\). Recall that \(m_0^i = \emptyset\) for all \(i \in I\). As before, let also \(g^0(m^0, x^0) = (g^0_0(m^0, x^0), \ldots, g^0_n(m^0, x^0))\), and define \(g^0_k(m^0, x^0)\) in the obvious way.

Let \(\Phi^{IE}(m^0, x^0, a^0, y^0)\) be as follows

- If \(a^0 = g^0(m^0, x^0)\) and \(y^0 = y(j)\) then \(m_{i-1,j}^t = (\bar{m}^j, \ldots, \bar{m}^j)\) with probability 1.
- If \(a^i = g^0(m^0, x^0)\) and \(y^i = y(i)\) then \(m_{i,j}^t = (\bar{m}^i, \ldots, \bar{m}^i)\) with probability 1.
- If \(a^0_k = g^0_k(m^0, x^0)\) and \(a^0_k \neq g^0_k(m^0, x^0)\) then \(m_{i,j}^t = (\bar{m}^{k_1}, \ldots, \bar{m}^{k,T})\) with probability 1.
- Otherwise \(m_{i,j}^t = (m^*, \ldots, m^*)\) with probability 1.

Our next case is \(t \geq 1\) and \(x^t = x(\kappa)\) with \(\kappa > \bar{\kappa}\). Let \(x(\ell_0, \ell^*)\) denote the realization of \(x^t\). For any player \((i, t)\), let \(\Phi^{IE}(m^i, x(\ell_0, \ell^*), a^t, y^t)\) be as follows\(^{T.14}\):

- If \(a^t = a(\ell^*)\) and \(m_i^t = \bar{m}^j\) then 
  \[
  m_{i,j}^{t+1} = (m^{j_1}, \ldots, m^{j,T}) \quad \text{with probability } 1.
  \]
- Otherwise
  \[
  m_{i,j}^{t+1} = (m^*, \ldots, m^*) \quad \text{with probability } 1.
  \]

We divide the case of \(t \geq 1\) and \(x^t = x(\kappa)\) with \(\kappa \leq \bar{\kappa}\) into several subcases, according to which message player \((i, t)\) has received. We begin with \(m_i^t = m^*\). Let \(x(\hat{\ell}, \cdots)\) denote the realization of \(x^t\). For any player \((i, t)\), with the understanding that \(m^{j,T}\) is a generic element of \(M(j, t+1)\), let \(\Phi^{IE}(m^*, x(\cdot, \hat{\ell}, \cdots), a^t, y^t)\) be...
as follows

\[
\begin{align*}
\text{if } \alpha' &= \alpha(\hat{t}) \text{ and } y' = y(j) \quad \text{then} \quad \left\{ \begin{array}{l}
m_{t+1} = (\hat{\nu}^j, \ldots, \hat{\nu}^j) \\
m_{j} = m_{j+1}^\tau \end{array} \right. \quad \text{with pr. } \frac{1}{[M(j, t + 1)]} \\
\text{if } \alpha' &= \alpha(\hat{t}) \text{ and } y' = y(i) \quad \text{then} \quad m_{t+1} = (\hat{\nu}^i, \ldots, \hat{\nu}^i) \quad \text{with probability } 1 \\
\text{if } \alpha_{-k}' = \alpha_{-k}(\hat{t}) \text{ and } \alpha_k' \neq \alpha_k(\hat{t}) \quad \text{then} \quad m_{t+1} = (m_{k,T}^i, \ldots, m_{k,T}^i) \quad \text{with probability } 1 \\
\text{otherwise} \quad m_{t+1} = (m^*, \ldots, m^*) \quad \text{with probability } 1 \\
\end{align*}
\]

The next subcase is that of \( m_1 = \hat{\nu}^j \). Let \( x(\cdots, j, \cdots) \) denote the realization of \( x^j \). With the understanding that \( j' \) is an element of \( I \) not equal to \( i \) and that \( m_{1, \tau}^j \) is a generic element of \( M(j', t + 1) \), let \( \Phi_{1,E}^j(m_{1, \tau}^j, x(\cdots, j, \cdots), a', y') \) be as follows

\[
\begin{align*}
\text{if } \alpha' &= \hat{\alpha}(j, i) \text{ and } y' = y(j') \quad \text{then} \quad \left\{ \begin{array}{l}
m_{t+1} = (\hat{\nu}^j, \ldots, \hat{\nu}^j) \\
m_{j+1} = m_{j+1}^\tau \end{array} \right. \quad \text{with pr. } \frac{1}{[M(j', t + 1)]} \\
\text{if } \alpha' &= \hat{\alpha}(j, i) \text{ and } y' = y(i) \quad \text{then} \quad m_{t+1} = (\hat{\nu}^i, \ldots, \hat{\nu}^i) \quad \text{with probability } 1 \\
\text{if } \alpha_{-k}' = \hat{\alpha}_{-k}(j, i) \text{ and } \alpha_k' \neq \hat{\alpha}_k(j, i) \quad \text{then} \quad m_{t+1} = (m_{k,T}^i, \ldots, m_{k,T}^i) \quad \text{with probability } 1 \\
\text{otherwise} \quad m_{t+1} = (m^*, \ldots, m^*) \quad \text{with probability } 1 \\
\end{align*}
\]

The next subcase is that of \( m_1 = m_{1, \tau}^j \in M(i, t) \). Let \( x(\cdots, i, \cdots) \) denote the realization of \( x^i \). With the understanding that \( m_{1, \tau}^j \) is a generic element of \( M(j, t + 1) \), let \( \Phi_{1,E}^j(m_{1, \tau}^j, x(\cdots, i, \cdots), a', y') \) be as follows

\[
\begin{align*}
\text{if } \alpha' &= \hat{\alpha}(i, i) \text{ and } y' = y(j) \quad \text{then} \quad \left\{ \begin{array}{l}
m_{t+1} = (\hat{\nu}^i, \ldots, \hat{\nu}^i) \\
m_{j+1} = m_{j+1}^\tau \end{array} \right. \quad \text{with pr. } \frac{1}{[M(j, t + 1)]} \\
\text{if } \alpha' &= \hat{\alpha}(i, i) \text{ and } y' = y(i) \quad \text{then} \quad m_{t+1} = (\hat{\nu}^i, \ldots, \hat{\nu}^i) \quad \text{with probability } 1 \\
\text{if } \alpha_{-k}' = \hat{\alpha}_{-k}(i, i) \text{ and } \alpha_k' \neq \hat{\alpha}_k(i, i) \quad \text{then} \quad m_{t+1} = (m_{k,T}^i, \ldots, m_{k,T}^i) \quad \text{with probability } 1 \\
\text{otherwise} \quad m_{t+1} = (m^*, \ldots, m^*) \quad \text{with probability } 1 \\
\end{align*}
\]

where we set \( m_{0,0} = \bar{m}^j \).

The final subcase to consider is that of \( m_1 = m_{1, \tau}^j \) for some \( k' \in I \). Let \( x(\cdots, k', \cdots) \) denote the realization of \( x^{k'} \). Let \( \Phi_{1,E}^j(\bar{m}^{k'}, x(\cdots, k', \cdots), a', y') \) be as follows

\[
\begin{align*}
\text{if } \alpha' &= a(k') \quad \text{then} \quad m_{t+1} = (\bar{m}_{\tau-1}^{k'}, \ldots, \bar{m}_{\tau-1}^{k'}) \quad \text{with probability } 1 \\
\text{if } \alpha_{-k}' = a_{-k}(k') \text{ and } \alpha_k' \neq a(k') \quad \text{then} \quad m_{t+1} = (m_{k,T}^{k'}, \ldots, m_{k,T}^{k'}) \quad \text{with probability } 1 \\
\text{otherwise} \quad m_{t+1} = (m^*, \ldots, m^*) \quad \text{with probability } 1 \\
\end{align*}
\]

where we set \( m_{0,0} = \bar{m}^{k'} \).

T.10. Proof of Theorem 3: Sequential Rationality

**Definition T.10.1:** Let \( \mathcal{I}_{i,t}^E \) denote the end-of-period-\( t \) collection of information sets that belong to player \( (i,t) \), with typical element \( \mathcal{I}_{i,t}^E \).
It is convenient to partition $\textbf{I}_t^{IE}$ into mutually disjoint exhaustive subsets on the basis of the associated beliefs of player $(i,t)$. The fact that they exhaust $\textbf{I}_t^{IE}$ can be checked directly from Definition T.9.2 above.

Let $\textbf{I}_t^{IE}(s) \subset \textbf{I}_t^{IE}$ be the collection of information sets in which player $(i,t)$ believes that $m_{i,t}^{t+1}$ is equal to $(m_1^* , \ldots , m^*)$ with probability one. These beliefs will be denoted by $\Phi^{IE}_i(s)$.

Let $\textbf{I}_t^{IE}(-i) \subset \textbf{I}_t^{IE}$ be the collection of information sets in which player $(i,t)$ believes that $m_{t-i}^{t+1}$ is equal to $(\tilde{m}_1^* , \ldots , \tilde{m}^*)$ with probability one. These beliefs will be denoted by $\Phi^{IE}_i(-i)$.

For every $j \in I$ not equal to $i$, let $\textbf{I}_t^{IE}(-j,t) \subset \textbf{I}_t^{IE}$ be the collection of information sets in which player $(i,t)$ believes that $m_{t-j}^{t+1}$ is equal to $(\tilde{m}_1^* , \ldots , \tilde{m}^*)$ with probability one, that $\Pr(m_{j,t}^{t+1} = m_{j,t}^{k}) > 0 \forall m_{j,t}^{k} \in M(j,t)$, and $\Pr(m_{j,t}^{t+1} \in M(j,t)) = 1$. These beliefs will be denoted by $\Phi^{IE}_i(-j,t)$.

For every $j \in I$ not equal to $i$, let $\textbf{I}_t^{IE}(-j,t+1) \subset \textbf{I}_t^{IE}$ be the collection of information sets in which player $(i,t)$ believes that $m_{t-j}^{t+1}$ is equal to $(\tilde{m}_1^* , \ldots , \tilde{m}^*)$ with probability one, $\Pr(m_{j,t}^{t+1} = m_{j,t}^{k}) = \| M(j,t+1) \|^{-1} \forall m_{j,t}^{k} \in M(j,t+1)$. These beliefs will be denoted by $\Phi^{IE}_i(-j,t+1)$.

For every $k \in I$, let $\textbf{I}_t^{IE}(k) \subset \textbf{I}_t^{IE}$ be the collection of information sets in which player $(i,t)$ believes that $m_{t-k}^{t+1}$ is equal to $(\tilde{m}_1^* , \ldots , \tilde{m}^*)$ with probability one. These beliefs will be denoted by $\Phi^{IE}_i(k)$.

For every $k \in I$, and every $\tau = \max\{T-t,1\}, \ldots , T$ let $\textbf{I}_t^{IE}(k,\tau) \subset \textbf{I}_t^{IE}$ be the collection of information sets in which player $(i,t)$ believes that $m_{t-k}^{t+1}$ is equal to $(m_1^k, \ldots , m^k)$ with probability one. These beliefs will be denoted by $\Phi^{IE}_i(k,\tau)$.

Definition T.10.2: Let the strategy profile $(g,\mu)$ described in Definitions B.8 and B.9 be given. Fix a period $t$ and an $n$-tuple of messages $m_t^{t+1} = (m_1^{t+1}, \ldots , m_n^{t+1})$, with $m_k^{t+1} \in M_k^{t+1}$ for every $k \in I$.

Clearly, the profile $(g,\mu)$ together with $m_t^{t+1}$ uniquely determine a probability distribution over action profiles over all future periods, beginning with $t+1$.

Therefore, we can define the expected discounted (from the beginning of period $t+1$) payoff to player $(i,t)$, given $(g,\mu)$ and $m_t^{t+1}$ in the obvious way. This will be denoted by $\hat{v}_i^t(m_t^{t+1})$. Moreover, since they play a special role in some of the computations that follow, we reserve two pieces of notation for two particular instances of $m_t^{t+1}$. The expression $\hat{v}_i^t(*)$ stands for $\hat{v}_i^t(m_t^{t+1})$ when $m_t^{t+1} = (m_1^*, \ldots , m^*)$. Moreover, for any $k \in I$, the expression $\hat{v}_i^k(t,\tau)$ stands for $\hat{v}_i^k(m_t^{t+1})$ when $m_{t-k}^{t+1} = (\tilde{m}_1^k, \ldots , \tilde{m}^k)$ and $m_{t-k}^{t+1} \in M(k,t+1)$.

Lemma T.10.1: For any $i \in I$, any $k \in I$, any $t$, and any $\tau = \max\{T-t,1\}, \ldots , T$, we have that

$$\hat{v}_i^t(*) = \frac{(1-\delta) \left[ q \hat{v}_i + (1-q) z_i \right]}{1 - \delta (1 - q)}$$  \hspace{1cm} (T.10.1)

and

$$\hat{v}_i^k(t,\tau) = \frac{(1-\delta) \left[ q \hat{u}_i^k + (1-q) z_i \right]}{1 - \delta (1 - q)}$$  \hspace{1cm} (T.10.2)

where $\hat{v}_i^t(*)$ and $\hat{v}_i^k(t,\tau)$ are as in Definition T.10.2, $\hat{v}_i$ is as in (B.6), $z_i$ is as in Remark B.4, $v_i^*$ is as in the statement of the Theorem, and $\hat{u}_i^k$ is as in (B.3).

\textit{T.15} See footnote T.14 above.
Proof: Assume first that \( t \geq T \). Using Definitions B.8 and B.9 we can write \( \bar{v}_t^i(*) \) and \( \bar{v}_t^i(k, \tau) \) recursively as

\[
\bar{v}_t^i(*) = q \left\{ (1 - \delta) \hat{v}_i + \delta \left[ (1 - \eta) \bar{v}_{t+1}^i(*) + \frac{\eta}{n} \sum_{k' \in I} \sum_{\tau=1}^{T} \frac{\bar{v}_{t+1}^{k'}(k', \tau)}{T} \right] \right\} + \left( 1 - q \right) \left( 1 - \delta \right) z_i + \delta \bar{v}_{t+1}^i(*)
\]

(T.10.3)

and

\[
\bar{v}_t^i(k, \tau) = q \left\{ (1 - \delta) \hat{u}_i^k + \delta \left[ (1 - \eta) \bar{v}_{t+1}^{i,k}(*, \tau) + \frac{\eta}{n} \sum_{k' \in I} \sum_{\tau=1}^{T} \frac{\bar{v}_{t+1}^{k'}(k', \tau)}{T} \right] \right\} + \left( 1 - q \right) \left( 1 - \delta \right) z_i + \delta \bar{v}_{t+1}^{i,k}(k, \tau)
\]

(T.10.4)

Since the strategy profile \((g, \mu)\) described in Definitions B.8 and B.9 is stationary for \( t \geq T \), we immediately have that \( \bar{v}_t^i(*) = \bar{v}_{t+1}^i(*) \) and, for any \( k \in I \) and any \( \tau = 1, \ldots, T \), \( \bar{v}_{t}^i(k, \tau) = \bar{v}_{t+1}^{i,k}(k, \tau) \). Hence we can solve (T.10.3) and (T.10.4) simultaneously for the NT + 1 variables \( \bar{v}_t^i(*) \) and \( \bar{v}_t^i(k, \tau) \) (\( k \in I \) and \( \tau = 1, \ldots, T \)). Using (B.7) this immediately gives (T.10.1) and (T.10.2), as required.

Proceeding by induction backwards from \( t = T \), it is also immediate to verify that the statement holds for any \( t < T \). The details are omitted for the sake of brevity. \( \square \)

Lemma T.10.2: Let the strategy profile \((g, \mu)\) and system of beliefs \( \Phi \) described in Definitions B.8, B.9, T.9.1 and T.9.2 be given. Then the end-of-period continuation payoffs for any player \( \langle i, t \rangle \) (discounted as of the beginning of period \( t + 1 \)) at any information set \( I_t^i \in \mathcal{I}_t^{I_E} \) (as categorized in Definition T.10.1) are as follows.\( ^{T.16} \)

\[
v_t^i(g, \mu| \Phi_t^{I_E}(*)) = \frac{(1 - \delta) \left[ q \hat{v}_i + (1 - q) z_i \right] + \delta q v_t^i}{1 - \delta (1 - q)}
\]

(T.10.5)

\[
v_t^i(g, \mu| \Phi_t^{I_E}(-i)) = \frac{(1 - \delta) \left[ q \hat{u}_i + (1 - q) z_i \right] + \delta q v_t^i}{1 - \delta (1 - q)}
\]

(T.10.6)

\[
v_t^i(g, \mu| \Phi_t^{I_E}(-j, t)) = \frac{(1 - \delta) \left[ q \hat{u}_j + (1 - q) z_j \right] + \delta q v_t^i}{1 - \delta (1 - q)} \quad \forall j \neq i
\]

(T.10.7)

\[
v_t^i(g, \mu| \Phi_t^{I_E}(E)) = q \hat{v}_i + (1 - q) z_i \quad \forall k \in I
\]

(T.10.8)

\[
v_t^i(g, \mu| \Phi_t^{I_E}(k, \tau)) = \left[ 1 - \left( \frac{\delta q}{1 - \delta (1 - q)} \right)^\tau \right] \left[ q \hat{u}_i + (1 - q) z_i \right] + \frac{\delta q}{1 - \delta (1 - q)} \left[ q \hat{w}_i^k + (1 - q) z_i \right] \quad \forall k \in I \quad \forall \tau = \max\{T - t, 1\}, \ldots, T
\]

(T.10.9)

where \( \hat{v}_i \) is as in (B.6), \( z_i \) is as in Remark B.4, \( v_t^i \) is as in the statement of the Theorem, \( \hat{u}_i^k \) is as in (B.3), and \( \hat{w}_i^k \) is as in (B.2).

\( ^{T.16} \) See our Point of Notation T.1.1 above.
Proof: Equations (T.10.5), (T.10.6) and (T.10.7) are a direct consequence of Definition T.10.1 and Lemma T.10.1.

Equation (T.10.8) follows directly from Definition T.10.1 and the description of the profile \((g, \mu)\) in Definitions B.8 and B.9.

Using the notation established in Definition T.10.2, consider the quantity \(\hat{v}_i^t(m_i^k, \ldots, m_i^\tau)\). Given the strategies described in Definitions B.8 and B.9 it is evident that this quantity does not depend on \(t\). Therefore, for any \(k \in I\) and \(\tau = \max\{T - t, 1\}, \ldots, T\), we can let \(\hat{v}_i(k, \tau) = \hat{v}_i^t(m_i^k, \ldots, m_i^\tau)\), for all \(t\). Clearly, using Definition T.10.1, we have that for all \(k, \tau\) and \(t\), \(v_i^t(g, \mu(\Phi_i^t(k, \tau))) = \hat{v}_i(k, \tau)\).

From the description of \((g, \mu)\) in Definitions B.8 and B.9, for any \(k \in I\) and for any \(\tau = 2, \ldots, T\), the quantity \(\hat{v}_i(k, \tau)\) obeys a difference equation as follows.

\[
\hat{v}_i(k, \tau) = q \left[ (1 - \delta)\hat{v}_i(k, \tau - 1) \right] + (1 - q) \left[ (1 - \delta)z_i + \delta\hat{v}_i(k, \tau) \right]
\]

(T.10.10)

Using again Definitions B.8 and B.9, the terminal condition for (T.10.10) is

\[
\hat{v}_i(k, 1) = q \left[ (1 - \delta)\hat{v}_i(k, 1) \right] + (1 - q) \left[ (1 - \delta)z_i + \delta\hat{v}_i(k, 1) \right]
\]

(T.10.11)

Solving (T.10.10) and imposing the terminal condition (T.10.11) now yields (T.10.9), as required.

Purely for expositional convenience, before completing the proof of sequential rationality at the message stage, we now proceed with the argument that establishes sequential rationality at the action stage.

Definition T.10.3: Recall that at the action stage, player \(\langle i, t \rangle\) chooses an action after having received a message \(m_i^t\) and having observed a realization \(x^t\) of the correlation device \(\hat{x}^t\).

Let \(\mathcal{I}^t_i\) denote period-\(t\) action-stage collection of information sets that belong to player \(\langle i, t \rangle\), with typical element \(I^t_i\). Clearly, each element of \(\mathcal{I}^t_i\) is identified by a pair \((m_i^t, x^t)\).

It is convenient to partition \(\mathcal{I}^t_i\) into mutually disjoint exhaustive subsets. The fact that they exhaust \(\mathcal{I}^t_i\) can be checked directly from Definition T.9.1 above.

Let \(\mathcal{I}^t_i(\ast) \subseteq \mathcal{I}^t_i\) be the collection of information sets in which player \(\langle i, t \rangle\) believes that \(m_i^t - 1\) is equal to \((m_1^*, \ldots, m_\ast^*)\) with probability one.\(^T.17\) These beliefs will be denoted by \(\Phi_i^t(\ast)\).

Let \(\mathcal{I}^t_i(-i) \subseteq \mathcal{I}^t_i\) be the collection of information sets in which player \(\langle i, t \rangle\) believes that \(m_i^t - 1\) is equal to \((\hat{m}_1^t, \ldots, \hat{m}_i^t)\) with probability one. These beliefs will be denoted by \(\Phi_i^t(-i)\).

For every \(j \in I\) not equal to \(i\), let \(\mathcal{I}^t_i(-j) \subseteq \mathcal{I}^t_i\) be the collection of information sets in which player \(\langle i, t \rangle\) believes that \(m_j^{t-1}\) is equal to \((\hat{m}_i, \ldots, \hat{m}_i)\) with probability one, that \(\Pr(m_j^t = m_j^{t-1}) > 0 \forall m_j^{t-1} \in M(j, t)\), and that \(\Pr(m_j^t \in M(j, t)) = 1.\(^T.18\) These beliefs will be denoted by \(\Phi_i^t(-j)\).

For every \(j \in I\) not equal to \(i\), and every \(\tau = \max\{T - t + 1, 1\}, \ldots, T\) let \(\mathcal{I}^t_i(j, \tau) \subseteq \mathcal{I}^t_i\) be the collection of information sets in which player \(\langle i, t \rangle\) believes that \(m_i^t - 1\) is equal to \((m_1^{t-1}, \ldots, m_\tau^{t-1})\) with probability one. These beliefs will be denoted by \(\Phi_i^t(j, \tau)\).

For every \(k \in I\), let \(\mathcal{I}^t_i(E) \subseteq \mathcal{I}^t_i\) be the collection of information sets in which player \(\langle i, t \rangle\) believes that \(m_i^t - 1\) is equal to \((\hat{m}_k, \ldots, \hat{m}_k)\) with probability one. These beliefs will be denoted by \(\Phi_i^t(E)\).

Lemma T.10.3: There exists a \(\hat{\delta} \in (0, 1)\) such that whenever \(\delta > \hat{\delta}\) the action-stage strategies described in Definition B.8 are sequentially rational given the beliefs described in Definition T.9.1 for every player \(\langle i, t \rangle\).\(^T.19\)

\(^T.17\) In the interest of brevity, we avoid an explicit distinction between the \(t = 0\) players and all others. What follows can be interpreted as applying to all players re-defining \(m_i^t\) to be equal to \(m_\ast\) for players \(i \in I, 0\).

\(^T.18\) See footnote T.13.

\(^T.19\) It should be understood that we are, for now, taking it as given that each player \(\langle i, t \rangle\) follows the prescriptions of the message-stage strategies described in Definition B.9. Of course, we have not demonstrated yet that this is in fact sequentially rational given the beliefs described in Definition T.9.2. We will come back to this immediately after the current lemma is proved.
Proof: Consider any information set \( I^*_B \in \{ I^B_i(\ast) \cup I^B_i(-i) \cup I^B_i(-j) \} \).

Using Definition B.8, Lemma T.10.2 and Definition T.10.3, it is immediate to check that, as \( \delta \to 1 \),
the limit expected continuation payoff to player \( \langle i, t \rangle \) from following the action-stage strategies described in
Definition B.8 at any of these information sets is

\[
v^*_i = q \hat{v}_i + (1 - q)z_i \quad (T.10.12)
\]

In the same way, it can be checked that, as \( \delta \to 1 \), the limit expected continuation payoff to player \( \langle i, t \rangle \) from
deviating at any of these information sets is

\[
q \tau^*_i + (1 - q)z_i \quad (T.10.13)
\]

Since by assumption \( \hat{v}_i > v^*_i \) this is of course sufficient to prove our claim for any information set
\( I^*_B \in \{ I^B_i(\ast) \cup I^B_i(-i) \cup I^B_i(-j) \} \).

Now consider any information set \( I^*_B \) either in \( I^B_i(j, \tau) \) or in \( I^B_i(\bar{j}) \) (with \( j \neq i \)).

Using Definition B.8, Lemma T.10.2 and Definition T.10.3, it is immediate to check that, as \( \delta \to 1 \),
the limit expected continuation payoff to player \( \langle i, t \rangle \) from following the action-stage strategies described in
Definition B.8 at any of these information sets is

\[
q \tau^*_i + (1 - q)z_i \quad (T.10.14)
\]

In the same way, it can be checked that, as \( \delta \to 1 \), the limit expected continuation payoff to player \( \langle i, t \rangle \) from
deviating at any of these information sets is exactly as in (T.10.13).

Since by assumption for any \( j \neq i \) we have that \( \tau^*_i > \tau^*_i \) this is of course sufficient to prove our claim for
any of these information sets.

To conclude the proof of the lemma, we now consider any information set \( I^*_B \in I^B_i(\bar{i}) \). Using Definition
B.8, Lemma T.10.2 and Definition T.10.3, it can be checked that the expected continuation payoff to player
\( \langle i, t \rangle \) from following the action-stage strategies described in Definition B.8 at any of these information sets is
bounded below by

\[
(1 - \delta)u_i + \delta \left[ q \tau^*_i + (1 - q)z_i \right] \quad (T.10.15)
\]

In the same way it can be readily seen that the expected continuation payoff to player \( \langle i, t \rangle \) from deviating
at any of these information sets is bounded above by

\[
(1 - \delta)u_i + \delta \left\{ 1 - \left( \frac{\delta q}{1 - \delta (1 - q)} \right)^T \left[ q \tau^*_i + (1 - q) z_i \right] + \left( \frac{\delta q}{1 - \delta (1 - q)} \right)^T \left[ q \tau^*_i + (1 - q) z_i \right] \right\} \quad (T.10.16)
\]

The difference given by (T.10.15) minus (T.10.16) can be written as

\[
(1 - \delta) \left\{ \frac{\delta q}{1 - \delta (1 - q)} \left( \frac{1 - \left( \frac{\delta q}{1 - \delta (1 - q)} \right)^T \left( \tau^*_i - \omega^*_i \right)}{(1 - \delta)} - (\tau^*_i - u_i) \right) \right\} \quad (T.10.17)
\]

\( T.20 \) See Definition T.10.3.
Consider now the term inside the curly brackets in (T.10.17). We have that
\[
\lim_{\delta \to 0} \left[ 1 - \left( \frac{\delta q}{1 - \delta (1 - q)} \right)^T (\Tilde{\pi}_i^t - \omega_i^t) \right] = (\Tilde{\pi}_i^t - \omega_i^t) = T(\Tilde{\pi}_i^t - \omega_i^t) - (\Tilde{\pi}_i - \omega_i)
\]  
(T.10.18)

Using (B.11), we know that the quantity on the right-hand side of (T.10.18) is strictly positive. Hence we can conclude our claim is valid at any information set \( T_i^B \in \mathcal{F}_i^B(\Tilde{I}) \).

**Lemma T.10.4:** Consider the notation we established in Definition T.10.2. For any given \( t \) and \( \tau = \max\{T - t, 1\} \), \( \ldots, T \) let \( \tilde{v}_t^1(m, m_i^t, \omega) \) denote \( v_t^1(m^{t+1}) \) when the vector \( m^{t+1} \) has the \( i \)-th component equal to a generic \( m \in M_i^{t+1} \) and \( m_i^{t+1} = (m_i^t, \ldots, m_i^{T-\tau}) \). As in the proof of Lemma T.10.2, let \( \tilde{v}_i(\tilde{z}, \tau) = \tilde{v}_t^1(m_i^t, \ldots, m_i^{T-\tau}) \).

Then there exists a \( \bar{\delta} \in (0, 1) \) such that whenever \( \delta > \bar{\delta} \) for every player \( i, t \), for every \( m \in M_i^{t+1} \), and for every \( \tau = \max\{T - t, 1\} \), \( 1 \), \( \ldots, T \)

\[
\tilde{v}_i(\tilde{z}, \tau) \geq \tilde{v}_t^1(m, m_i^{t+1})
\]  
(T.10.19)

**Proof:** We prove the claim for the case \( t \geq T \). The treatment of \( t < T \) has some completely non-essential complications due to the fact that the players’ message spaces increase in size for the first \( T \) periods. The details are omitted for the sake of brevity.

We now introduce a new random random variable \( \tilde{w} \), independent of \( \tilde{x} \) and \( y \) (see Definitions B.4 and B.5), and uniformly distributed over the finite set \( \{1, \ldots, T\} \). This will be used in the rest of the proof of the lemma to keep track of the “private” randomization across messages that members of dynasty \( i \) may be required to perform (see Definition B.9). Just as we did for the action-stage and the message-stage correlation devices, we consider countably many independent “copies” of \( \tilde{w} \), one for each time period, denoted by \( \tilde{w}^t \), with typical realization \( w^t \).

To keep track of all “future randomness” looking ahead for \( t' = 1, 2, \ldots \) periods from \( t \), it will also be convenient to define the random vectors \( \tilde{s}^{t, t'} \):

\[
\tilde{s}^{t, t'} = [(\tilde{x}^{t+1}, \tilde{y}^{t+1}, \tilde{w}^{t+1}), \ldots, (\tilde{x}^{t+t'}, \tilde{y}^{t+t'}, \tilde{w}^{t+t'})]
\]  
(T.10.20)

A typical realization of \( \tilde{s}^{t, t'} \) will be denoted by \( s^{t, t'} = [(x^{t+1}, y^{t+1}, w^{t+1}), \ldots, (x^{t+t'}, y^{t+t'}, w^{t+t'})] \). The set of all possible realizatons of \( \tilde{s}^{t, t'} \) (which obviously does not depend on \( t \)) is denoted by \( S^{t, t'} \).

Recall that the profile \( (g, \mu) \) described in Definitions B.8 and B.9 is taken as given throughout. Now suppose that in period \( t \), player \( (i, t) \) sends a generic message \( m \in M_i^{t+1} \) and that \( m_i^{t+1} = (m_i^t, \ldots, m_i^{T-\tau}) \). Then, given any realization \( s^{t, t'} \) we can compute the actual action profile played by all players \( (k \in I, t + t') \). This will be denoted by \( a^{t+t'}(m, m_i^t, s^{t, t'}) \). Similarly, we can compute the profile of messages \( m_i^{t+1} \) received by all players \( (j \neq i, t + t') \). This \( n - 1 \)-tuple will be denoted by \( m^{t+t'}(m, m_i^t, s^{t, t'}) \).

Recall that the messages received by all time- \( t + t' \) players are the result of choices and random draws that take place on or before period \( t + t' - 1 \). Therefore it is clear that if we are given two realizations \( \tilde{s}^{t, t'} = [s^{t-t-1}, (\tilde{x}^{t+t'}, \tilde{y}^{t+t'}, \tilde{w}^{t+t'})] \) and \( \tilde{s}^{t, t'} = [s^{t-t-1}, (\tilde{x}^{t+t'}, \tilde{y}^{t+t'}, \tilde{w}^{t+t'})] \), then it must be that

\[
m^{t+t'}(m, m_i^t, \tilde{s}^{t, t'}) = m^{t+t'}(m, m_i^t, \tilde{s}^{t, t'})
\]  
(T.10.21)

Notice next that from the description of the profile \( (g, \mu) \) in Definitions B.8 and B.9 it is also immediate to check that for any \( t' \), any \( m \in M_i^{t+1} \) and any realization \( s^{t, t'} \) the message profile \( m^{t+t'}(m, m_i^t, s^{t, t'}) \) can only take one out of two possible forms. Either we have \( m^{t+t'}(m, m_i^t, s^{t, t'}) = (m_i^t, \ldots, m_i^t) \) or it must be that \( m^{t+t'}(m, m_i^t, s^{t, t'}) = (m_i^t, \ldots, m_i^t) \) for some \( \tau' = 1, \ldots, T \).
Lastly, notice that, given an arbitrary message $m \in M_{i+1}$ we can write
\[
\bar{v}_i^s(m, \bar{m}^i) = (1 - \delta) \sum_{t' = 1}^{\infty} \sum_{s^{t'} \in S^{t'}} \Pr(s^{t'} = s^{t'}) u_i [a^{t' + \bar{s}}(m, \bar{m}^i, s^{t'})] 
\]  
(T.10.22)

Since the strategies described in Definitions B.8 and B.9 are stationary for $t \geq T$, and the distribution of $s^{t'}$ is independent of $t$, it is evident from (T.10.22) that $\bar{v}_i^s(m, \bar{m}^i)$ does not depend on $t$. From now on we drop the superscript and write $\bar{v}_i(m, \bar{m}^i)$.

We now proceed with the proof of inequality (T.10.19) of the statement of the lemma. In order to do so, from now on we fix a particular $t = i$, $m = \bar{m}$ and $\tau = \hat{\tau}$, and we prove (T.10.19) for these fixed values of $t$, $m$ and $\tau$. Since the lower bound on $\delta$ that we will find will clearly not depend on $t$, and since there are finitely many values that $m$ and $\tau$ can take, this will be sufficient to prove the claim.

Inequality (T.10.19) in the statement of the lemma is trivially satisfied (as an equality) if $m = \bar{m}^i$. From now on assume that $\bar{m} \neq \bar{m}^i$.

Given any $t' = 1, 2, \ldots$, we now partition the set of realizations $S_i$ into five disjoint exhaustive subsets: $S'_1$, $S'_2$, $S'_3$, $S'_4$, and $S'_5$. This will allow us to decompose the right-hand side of (T.10.22) in a way that will make possible the comparison with (a similar decomposition of) the left-hand side of (T.10.19) as required to prove the lemma.

Let
\[
S'_1 = \{ s^{t'} \mid m^{t'+i} \neq \bar{m}^i \} 
\]  
and notice that if $t' \leq \hat{\tau}$ then $S'_1 = S'$. Let $t'$ be such that $t' \leq \hat{\tau}$ and let
\[
S'_2 = \{ s^{t'} \mid m^{t'+i} (\bar{m}, \bar{m}^i, s^{t'}) = (\bar{m}^i, \ldots, \bar{m}^i) \text{ and } u_i (a^{t'+i} (\bar{m}, \bar{m}^i, s^{t'})) \leq u_i (a^{t'+i} (\bar{m}^i, \bar{m}^i, s^{t'})) \} 
\]  
(T.10.23)

and
\[
S'_3 = \{ s^{t'} \mid m^{t'+i} (\bar{m}, \bar{m}^i, s^{t'}) = (\bar{m}^i, \ldots, \bar{m}^i) \text{ and } u_i (a^{t'+i} (\bar{m}, \bar{m}^i, s^{t'})) > u_i (a^{t'+i} (\bar{m}^i, \bar{m}^i, s^{t'})) \} 
\]  
(T.10.24)

Notice that if the first condition in (T.10.24) holds, then $m^{t'+i} (\bar{m}^i, \bar{m}^i, s^{t'}) = (\bar{m}^i, \ldots, \bar{m}^i)$. Therefore, $S'_1$ and $S'_2$ and $S'_3$ are disjoint.

Next, let any $s^{t'} \in S'_3$ with $t'' < t'$ be given and define
\[
S'_4 (s^{t''}) = \{ s^{t'} \mid s^{t'} = (s^{t''}, s^{t''}) \text{ for some } s^{t''}, t' \text{ and } | \{ t' \mid (t'' + 1, \ldots, t' - 1) \} \times \{ x^{t'} = x^{(k)} \text{ with } k \leq \hat{\tau} \} \times T - 1 \} 
\]  
(T.10.25)

Now let
\[
S'' = \bigcup_{t'' \in S''} S'_4 (s^{t''}) 
\]  
(T.10.26)

From the strategies described in Definitions B.8 and B.9 it can be checked that if $s^{t'} \in S'_4$ then $m^{t'+i} (\bar{m}, \bar{m}^i, s^{t'}) = (\bar{m}^i, \ldots, \bar{m}^i)$ for some $\tau'$ and $m^{t'+i} (\bar{m}^i, \bar{m}^i, s^{t'}) = (\bar{m}^i, \ldots, \bar{m}^i)$. Therefore, it is clear that $S''$ is disjoint from $S'_1$, $S'_2$, and $S'_3$. 

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The last set in the partition of $S'_t$ is defined as the residual of the previous four.

$$S'_5 = S'_t / \{ S'_1 \cup S'_2 \cup S'_3 \cup S'_4 \}$$  \hspace{1cm} \text{(T.10.28)}$$

Using (T.10.22), we can now proceed to compare the two sides of inequality (T.10.19) of the statement of the lemma for the five distinct (conditional) cases $\bar{s}^{t',t'} \in S'_1$ through $\bar{s}^{t',t'} \in S'_5$. Notice first of all that when $\bar{s}^{t',t'} \in S'_2$, we know immediately from (T.10.24) that there is nothing to prove.

We begin with $\bar{s}^{t',t'} \in S'_1$. Notice first of all that if we fix any $\bar{s}^{t',t'} \in S'_1$, then it follows from (T.10.21) and (T.10.23) that any $\bar{s}^{t',t'}$ of the form $[\bar{s}^{t',t'-1}, \bar{s}^{t'-1,t'}]$ (where $\bar{s}^{t'-1}$ are the first $t' - 1$ triples of $\bar{s}^{t',t'}$) is in fact in $S'_1$.

Using, (T.10.23) and Definitions 3, B.8 and B.9 we get

$$\sum_{s^{t'-1,t'} \in S^t} \Pr(\bar{s}^{t'-1,t'} = s^{t'-1,t'}) u_i(\bar{a}^{t+t'}(\bar{m}^{i,\hat{t}}, \bar{m}^{i,\hat{t}}, [\bar{s}^{t'-1}, s^{t'-1,t'}])) = q \bar{x}_i + (1 - q) z_i \geq \sum_{s^{t'-1,t'} \in S^t} \Pr(\bar{s}^{t'-1,t'} = s^{t'-1,t'}) u_i(\bar{a}^{t+t'}(\bar{m}, m^{i,\hat{t}}, s^{t',t'}))$$  \hspace{1cm} \text{(T.10.29)}$$

Therefore, since the $\bar{s}^{t',t'}$ that we fixed is an arbitrary element of $S'_1$, we can now conclude that

$$\sum_{s^{t',t'} \in S'_1} \Pr(\bar{s}^{t',t'} = s^{t',t'}) u_i(\bar{a}^{t+t'}(\bar{m}^{i,\hat{t}}, \bar{m}^{i,\hat{t}}, s^{t',t'})) \geq \sum_{s^{t',t'} \in S'_1} \Pr(\bar{s}^{t',t'} = s^{t',t'}) u_i(\bar{a}^{t+t'}(\bar{m}, m^{i,\hat{t}}, s^{t',t'}))$$  \hspace{1cm} \text{(T.10.30)}$$

Now fix any $\bar{s}^{t',t'} \in S'_1$. Using, (T.10.25), (T.10.26) and (T.10.27), and Definitions B.8 and B.9 we get that the difference given by

$$\Pr(\bar{s}^{t',t'} = \pi^{t',t'}) u_i(a^{t+t'}(\bar{m}^{i,\hat{t}}, \bar{m}^{i,\hat{t}}, \pi^{t',t'})) + \sum_{t'' = t+1}^\infty \delta(t'' - t') \sum_{s^{t'',t'} \in S'_t(\pi^{t',t'})} \Pr(\bar{s}^{t'',t'} = \pi^{t',t'}) u_i(a^{t+t''}(\bar{m}^{i,\hat{t}}, \bar{m}^{i,\hat{t}}, \pi^{t'',t'}))$$  \hspace{1cm} \text{(T.10.31)}$$

minus

$$\Pr(\bar{s}^{t',t'} = \pi^{t',t'}) u_i(a^{t+t'}(\bar{m}, \bar{m}^{i,\hat{t}}, \pi^{t',t'})) + \sum_{t'' = t+1}^\infty \delta(t'' - t') \sum_{s^{t'',t'} \in S'_t(\pi^{t',t'})} \Pr(\bar{s}^{t'',t'} = \pi^{t',t'}) u_i(a^{t+t''}(\bar{m}, \bar{m}^{i,\hat{t}}, \pi^{t'',t'}))$$  \hspace{1cm} \text{(T.10.32)}$$

is greater or equal to

$$\Pr(\bar{s}^{t',t'} = \pi^{t',t'}) \left\{ \frac{\delta q \left[ 1 - \left( \frac{\delta}{1 - \delta (1 - q)} \right)^T \left( \bar{x}_i - \bar{u}_i \right) \right]}{(1 - \delta)} - (\bar{x}_i - \bar{u}_i) \right\}$$  \hspace{1cm} \text{(T.10.33)}$$

Notice now that we know that the quantity in (T.10.33) is in fact positive for $\delta$ sufficiently close to 1. This is simply because the term in curly brackets in (T.10.33) is the same as the right-hand side of (T.10.18). Therefore, we have dealt with any $\bar{s}^{t',t'} \in S'_1$ and with all its relevant “successors” of the form $S'_1(\bar{s}^{t',t'})$. Since $t'$ is arbitrary, by (T.10.27), this exhausts $S'_1$ and $S'_t$ for all possible values of $t'$.

Finally, we deal with $\bar{s}^{t',t'} \in S'_5$. Notice first of all that if we fix any $\bar{s}^{t',t'} \in S'_5$, then it follows from (T.10.21) and (T.10.28) that any $\bar{s}^{t',t'}$ of the form $[\bar{s}^{t',t'-1}, \bar{s}^{t'-1,t'}]$ (where $\bar{s}^{t'-1}$ are the first $t' - 1$ triples of $\bar{s}^{t',t'}$)
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\( \pi^{t,t'} \) is in fact in \( S^t_{\delta} \).

Using \eqref{eq:10.28} and Definitions B.8 and B.9 we get

\[
\sum_{s^{t-1,t'} \in S^t} \Pr(s^{t-1,t'} = s^{t-1,t'}) u_i(a_i^{t+t'}(m_i^{t'}, m_{-i}^{t'}, [s^{t,t'-1}, s^{t-1,t'}])) = q_{\mu_i} + (1-q)z_i > q_{\mu_i} + (1-q)z_i \geq \sum_{s^{t-1,t'} \in S^t} \Pr(s^{t-1,t'} = s^{t-1,t'}) u_i(a_i^{t+t'}(\tilde{m}, m_{-i}^{t'}, [s^{t,t'-1}, s^{t-1,t'}])) \tag{10.34}
\]

Therefore, since the \( \tilde{s}^{t,t'} \) that we fixed is an arbitrary element of \( S^t_{\delta} \), we can now conclude that

\[
\sum_{s^{t,t'} \in S^t_{\delta}} \Pr(s^{t,t'} = s^{t,t'}) u_i(a_i^{t,t'}(m_i^{t'}, m_{-i}^{t'}, s^{t,t'})) \geq \sum_{s^{t,t'} \in S^t_{\delta}} \Pr(s^{t,t'} = s^{t,t'}) u_i(a_i^{t,t'}(\tilde{m}, m_{-i}^{t'}, s^{t,t'})) \tag{10.35}
\]

Hence, the proof of the lemma is now complete. \( \blacksquare \)

**Remark 10.1:** Let the strategy profile \((\mu, \mu)\) described in Definitions B.8 and B.9 be given. Consider a player \((i, t)\), and a realization of future uncertainty \( s^{t,t'} \) as defined in the proof of Lemma T.10.4.

Let any message \( m \in M_i^{t+1} \) be given, and fix any information set \( \mathcal{I}^{t,E}_i \) and associated beliefs \( \Phi^{t,E}_i(\cdot) \).

It is then clear from Definitions B.8 and B.9 and T.10.1, that for any \( t' \) the action that player \((i, t)\) expects player \((i, t + t')\) to take is uniquely determined by \( m \), \( s^{t,t'} \) and \( \mathcal{I}^{t,E}_i \).

For the rest of the argument we will denote this by \( a_i^{t,t'}(m, s^{t,t'}, \mathcal{I}^{t,E}_i) \).

**Lemma 10.5:** There exists a \( \delta \in (0,1) \) such that whenever \( \delta > \delta \) the message-stage strategies described in Definition B.9 are sequentially rational given the beliefs described in Definition T.9.2 for every player \((i, t)\).

**Proof:** Consider any information set \( \mathcal{I}^{t,E}_i \in \mathcal{I}^{t,E}_i(z, \tau) \), where \( \mathcal{I}^{t,E}_i(z, \tau) \) is as in Definition T.10.1. It is then evident from Lemma T.10.4 and from the beliefs \( \Phi^{t,E}_i(z, \tau) \) described in Definition T.10.1 that for \( \delta \) sufficiently close to 1, the message strategies described in Definition B.9 are sequentially rational at any such information set.

From now on, consider any information set \( \mathcal{I}^{t,E}_i \not\in \mathcal{I}^{t,E}_i(z, \tau) \). Let \( m \in M_i^{t+1} \) be the message that player \((i, t)\) should send according to the strategy \( \mu_i \), and let \( \tilde{m} \) be any other message in \( M_i^{t+1} \). Consider a particular realization \( \pi^{t,t'} \), and for any \( t'' \in \{1, \ldots, t-1\} \), let \( \pi^{t,t'} \) denote the first \( t'' \) triples of \( \pi^{t,t'} \).

Next, assume that \( a_i^{t+t'}(m, \pi^{t,t'}, \mathcal{I}^{t,E}_i) \neq a_i^{t+t'}(\tilde{m}, \pi^{t,t'}, \mathcal{I}^{t,E}_i) \), and that either \( t' = 1 \), or alternatively that \( a_i^{t+t'}(m, \pi^{t,t'}, \mathcal{I}^{t,E}_i) = a_i^{t+t'}(\tilde{m}, \pi^{t,t'}, \mathcal{I}^{t,E}_i) \) for every \( t'' \in \{1, \ldots, t-1\} \).

Clearly, in periods \( \{t+1, \ldots, t'-1\} \), conditional on \( \pi^{t,t'} \), the payoff to player \((i, t)\) is unaffected by the deviation to \( \tilde{m} \). Now consider the payoff to player \((i, t)\), conditional on \( \pi^{t,t'} \), from the beginning of period \( t' \) on, for simplicity discounted from the beginning of period \( t' \). If player \((i, t)\) sends message \( m \) as prescribed by \( \mu_i \), and \( \delta \) is close enough to 1, the payoff in question is bounded below by

\[
(1 - \delta)u_i + \delta(q_{\pi_i} + (1-q)z_i) \tag{10.36}
\]

Now consider the payoff to player \((i, t)\) if he sends message \( \tilde{m} \), conditional on \( \pi^{t,t'} \), from the beginning of period \( t' \) on, for simplicity discounted from the beginning of period \( t' \). In period \( t' \) the action played cannot yield him more than \( \pi_i \). From Lemma T.10.4, we know that, for \( \delta \) close enough to 1, from the beginning of period \( t'+1 \) the payoff is bounded above by \( \tilde{v}_i(\tilde{z}, T) \). Hence, for \( \delta \) close enough to 1, using \eqref{eq:10.9} the payoff in question is bounded above by

\[
\delta u_i + (1 - \delta) \left[ 1 - \left( \frac{\delta q}{1 - \delta(1-q)} \right) \right]^T \left[ q_{\pi_i} + (1-q)z_i \right] + \left( \frac{\delta q}{1 - \delta(1-q)} \right) \left[ q_{\pi_i} + (1-q)z_i \right] \tag{10.37}
\]
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Notice now that the quantity in (T.10.36) is the same as the quantity in (T.10.15), and the quantity in (T.10.37) is in fact the same as the quantity in (T.10.16). Hence, exactly as in the proof of Lemma T.10.3, we know that, for \( \delta \) sufficiently close to 1, the quantity in (T.10.36) is greater than the quantity in (T.10.37). This is clearly enough to conclude the proof.

T.11. Proof of Theorem 3: Consistency of Beliefs

Remark T.11.1: Let \((g_\varepsilon, \mu_\varepsilon)\) be the completely mixed strategy profile of Definitions B.11 and B.12. It is then straightforward to check that as \( \varepsilon \to 0 \) the profile \((g_\varepsilon, \mu_\varepsilon)\) converges pointwise (in fact uniformly) to the equilibrium strategy profile described in Definitions B.8 and B.9, as required.

Lemma T.11.1: The strategy profile \((g, \mu)\) described in Definitions B.8 and B.9 and the beginning-of-period beliefs described in Definition T.9.1 are consistent.

Proof: When \( t = 0 \), there is nothing to prove. Assume \( t \geq 1 \). We consider two cases. First assume that player \((i, t)\) receives message \( m \in \{m^*\} \cup \bar{M}_{-i} \cup \bar{M}(i, t) \). Clearly, this is on the equilibrium path generated by the profile of strategies \((g, \mu)\) described in Definitions B.8 and B.9. Therefore, consistency in this case simply requires checking that the beginning-of-period beliefs described in Definition T.9.1 are obtained via Bayes’ rule from the profile \((g, \mu)\) and the action-stage strategies described in Definition B.8.

Now assume that player \((i, t)\) receives message \( m \not\in \{m^*\} \cup \bar{M}_{-i} \cup \bar{M}(i, t) \). From Definition T.9.1 it is immediate to check that in this case player \((i, t)\) assigns probability one to the event that \( m_{t-i} = (m, \ldots, m) \). Notice however, that the profile \((g, \mu)\) is such that a single deviation by one player at the action stage is sufficient to generate the message profile \( m^i = (m, \ldots, m) \). Therefore, upon observing \( m \not\in \{m^*\} \cup \bar{M}_{-i} \cup \bar{M}(i, t) \) the probability that \( m_{t-i} = (m, \ldots, m) \) is an infinitesimal in \( \varepsilon \) of order no higher than 2. \( T.21 \) This needs to be compared with the probability that \( m_{t-i} \neq (m, \ldots, m) \) and \( m_{t} = m \). The latter event is impossible given the profile \((g, \mu)\) unless a deviation at the message stage has occurred at some point. Therefore its probability is an infinitesimal in \( \varepsilon \) of order no lower than \( 2n + 1 \). This is obviously enough to prove the claim.

Lemma T.11.2: The strategy profile \((g, \mu)\) described in Definitions B.8 and B.9 and the end-of-period beliefs described in Definition T.9.2 are consistent.

Proof: The case \( t = 0 \) is trivial. Assume \( t \geq 1 \), and consider any player \((i, t)\) after having observed \((m^i, x^t, a^t, y^t)\).

We deal first with the case in which \( x^t = x(\kappa) \) with \( \kappa > \bar{\kappa} \). Let \( x(\ell_{00}, \ell^{*}) \) denote the realization \( x^t \). In this case, the action-stage strategies described in Definition B.8 prescribe that every player \((k \in I, t)\) should play \( a^t_1(\ell^{*}) \). Therefore, if the observed action profile \( a^t \) is equal to \( a(\ell^{*}) \), player \((i, t)\) does not revise his beginning-of-period beliefs during period \( t \). Hence consistency in this case follows immediately from the profile \( \mu \) and from the consistency of beginning-of-period beliefs, which of course was proved in Lemma T.11.1. Notice now that if \( a^t \neq a(\ell^{*}) \), then the message strategies described in Definition B.9 prescribe that each player \((k \in I, t)\) should send a message that does not depend on the message \( m^i \) he received. Hence, in this case consistency is immediate from Definition T.9.2 and the profile \( \mu \).

We now turn to the case in which \( x^t = x(\kappa) \) with \( \kappa \leq \bar{\kappa} \). Here, it is necessary to consider several subcases, depending on the message \( m \) received by player \((i, t)\). Assume first that \( m \not\in \bar{M}_{-i} \cup \bar{M}(i, t) \). Then for any possible triple \((x^t, a^t, y^t)\) we have that

\[
\lim_{\varepsilon \to 0} \Pr(m_{t-i} = (m, \ldots, m) \mid m^i_t = m, x^t, a^t, g_\varepsilon, \mu_\varepsilon) = 1
\]

(\text{T.11.1})

\(^{T.21}\) See footnote T.2 above for a specification of our (standard) use of terminology concerning the orders of infinitesimals.
To see this consider two sets of possibilities. First, \( m = m^* \), \( x^t = x(\cdot, \hat{\ell}, \cdots, \cdot) \), and \( a^t = (a_1(\hat{\ell}), \ldots, a_n(\hat{\ell})) \). Then play is as prescribed by the equilibrium path generated by the profile \((g, \mu)\), and from Definitions B.8 and B.9 there is nothing more to prove. For all other possibilities, notice that the event \( m^t = (m, \ldots, m) \) is consistent with any \( a^t \) together with \( n \) deviations at the action stage of the second type described in Definition B.11. Therefore, for any \( a^t \), the probability of \( m^t = (m, \ldots, m) \) and \( a^t \) is an infinitesimal in \( \varepsilon \) of order no higher than \( 2n \). On the other hand, from Definition B.12 it is immediate that the probability that \( m^i_{\ell-1} \neq (m, \ldots, m) \) (since it requires at least one deviation at the message stage) is an infinitesimal in \( \varepsilon \) of order no lower than \( 2n + 1 \). Hence (T.11.1) follows. From (T.11.1) it is a matter of routine to check the consistency of end-of-period beliefs from using the profile \((g, \mu)\). We omit the details.

Still assuming that \( x^t = x(\kappa) \) with \( \kappa \leq \pi \), now consider the case \( m = \tilde{m}_j \in \tilde{M}_{\ell-1} \). In this case we can show that

\[
\lim_{\varepsilon \to 0} \Pr(m^i_{\ell-1} = (\tilde{m}_j, \ldots, \tilde{m}_j) \mid m^t_i = \tilde{m}_j, x^t, a^t, g_\varepsilon, \mu_\varepsilon) = 1
\]

(T.11.2)

using an argument completely analogous to the one we used for (T.11.1). The details are omitted. As in the previous case, from (T.11.2) it is a matter of routine to check the consistency of end-of-period beliefs from using the profile \((g, \mu)\).

The last case remaining is \( x^t = x(\kappa) \) with \( \kappa \leq \pi \) and \( m = m^{i,\tau} \). In this case we have that

\[
\lim_{\varepsilon \to 0} \Pr(m^i_{\ell-1} = (\tilde{m}_i, \ldots, \tilde{m}_i) \mid m^t_i = m^{i,\tau}, x^t, a^t, g_\varepsilon, \mu_\varepsilon) + \\
\lim_{\varepsilon \to 0} \Pr(m^i_{\ell-1} = (m^{i,\tau}, \ldots, m^{i,\tau}) \mid m^t_i = m^{i,\tau}, x^t, a^t, g_\varepsilon, \mu_\varepsilon) = 1
\]

(T.11.3)

Again, the argument is completely analogous to the one used for (T.11.1) and (T.11.2), and the details are omitted. Now take (T.11.3) as given and let \( x^t = (\cdots, i_\ell, \cdots) \).

Suppose next that \( a^i_{\ell-1} = \tilde{a}^i_{\ell-1}(i_\ell) \). Then player \((i, t)\) does not revise his beginning-of-period beliefs, and hence, using the profile \( \mu \) and Lemma T.11.1 it is immediate to check that his end-of-period beliefs are consistent in this case.

Now suppose that for some \( j \neq i \) we have that \( a^j_\ell \neq \tilde{a}^j_\ell(i_\ell) \) and \( a^i_{\ell-1} = \tilde{a}^i_{\ell-1}(i_\ell) \). Consistency of beliefs in this case requires showing that the first element in the sum in (T.11.3) is equal to 1. Of course given (T.11.3) it suffices to compare the probabilities of the two events \( m^i_{\ell-1} = (\tilde{m}_i, \ldots, \tilde{m}_i) \) and \( m^i_{\ell-1} = (m^{i,\tau}, \ldots, m^{i,\tau}) \). The first is compatible with a single deviation at the action stage on the part of player \((j, t)\). Therefore its probability is an infinitesimal in \( \varepsilon \) of order no higher than 2. The latter requires an action-stage deviation in some period \( t' < t \) (order 2 in \( \varepsilon \)), and \( n - 2 \) action-stage deviations in period \( t \) (order 1 each). Hence, player \((i, t)\) has consistent beliefs if he assigns probability 1 to \( m^i_{\ell-1} = (\tilde{m}_i, \ldots, \tilde{m}_i) \). The consistency of his end-of-period beliefs can then be checked from the profile \( \mu \).

Now suppose that for some \( j \neq i \) we have that \( a^j_\ell \neq a^j_\ell(i_\ell) \) and \( a^i_{\ell-1} = a^i_{\ell-1}(i_\ell) \). Consistency of beliefs in this case requires showing that the second element in the sum in (T.11.3) is equal to 1. Of course given (T.11.3) it suffices to compare the probabilities of the two events \( m^i_{\ell-1} = (\tilde{m}_i, \ldots, \tilde{m}_i) \) and \( m^i_{\ell-1} = (m^{i,\tau}, \ldots, m^{i,\tau}) \). The first requires \( (n - 2) \) deviations at the action-stage of period \( t \), each of order 2 in \( \varepsilon \). Since \( n \geq 4 \), this is therefore an infinitesimal in \( \varepsilon \) of order no lower than 4. The second is consistent with a deviation of order 2 in \( \varepsilon \) at the action-stage of some period \( t' < t \), together with a deviation of order 1 in \( \varepsilon \) at the action stage of period \( t \). Therefore its probability is an infinitesimal in \( \varepsilon \) of order no higher than 3. Hence, player \((i, t)\) has consistent beliefs if he assigns probability 1 to \( m^i_{\ell-1} = (\tilde{m}_i, \ldots, \tilde{m}_i) \). The consistency of his end-of-period beliefs can then be checked from the profile \( \mu \). The same argument applies to show the consistency of his end-of-period beliefs when \( a^i_{\ell-1} = a^i_{\ell-1}(i_\ell) \). We omit the details.

In all other possible cases for \( a^t \), the messages sent by all players \((j \neq i, t)\) do not in fact depend on \( a^t \), provided that \( m^t_i \) is either \( \tilde{m}_i \) or \( m^{i,\tau} \). Given (T.11.3), the consistency of the end-of-period beliefs of player \((i, t)\) can then be checked directly from the profile \( \mu \).
T.12. Proof of Theorem 3

Given any $v^* \in \text{int}(V)$ and any $\delta \in (0, 1)$, using (B.8), (B.7) and the strategies and correlation devices described in Definitions B.4, B.5, B.8 and B.9 clearly implement the payoff vector $v^*$.

From Lemmas T.10.3 and T.10.5 we know that there exists a $\hat{\delta} \in (0, 1)$ such that whenever $\delta > \hat{\delta}$ each strategy in the profile described in Definitions B.8 and B.9 is sequentially rational given the beliefs described in Definitions T.9.1 and T.9.2.

From Lemmas T.11.1 and T.11.2 we know that the strategy profile described in Definitions B.8 and B.9 and the beliefs described in Definitions T.9.1 and T.9.2 are consistent.

Hence, using Lemma T.2.1, the proof of Theorem 3 is now complete. ■