The Folk Theorem in Dynastic Repeated Games

LUCA ANDERLINI
(Georgetown University)

DINO GERARDI
(Yale University)

ROGER LAGUOFF
(Georgetown University)

December 2004

Abstract. A canonical interpretation of an infinitely repeated game is that of a "dynastic" repeated game: a stage game repeatedly played by successive generations of finitely-lived players with dynastic preferences. These two models are in fact equivalent when the past history of play is observable to all players.

In our model all players live one period and do not observe the history of play that takes place before their birth, but instead receive a private message from their immediate predecessors.

Under very mild conditions, when players are sufficiently patient, all feasible payoff vectors (including those below the minmax) can be sustained as a Sequential Equilibrium of the dynastic repeated game with private communication. The result applies to any stage game for which the standard Folk Theorem yields a payoff set with a non-empty interior.

We are also able to characterize entirely when a Sequential Equilibrium of the dynastic repeated game can yield a payoff vector not sustainable as a Subgame Perfect Equilibrium of the standard repeated game. For this to be the case it must be that the players’ equilibrium beliefs violate a condition that we term “Inter-Generational Agreement.”

JEL Classification: C72, C73, D82.

Keywords: Dynastic Repeated Games, Private Communication, Folk Theorem.

Address for Correspondence: Luca Anderlini, LA2@GEORGETOWN.EDU, Department of Economics, Georgetown University, 37th and O Streets, Washington DC 20057, USA.

*We are grateful to Leonardo Felli, Stephen Morris, Michele Piccione and Andrew Postlewaite, and to seminar audiences at Bocconi, Chicago, Columbia, Georgetown, Rutgers, SUNY-Albany, Venice and Yale for helpful feedback. All errors are our own.
1. Introduction

1.1. Motivation

The theory of infinitely repeated games is a cornerstone of modern economic analysis. Besides its tremendous effect on the theory of games itself, it has pervaded economic applications that stretch from industrial organization, to political economy, to development economics.

However, it seems self-evident that an infinitely repeated game cannot be interpreted literally as a stage game played infinitely many times by infinitely-lived players.

One interpretation that does not collide directly with the obvious fact that players are finitely-lived is that of repeated play that terminates in every period with (say constant) positive probability. The repeated interaction eventually ends with probability one, and hence does not require the players to literally live forever.

The stochastic termination interpretation is also hard to take literally, however. First of all, it requires that players live indefinitely long, albeit with vanishingly small probability. Second, it is at odds with a common limit operation that is carried out in the analysis of repeated games; namely that the players get closer and closer to becoming infinitely patient. Unless the context warrants interpreting this limit as an ever increasing frequency of play, it corresponds to a termination probability that shrinks to zero, and yields a life expectancy for the players that becomes unboundedly large.

A more compelling interpretation of a repeated game is that of “dynastic” repeated game: a stage game repeatedly played by successive generations of finitely-lived players with dynastic preferences. Whether the length of players’ lives is stochastic or deterministic, the model does not extend outside the bounds of the physically possible. Even the case of almost-infinite patience in the standard model is re-interpreted as a world with almost-complete altruism (towards successors in the same dynasty) and hence may seem more or less plausible, depending on circumstances.

Suppose that the altruistic component of players’ preferences replicates the temporally discounted preferences of infinitely-lived players. Suppose also that all players observe the past history of play. Then dynastic repeated games are an extremely robust way to re-interpret the standard model. An application of the one-shot deviation principle is enough to show that the standard repeated game and the dynastic game are equivalent, both in terms of Subgame-Perfect Equilibrium (henceforth SPE) strategies and consequently of equilibrium payoffs. This is so regardless of the “demographic structure” of the dynastic game. In other words, the equivalence holds regardless of the process governing birth and death of successive generations of players, provided that extinction of any dynasty is ruled out.

Given the appeal of the dynastic interpretation of the standard model, it is clearly important to question closely the plausibility of the basic structure of the dynastic game with “full memory” described above. In our view, the feature of the model that stands out as deserving skepticism is the full memory of the players. Why should the players be able to
observe the history of play that took place before their birth (or more loosely, before they began participating in the game)? At best they will be able to observe a “footprint” of the previous history of play which will inevitably fall short of full precision. At worst, they will observe nothing at all, and will have to rely on communication with their predecessors for information about what happened in the past.

In fact, in many situations of economic interest, postulating that communication is the only way to convey information about the past seems a compelling assumption. For instance, if the dynasties are competing firms in an oligopolistic market, whose behavior is controlled by successive generations of managers it hardly seems plausible that the current managers would have access to a physical record of the past history of play. They may of course have access to accounts and a host of other past records. But it seems compelling to say that these are better modeled as messages from past players, rather than direct observation of the past.

In this paper we characterize the set of Sequential Equilibria (henceforth SE) of dynastic repeated games in which all players live one period, and have to rely on private messages from their immediate predecessors about the previous history of play. Our results would remain intact if public messages were allowed alongside private ones.

Of course, one would like to know what happens when the demographics of the model are generalized, and what would be the effects of the presence of an imperfect but non-null physical footprint of the past history of play, which we assume away. We return to these and other issues in Section 10 below, which concludes the paper.

1.2. Results

The Folk Theorem of repeated games is a stubbornly robust result. For instance, to cite but a few contributions, the result survives more or less intact when incomplete (Fudenberg and Maskin 1986) or imperfect public (Fudenberg, Levine, and Maskin 1994) information is allowed, or when the players have bounded memory (Sabourian 1998).

Considerable effort has gone into introducing considerations that reduce the equilibrium set of a repeated game. For instance, depending on the stage game, the set of equilibrium payoffs is known to shrink by varying degrees when complexity costs are (lexicographically) taken into account (Rubinstein 1986, Abreu and Rubinstein 1988, Piccione 1992, Piccione and Rubinstein 1993), when strategies and beliefs are restricted to be Turing-computable (Anderlini and Sabourian 1995, Anderlini and Sabourian 2001), or when asynchronous choice is allowed (Lagunoff and Matsui 1997).

In all the contributions of which we are aware, the equilibrium sets of the repeated game share a common feature. It is never the case that a player’s equilibrium payoff can be below his minmax value. By contrast, we show that, under very general conditions, in a dynastic

---

1With the notable exception of Takahashi and Wen (2003) whose benchmark “effective minmax” is determined by the fact that they consider asynchronously repeated games.
repeated game with private communication there are equilibria that yield the players (possibly even all players) a payoff below their minmax value. In short, we find that in a very broad class of games (which includes the $n \geq 3$-player version of the Prisoners’ Dilemma), as the players become more and more altruistic (patient), all (interior) payoff vectors in the convex hull of the payoffs of the stage game can be sustained as an SE of the dynastic repeated game with private communication. For some stage games, the ability to sustain all (interior) payoffs also implies that, in some of the SE of the dynastic repeated game with private communication, one or more players will receive payoffs that are higher than the maximum they could receive in any SPE of the standard repeated game.

A closer look at the SE that prove the assertion we have just made reveals an interesting fact. There is an obvious sense in which, following some histories of play, in equilibrium, beliefs are “mismatched” across players of the same cohort. This phenomenon, in turn, can be traced back to the fact that following some histories of play, the structure of the equilibrium does not permit a player to communicate to his successor some information about the past that is relevant for future play. This may happen while the other players do not face the same difficulty, and hence generate the mismatch in beliefs. Below, we formalize this notion via a condition that we term “Inter-Generational Agreement.” The mismatch in beliefs is equivalent to saying that an SE violates Inter-Generational Agreement.

We are able to show that the mismatching phenomenon we have just described characterizes fully the difference between the set of SE of the dynastic repeated game and the set of SPE of the standard repeated game. Any SPE of the standard repeated game can be replicated as an SE of the dynastic repeated game that displays Inter-Generational Agreement. Conversely any SE of the dynastic repeated game that yields a payoff vector that is not sustainable as an SPE of the standard repeated game must violate Inter-Generational Agreement.

2. Outline

The outline of the rest of the paper is as follows. In Section 3 we lay down the notation and details of the Standard Repeated Game and of the Dynastic Repeated Game. In Section 4 we define what constitutes an SE for the Dynastic Repeated Game. In Section 5 we present our first result asserting that all payoffs that can be sustained as SPE of the Standard Repeated Game can be sustained as SE of the Dynastic Repeated Game. Section 6 is devoted to our first “extended” Folk Theorem for the Dynastic Repeated Game. This result applies to the case of three or more dynasties. In Section 7 we present our second extended Folk Theorem. The result we state there applies to the case of four dynasties or more. In Section 8 we report a result that completely characterizes the features of an SE of the Dynastic Repeated Game that make it possible to sustain payoff vectors that are not sustainable as SPE of the Standard Repeated Game. Section 9 reviews some related literature, and Section 10 concludes.
For ease of exposition, and for reasons of space, no formal proofs appear in the main body of the paper. The main ingredients (correlation devices, strategies and trembles) for the proofs of our two extended Folk Theorems (Theorems 2 and 3) appear in Appendix A and Appendix B respectively. The complete proof of Theorem 4 appears in Appendix C. In the numbering of equations, Lemmas etc. a prefix of “A,” “B” or “C” or means that the item is located in the corresponding Appendix.

A technical addendum to the paper contains the rest of the formal proofs. In particular, the technical addendum contains the proof of Theorem 1 and the analysis of “consistency” and “sequential rationality” that closes the proofs of Theorems 2 and 3. In the numbering of equations, Lemmas etc. a prefix of “T.X” means that the item is located in Section X of the technical addendum.

3. The Model

3.1. A Standard Repeated Game

We first describe a standard, n-player repeated game. We will then augment this structure to describe the dynastic repeated game with communication from one cohort to the next. The standard repeated game structure is of course familiar. We set it up below simply to establish the basic notation.

Since in our main results below we will make explicit use of public randomization devices we build these in our notation right from that start.

The stage game is described by the array \( G = (A, u; I) \) where \( I = \{1, \ldots, n\} \) is the set of players, indexed by \( i \). The \( n \)-fold cartesian product \( A = \times_{i \in I} A_i \) is the set of pure action profiles \( a = (a_1, \ldots, a_n) \in A \), assumed to be finite. Stage game payoffs are defined by \( u = (u_1, \ldots, u_n) \) where \( u_i : A \to \mathbb{R} \) for each \( i \in I \). Let \( \sigma_i \in \Delta(A_i) \) denote a mixed strategy for \( i \), with \( \sigma \) denoting the profile \( (\sigma_1, \ldots, \sigma_n) \).

The symbol \( \sigma_i(a_i) \) represents the probability of pure action \( a_i \) given by \( \sigma_i \), so that for any \( a = (a_1, \ldots, a_n) \), with a minor abuse of notation, we can let \( \sigma(a) = \Pi_{i \in I} \sigma_i(a_i) \) denote the probability of pure action profile \( a \) that \( \sigma \) induces. The corresponding payoff to player \( i \), is defined in the usual way: \( u_i(\sigma) = \sum_{a \in A} \sigma(a) u_i(a) \).

Dropping the \( i \) subscript and writing \( u(\sigma) \) gives the entire profile of payoffs.

Throughout the rest of the paper, we denote by \( V \) the convex hull of the set of payoff vectors from pure strategy profiles in \( G \). We let \( \text{int} V \) denote the (relative) interior of \( V \).\(^5\)

---

\(^2\)The technical addendum is available via [http://www.georgetown.edu/faculty/la2/Folktheorem.htm](http://www.georgetown.edu/faculty/la2/Folktheorem.htm).

\(^3\)For ease of reference, pages and footnotes are also numbered separately in the technical addendum. Their numbers have a prefix “T.”

\(^4\)As standard, here and throughout the paper and the technical addendum, given any finite set \( Z \), we let \( \Delta(Z) \) be the set of all probability distributions over \( Z \).

\(^5\)Of course, when \( V \) has full dimension (\( n \)) the relative interior of \( V \) coincides with the interior of \( V \). In general, \( \text{int} V \) is simply the set of payoff vectors that can be achieved placing strictly positive weight on all payoff vectors obtained from pure action profiles in \( G \). We use the qualifier “relative” since we will not be

...
We assume that all players observe the outcome of a public randomization device. We model this as a random variable $\tilde{x}$ taking values in a finite set $X$, so that its distribution is a point in $\Delta(X)$. Each player can condition his choice of mixed strategy on the realization of $\tilde{x}$, denoted by $x$. Player $i$ chooses a mixed strategy $\sigma_i \in \Delta(A_i)$ for each $x \in X$.

In the repeated game, time is discrete and indexed by $t = 0, 1, 2, \ldots$ The pure (realized) action profile played at time $t$ is denoted by $a^t$. In each period there is a new public randomization device $\tilde{x}^t$. These devices are i.i.d. across periods; we write $\tilde{x}$ to indicate the random variable of which all the $\tilde{x}$’s are independent “copies.” We refer to $\tilde{x}$ as the action-stage correlation device. A history of length $t \geq 1$, denoted by $h^t$, is an object of the form $(x^0, a^0, \ldots, x^{t-1}, a^{t-1})$. The “initial” history $h^0$ is of course the “null” history $\emptyset$. The set of all possible histories of length $t$ is denoted by $H^t$, while $H = \cup_{t=0}^{\infty} H^t$ denotes the collection of all possible histories of play.

A strategy for player $i$ in the repeated game is denoted by $g_i$ and can be thought of as a collection $(g^0_i, g^1_i, \ldots, g^n_i, \ldots)$ with each $g^t_i$ a function from $H^t \times X$ into $\Delta(A_i)$. The profile of repeated game strategies is $g = (g_1, \ldots, g_n)$, while $g^t$ indicates the time-$t$ profile $(g^0_1, \ldots, g^n_i)$. Given a profile $g$, recursing forward as usual, we obtain a probability distribution over action profiles $a^0$ played in period 0, then a probability distribution over profiles $a^t$ to be played in period 1, then in period 2, and so on without bound, so that we have a distribution over the profile of actions to be played in every $t \geq 0$. Of course, this forward recursion yields a probability distribution $\mathcal{P}(g)$ over the set of all possible sequences $(a^0, \ldots, a^t, \ldots)$ (which can be seen as a probability distribution over the set of possible actual histories of play $H$).

The players’ common discount factor is denoted by $\delta \in (0, 1)$. Given a profile $g$, player $i$’s expected payoff in the repeated game is denoted by $v_i(g)$ and is given by:

$$v_i(g) = E_{\mathcal{P}(g)} \left\{ (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(a^t) \right\} \quad (1)$$

Given a profile $g$ and a particular finite history $h^t$, we recurse forward as above to find the distribution over infinite sequences $(a^0, \ldots, a^t, \ldots)$, this time conditional on history $h^t$ and the realization $x^t$ having taken place. We denote this distribution by $\mathcal{P}(g, h^t, x^t)$. This allows making explicit assumptions on the dimensionality of $V$, although the hypotheses of many of our results will imply that $V$ should satisfy certain dimensionality conditions that we will point out below.

It should be made clear at this stage that at no point in the paper do we assume that mixed strategies are observable. Whatever (mutually independent) devices the players use to achieve a desired randomization among pure actions in $G$, given a realization of $\tilde{x}$, remain unobservable to other players.

Throughout, we restrict attention to action-stage correlation devices that have full support. That is, we assume that $\tilde{x}$ takes all values in the finite set $X$ with strictly positive probability. This seems reasonable, and keeps us away of a host of unnecessary technicalities.

Clearly, $v_i(g)$ depends on $\delta$ as well. To lighten the notation, this will omitted whenever doing so does not cause any ambiguity.
us to define the *continuation* payoff to player $i$, conditional on the pair $(h^t, x^t)$. This will be denoted by $v_i(g|h^t, x^t)$ and is given by

$$
v_i(g|h^t, x^t) = (1 - \delta) \left\{ u_i(g'(h^t, x^t)) + \mathbb{E}_{P(g,h^t,x^t)} \left[ \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u_i(a^\tau) \right] \right\}
$$

(2)

As before, dropping the subscript $i$ in either (1) or (2) will indicate the entire profile.

At this point we note that, as is standard, given (1) and (2) we know that the continuation payoffs in the repeated game follow a recursive relationship. In particular we have that

$$
v_i(g|h^t, x^t) = (1 - \delta) u_i(g'(h^t, x^t)) + \delta \mathbb{E}_{g'(h^t, x^t), \tilde{x}^{t+1}} \{ v_i(g|h^t, x^t, g'(h^t, x^t), \tilde{x}^{t+1}) \}
$$

(3)

where $(h^t, x^t, g'(h^t, x^t), \tilde{x}^{t+1})$ is the (random) history generated by the concatenation of $(h^t, x^t)$ with the realization of the mixed strategy profile $g'(h^t, x^t)$ and the realization of $\tilde{x}^{t+1}$. The expectation is then taken with respect to the realization of the mixed strategy profile $g'(h^t, x^t)$ and of $\tilde{x}^{t+1}$.

A *Subgame-Perfect Equilibrium*, $g^*$, for the repeated game is defined in the usual way: for each $i$, and each finite history $(h^t, x^t)$, and each strategy $g_i$ for $i$, we require that $v_i(g^*|h^t, x^t) \geq v_i(g_i, g^*_-|h^t, x^t)$.

We denote by $G^S(\delta, \tilde{x})$ the set of SPE strategy profiles, and by $E^S(\delta, \tilde{x})$ the set of SPE payoff profiles of the repeated game when correlation device $\tilde{x}$ is available and the common discount factor is $\delta$. We also let $E^S(\delta) = \bigcup_{\tilde{x}} E^S(\delta, \tilde{x})$ with the union being taken over the set of all possible finite random variables that may serve as correlation devices.

The standard model of repeated play we have just sketched out may be found in a myriad of sources. See, for example, Fudenberg and Maskin (1986) and the references contained therein. Hereafter, we refer to the standard model above as the *standard repeated game*.

We conclude our discussion of the standard repeated game noting that the one-shot deviation principle gives an immediate way to check, subgame by subgame, whether a profile $g^*$ is an SPE. Since it is completely standard, we state the following without proof.

**Remark 1.** *One-Shot Deviation Principle:* A profile of strategies $g^*$ is in $G^S(\delta, \tilde{x})$ if and only if for every $i \in I$, $h^t \in H$, $x^t \in X$ and any $\sigma_i \in \Delta(A_i)$ we have that

$$
v_i(g^*|h^t, x^t) \geq v_i(\sigma_i, g_i^{-t*, g^*_-}|h^t, x^t)
$$

(4)

where, in keeping with the notational convention we adopted above, $g_i^{-t*}$ stands for the profile $(g_i^0, \ldots, g_i^{t-1*}, g_i^{t+1*}, \ldots)$.

---

As is standard, here, and throughout the rest of the paper and the technical addendum, a subscript or a superscript of $-i$ indicates an array with the $i$-th element taken out.
3.2. The Dynastic Repeated Game: Full Memory

The first dynastic repeated game that we describe is a straw-man. It turns out to be equivalent, in both payoffs and strategies, to the standard repeated game. It fulfills purely an expository function in our story.

Assume that each \( i \in I \) indexes an entire progeny of individuals. We refer to each of these as a dynasty. Individuals in each dynasty are assumed to live one period. At the end of each period \( t \) (the beginning of period \( t+1 \)), a new individual from each dynasty — the date \((t+1)\)-lived individual — is born and replaces the date \( t \) lived individual in the same dynasty. Hence, \( u_i(a^t) \) now refers to the payoff directly received by the date \( t \) individual in dynasty \( i \).

Each date \( t \) individual is altruistic in the sense that his payoff includes the discounted sum of the direct payoffs of all future individuals in the same dynasty. The weight given to his own direct payoff is \( 1 - \delta \), while the weight given to the discounted sum of payoffs of his line of offsprings is \( \delta \). All cohorts in all dynasties can observe directly the past history of play. So all individuals in cohort \( t \), can observe \( h^t = (x^0, a^0, \ldots, x^{t-1}, a^{t-1}) \). It follows that \( g^t_i \) as above can now be interpreted as the strategy of individual \( i \) in cohort \( t \) (henceforth, simply “player \( \langle i, t \rangle \)”).

Therefore, in the full memory dynastic repeated game, given a profile of strategies \( g \) (now interpreted as an array giving a strategy for each player \( \langle i, t \rangle \)) the overall payoff to player \( \langle i, t \rangle \) conditional on history \( (h^t, x^t) \) can be written exactly as in (2) above.

Denote by \( G^F(\delta, \tilde{x}) \) the set of SPE strategy profiles, and by \( E^F(\delta, \tilde{x}) \) the set of SPE payoff profiles of the repeated game when the correlation device \( \tilde{x} \) is available and the common discount factor is \( \delta \). Now observe that from Remark 1 we know that a profile \( g \) constitutes an SPE of the standard repeated game if and only if when we interpret \( g \) as a strategy profile in the full memory dynastic repeated game, no player \( \langle i, t \rangle \) can gain by unilaterally deviating. Hence the equilibrium sets of the standard repeated game and of the full memory dynastic repeated game must be the same. Purely for completeness we state the following formally, but we omit any further proof.

**Theorem 0.** Dynastic Interpretation of the Standard Repeated Game: Let any stage game \( G \), any \( \delta \in (0,1) \) and any correlation device \( \tilde{x} \) be given. Then the standard repeated game and the full memory dynastic repeated game are equivalent in the sense that \( G^S(\delta, \tilde{x}) = G^F(\delta, \tilde{x}) \) and \( E^S(\delta, \tilde{x}) = E^F(\delta, \tilde{x}) \).

3.3. The Dynastic Repeated Game: Private Communication

We are now ready to drop the assumption that individuals in the \( t \)-th cohort observe the previous history of play. The players’ dynastic preferences are exactly as in the full memory dynastic repeated game described in Subsection 3.2 above.
Anderlini and Lagunoff (2005) analyzed the case in which each individual in the \( t \)-cohort has to rely on \textit{publicly observable messages} from the previous cohort for any information about the previous history of play.\(^{10}\)

In this paper, we assume that each player \( \langle i, t \rangle \) receives a \textit{private message} from player \( \langle i, t - 1 \rangle \) about the previous history of play. It should be noted at this point that our results would survive intact if we allowed \textit{public messages as well as private ones}. All the equilibria we construct below would still be viable, with the public messages ignored. We return to this issue in Section 10 below.\(^{11}\)

For simplicity, the set of messages \( M_{t+1} \) available to each player \( \langle i, t \rangle \) to send to player \( \langle i, t + 1 \rangle \) is the set \( H_{t+1} \) of finite histories of length \( t + 1 \) defined above.\(^{12}\) The message sent by player \( \langle i, t \rangle \) will be denoted by \( m_{t+1}^i \), so that player \( \langle i, t \rangle \) is the \textit{recipient} of message \( m_i^t \).

We also introduce a new public randomization device in every period, which is observable to all players in the \( t \)-th cohort at the stage in which they select the messages to be sent to the \( t + 1 \)-th cohort.\(^{13}\)

We let a finite random variable \( \tilde{y}_t \) taking values in a finite set \( Y \) be realized in every period. A typical realization of \( \tilde{y}_t \) is denoted by \( y_t \). We take these random variables to be mutually independent across any two time periods. We write \( \tilde{y} \) to indicate the finite random variable of which all the \( \tilde{y}_t \) are independent “copies,” and we refer to \( \tilde{y} \) as the message-stage correlation device.\(^{14}\)

To summarize, we take the time-line of events, observations and decisions within each

---

\(^{10}\)See Section 9 below for a brief discussion of the results in Anderlini and Lagunoff (2005) and other related papers.

\(^{11}\)Dealing explicitly with both private and public messages would be cumbersome, and make our results considerably less transparent. Analyzing the model with private messages only is the most economical way to put our main point across, and hence this is how we proceed.

\(^{12}\)As will be apparent from our proofs, smaller messages spaces — even ones that stay bounded in size through time — would suffice for our purposes. Taking the message spaces to coincide with the sets of previous histories of play seems the natural canonical modeling choice.

\(^{13}\)Three points are worth emphasizing at this stage. First of all, in general, we consider the availability of a correlation device (a publicly observable finite random variable) a weak assumption in just about any game. To reiterate an obvious fact, the device simply has to be available for the players to observe; whether or not the players decide to condition their play on its realization is an attribute of the \textit{equilibrium} of the model. Second, the assumption we are now making that a second correlation device is available seems a natural one in the model we are setting up. The players in the \( t \)-th cohort each take two successive decisions: what to play and then what to say. Given that a correlation device is available when play is decided, it seems natural, in fact compelling, that the circumstances should be the same when messages are sent to the next cohort of players. Third, and most importantly, all our theorems survive literally unchanged (with the proof of Theorem 2 being the only one that needs modification) if we assume that both correlation devices are simultaneously observable to all players in the \( t \)-th cohort at the time they are called upon to choose their actions in the stage game. In this case, clearly a \textit{single} correlation device would suffice.

\(^{14}\)Throughout, we restrict attention to message-stage correlation devices that have \textit{full support}. That is, we assume that \( \tilde{y} \) takes \textit{all} values in the finite set \( Y \) with strictly positive probability. See also footnote 7 above.
time period $t$ to be as follows. At the beginning of each time period $t$, each player $\langle i, t \rangle$ receives a private message $m_i^t \in H^t$ from player $\langle i, t - 1 \rangle$. Next, the random variable $\bar{x}^t$ is realized. Its realization $x^t$ is observed by all players $\langle i, t \rangle$ in the $t$-th cohort. After observing $x^t$, each $\langle i, t \rangle$ in the $t$-th cohort selects a mixed strategy $\sigma_i^t \in \Delta(A_i)$ for the stage game $G$. These choices are simultaneous. Subsequently, all players $\langle i, t \rangle$ in the $t$-th cohort observe the realized action profile $a^t$, which is of course the realization of the mixed strategy profile $\sigma^t$. After the profile $a^t$ is observed, the random variable $\bar{y}^t$ is realized and all players $\langle i, t \rangle$ in the $t$-th cohort observe its realization $y^t$. Finally, after observing $y^t$, each player $\langle i, t \rangle$ in the $t$-th cohort selects a probability distribution $\phi_i^t \in \Delta(H^{t+1})$ over messages $m_i^{t+1}$ in the set $H^{t+1}$.

The realized message from this distribution is then sent to player $\langle i, t+1 \rangle$, who observes it. In terms of notation, we distinguish between the action strategy of player $\langle i, t \rangle$, denoted by $g_i^t$, and the message strategy of player $\langle i, t \rangle$, denoted by $\mu_i^t$. Since we take the space of possible messages that player $\langle i, t \rangle$ may receive to be $M_i^t = H^t$, formally $g_i^t$ is exactly the same object as in the standard repeated game of Subsection 3.1 and the full memory dynastic repeated game of Subsection 3.2 above. In particular $g_i^t$ takes as input a message $m_i^t \in H^t$ and a value $x^t \in X$ and returns a mixed strategy $\sigma_i^t \in \Delta(A_i)$.

The message strategy $\mu_i^t$ of player $\langle i, t \rangle$ takes as inputs a message $m_i^t$, the realization $x^t$, the realized action profile $a^t$, the realized value $y^t$, and returns the probability distribution $\phi_i^t$ over messages $m_i^{t+1} \in H^{t+1}$. In what follows, we will often write $\mu_i^t(m_i^t, x^t, a^t, y^t)$ to indicate the (mixed) message $\phi_i^t \in \Delta(H^{t+1})$ that player $\langle i, t \rangle$ sends player $\langle i, t + 1 \rangle$ after observing the quadruple $(m_i^t, x^t, a^t, y^t)$.

In denoting profiles and sub-profiles of strategies we extend the notational conventions we established for the standard repeated game and the full memory dynastic repeated game. In other words we let $g_i$ denote the $i$-th dynasty profile $(g_0^i, g_1^i, \ldots, g_n^i, \ldots)$, while $g^t$ will indicate the time $t$ profile $(g_1^t, \ldots, g_n^t)$ and $g$ the entire profile of action strategies $(g_1, \ldots, g_n)$. Similarly, we set $\mu_i = (\mu_1^i, \mu_1^i, \ldots, \mu_i^i, \ldots)$, as well as $\mu^t = (\mu_1^t, \ldots, \mu_n^t)$ and $\mu = (\mu_1, \ldots, \mu_n)$. The pair $(g, \mu)$ therefore entirely describes the behavior of all players in the dynastic repeated game with private communication. Since no ambiguity will ensue, from now on we refer to this game simply as the dynastic repeated game.

4. Sequential Equilibrium

It is reasonably clear what one would mean by the statement that a pair $(g, \mu)$ constitutes a Weak Perfect Bayesian Equilibrium (WPBE) of the dynastic repeated game described above.

---

15 When $t = 0$, we assume that player $\langle i, 0 \rangle$ receives the “null” message $m_i^0 = \emptyset$.

16 Notice that we are excluding the realized value of $y^t$ from the set of histories $H^{t+1}$, and hence from the message space of player $\langle i, t \rangle$. This is completely without loss of generality. All our results remain intact if the set of messages available to each player $\langle i, t \rangle$ is augmented to include the realized values $\{y^t\}_{t=0}^\tau$. A formal proof of this claim can be obtained as a minor adaptation of the proof of Lemma T.2.1 in the technical addendum.
However, it is also clear that the off-the-equilibrium-path beliefs of the players may be crucial in sustaining a given profile as a WPBE. To avoid “unreasonable” beliefs, the natural route to take is to restrict attention further, to the set of SE of the model. This is a widely accepted benchmark, in which beliefs are restricted so as to be consistent with a fully-fledged, common (across players) “theory of mistakes.” We return to a discussion of this point in Section 10 below.

The original definition of Sequential Equilibrium (Kreps and Wilson 1982), of course, does not readily apply to our dynastic repeated game since we have to deal with infinitely many players.

As it turns out only a minor adaptation of the SE concept is needed to apply it to our set-up. We spell this out in detail for the sake of completeness and because we believe it makes the proofs of many of our results below considerably more transparent.

We begin with the observation that the beliefs of player $\langle i, t \rangle$ can in fact be boiled down to a simpler object than one might expect at first sight, because of the structure of the dynastic repeated game. Upon receiving message $m_t^i$, in principle, we would have to define the beliefs of player $\langle i, t \rangle$ over the entire set of possible past histories of play. However, when player $\langle i, t \rangle$ is born, an entire cohort of new players replaces the $t-1$-th one, and hence the real history leaves no trace other than the messages $(m_{t-1}^1, \ldots, m_{t-1}^n)$ that have been sent to cohort $t$. It follows that, without loss of generality, after player $\langle i, t \rangle$ receives message $m_t^i$ we can restrict attention to his beliefs over the $n-1$-tuple $m_{t-1}^i$ of messages received by other players in cohort $t$. This probability distribution, specifying the beginning-of-period beliefs of player $\langle i, t \rangle$, will be denoted by $\Phi_{tB}^i(m_t^i)$ throughout the rest of the paper and the technical addendum.

Consider now the two classes of information sets at which player $\langle i, t \rangle$ is called upon to play: the first defined by a pair $(m_t^i, x_t)$ when he has to select a mixed strategy $\sigma_t^i$, and the second defined by a quadruple $(m_t^i, x_t, a_t, y_t)$ when he has to select a probability distribution $\phi_t^i$ over the set of messages $H^{t+1}$.

The same argument as above now suffices to show that at the $(m_t^i, x^t)$ information set we can again restrict attention to he beliefs of player $\langle i, t \rangle$ over the $n-1$-tuple $m_{t-1}^i$ of messages received by other players in cohort $t$. Moreover, since all players observe the same $x^t$ and this realization is independent of what happened in the past, player $\langle i, t \rangle$ beliefs over $m_{t-1}^i$ must be the same as when he originally received message $m_t^i$.

\footnote{It should be made clear that the beliefs of player $\langle i, t \rangle$ over $m_{t-1}^i$ will in fact depend on the relative likelihoods of the actual histories of play that could generate different $n-1$-tuples $m_{t-1}^i$. What we are asserting here is simply that once we know the player’s beliefs over $m_{t-1}^i$, we have all that is necessary to check that his behavior is optimal given his beliefs.}
Finally, at the information set identified by the quadruple \((m_i^t, x^t, a^t, y^t)\), we can restrict attention to the beliefs of player \(\langle i, t \rangle\) over the \(n - 1\)-tuple \(m_{i-1}^t\) of messages that the other players in cohort \(t\) are about to send to cohort \(t + 1\). Just as before, since all players are replaced by a new cohort and time-\(t\) payoffs have already been realized, this is all that could ever matter for the payoff to player \(\langle i, t \rangle\) from this point on. This probability distribution, specifying the end-of-period beliefs of player \(\langle i, t \rangle\), will be denoted by \(\Phi_i^{tE}(m_i^t, x^t, a^t, y^t)\) throughout the rest of the paper and the technical addendum. When the dependence of this distribution on \((m_i^t, x^t, a^t, y^t)\) can be omitted from the notation without causing any ambiguity we will write it as \(\Phi_i^{tE}\). The notation \(\Phi_i^{tE}(\cdot)\) will indicate the entire array of possible probability distributions \(\Phi_i^{tE}(m_i^t, x^t, a^t, y^t)\) as the quadruple \((m_i^t, x^t, a^t, y^t)\) varies.

During the proofs of our main results, we will also need to refer to the (revised) end-of-period beliefs of player \(\langle i, t \rangle\) on the \(n - 1\)-tuple of messages \(m_{i-1}^t\) after he observes not only \(m_i^t\), but also \((x^t, a^t, y^t)\). These will be indicated by \(\Phi_i^{tR}(m_i^t, x^t, a^t, y^t)\), with the arguments omitted when this does not cause any ambiguity. The notation \(\Phi_i^{tR}(\cdot)\) will indicate the entire array of possible probability distributions \(\Phi_i^{tR}(m_i^t, x^t, a^t, y^t)\) as the quadruple \((m_i^t, x^t, a^t, y^t)\) varies.

Throughout the rest of the paper we refer to the array \(\Phi = \{\Phi_i^{tB}(\cdot), \Phi_i^{tE}(\cdot)\}_{t \geq 0, i \in I}\) as a system of beliefs. Following standard terminology we will also refer to a triple \((g, \mu, \Phi)\), a strategy profile and a system of beliefs, as an assessment. Also following standard terminology, we will say that an assessment \((g, \mu, \Phi)\) is consistent if the system of beliefs \(\Phi\) can be obtained (in the limit) using Bayes’ rule from a sequence of completely mixed strategies that converges pointwise to \((g, \mu)\). Since this is completely standard, for reasons of space we do not specify any further details here.

**Definition 1.** Sequential Equilibrium: An assessment \((g, \mu, \Phi)\) constitutes an SE for the dynastic repeated game if and only if \((g, \mu, \Phi)\) is consistent, and for every \(i \in I\) and \(t \geq 0\) strategy \(g_i^t\) is optimal for player \(\langle i, t \rangle\) given beliefs \(\Phi_i^{tB}(\cdot)\), and strategy \(\mu_i^t\) is optimal for the same player given beliefs \(\Phi_i^{tE}(\cdot)\).

We denote by \(G^D(\delta, \tilde{x}, \tilde{y})\) the set of SE strategy profiles, and by \(E^D(\delta, \tilde{x}, \tilde{y})\) the set of SE payoff profiles of the dynastic repeated game when the correlation devices \(\tilde{x}\) and \(\tilde{y}\) are available and common discount factor is \(\delta\). As previously, we let \(E^D(\delta) = \bigcup_{\tilde{x}, \tilde{y}} E^D(\delta, \tilde{x}, \tilde{y})\), with the union ranging over all possible pairs of finite random variables \(\tilde{x}\) and \(\tilde{y}\).

### 5. A Basic Inclusion

The first question we ask is whether all SPE of the standard repeated game survive as SE of the dynastic repeated game. The answer is affirmative.

We pursue this question separately from our limiting (as \(\delta \to 1\)) Folk Theorems for the dynastic repeated game presented below for several distinct reasons.
First of all, Theorem 1 below does not require any assumptions on the stage game. Second, it asserts that the SPE of the standard repeated game survive as SE of the dynastic repeated game regardless of the discount factor $\delta$. Third, Theorem 1 asserts that the actual strategies that form an SPE in the standard game will be (the action component of) some SE of the dynastic repeated game, thus going beyond any statement concerning equilibrium payoffs.

Theorem 1 also fulfils a useful expository function. Running through an intuitive outline of its proof helps an initial acquaintance with some of the mechanics of the SE of the dynastic repeated game. Before going any further, we proceed with a definition and a formal statement of the result.

**Definition 2.** **Truthful Message Strategies:** A communication strategy $\mu_t^i$ for player $\langle i, t \rangle$ in the dynastic repeated game is said to be truthful if and only if $\mu_t^i(m_t^i, x^t, a^t, y^t) = (m_t^i, x^t, a^t)$ for all $m_t^i, x^t, a^t$ and $y^t$. The profile $\mu$ is called truthful if all its components $\mu_t^i$ are truthful.

**Theorem 1.** **Basic Inclusion:** Fix a $\delta$, an $\tilde{x}$ and any profile $g^* \in G^S(\delta, \tilde{x})$. Then for every finite random variable $\tilde{y}$ there exists a profile $\mu^*$ of truthful message strategies such that $(g^*, \mu^*) \in G^D(\delta, \tilde{x}, \tilde{y})$.

It follows that $E^S(\delta) \subseteq E^D(\delta)$ for every $\delta \in (0, 1)$. In other words, the set of SE payoffs for the dynastic repeated game contains the set of SPE payoffs of the standard repeated game.

The argument we use to prove Theorem 1 in the technical addendum (Sections T.3 and T.4) is not hard to outline. Begin with an action-stage strategy profile $g^* \in G^S(\delta)$. Now consider a message-stage strategy profile $\mu^*$ that is “truthful” in the sense of Definition 2.

Now suppose that each player $\langle i, t \rangle$, upon receiving any message $m_t^i$, on or off the equilibrium path, believes that all other time-$t$ players have received exactly the same message as he has. Then it is not hard to see that since the action-stage strategy profile $g^*$ is an SPE of the standard repeated game, it will not be profitable for any player $\langle i, t \rangle$ to deviate from the prescriptions of either $g_t^i$ or $\mu_t^i$ in the dynastic repeated game.

So, why should player $\langle i, t \rangle$ hold such beliefs in an SE? Suppose for instance that player $\langle i, t \rangle$ receives a message that specifies a history of play that is “far away” from the equilibrium path, say a node that requires 1,000 past action-stage deviations to be reached. Of course he needs to weigh the possibility that he really is at such node against, for instance, the possibility that his immediate predecessor has deviated from $\mu_t^{-1}i$ and has sent him the message he observes, but no action-stage deviation has ever occurred.

---

18 Notice that we are defining as truthful a message strategy that ignores the value of $y^t$. This is consistent with the fact that we are excluding the realizations of $\tilde{y}^t$ from the set of possible messages. As we remarked before, all our results would be unaffected if these were included in the players’ message spaces. See also footnote 16 above.
In an SE, these relative likelihood considerations are embodied in the sequence of completely mixed strategies that support the beliefs, in the limit, via Bayes’ rule. The core of the argument behind Theorem 1 is to show that the sequence of completely mixed strategies converging to the equilibrium strategies can be constructed in such a way that the likelihood of a single past deviation from equilibrium at the message stage compared to the likelihood of all players in every previous cohort deviating at the action stage shrinks to zero in the limit. Of course, there is more than one way to achieve this. Our formal argument in the technical addendum relies on “trembles” defining the completely mixed strategies with the following structure. The probability of deviating at the message stage stays constant (at $\varepsilon \to 0$) through time. On the other hand, the order of the infinitesimal of the probability of $n$ deviations at the action stage decreases exponentially through time.\(^{19}\) In this way it is possible to ensure that the probability that all players in every cohort deviate at the action stage shrinks to zero slower than does the probability of a single-player deviation at the message stage. Hence, the beliefs we have described above are consistent, and Theorem 1 follows.

6. Three Dynasties or More

In this Section we present our first result asserting that the set of payoffs that are possible in a SE of the dynastic game is larger than the set of SPE payoffs of the standard repeated game, in the limit as $\delta$ approaches 1. The increase is in fact quite dramatic.

We postpone any further discussion and proceed with a formal statement of our next result and a couple of remarks on its scope. The proof of Theorem 2 begins in Appendix A, and continues in the technical addendum to the paper (Sections T.5 through T.8).

**Theorem 2.** *Dynastic Folk Theorem: Three Dynasties or More:* Let any stage game $G$ with three or more players be given. Assume that $G$ is such that we can find two pure action profiles $a^*$ and $a'$ in A with

$$u_i(a^*) > u_i(a') > u_i(a_i^*, a_{-i}') \quad \forall i \in I \quad (5)$$

Then for every $v \in \text{int}V$ there exists a $\delta \in (0, 1)$ such that $\delta > \tilde{\delta}$ implies $v \in \mathcal{E}^D(\delta)$.

**Remark 2.** *Boundary Points:* The statement of Theorem 2 is framed in terms of interior payoff vectors $v \in \text{int}V$ mostly to facilitate the comparison with the statement of Theorem 3 below dealing with the case of four dynasties or more. However, from the proof of Theorem 2 it is immediately apparent that a stronger statement is in fact true. The condition that $v \in \text{int}V$ can be replaced with the weaker requirement that $v$ is any weighted average of payoff

\(^{19}\)In particular, in equation T.3.3 in the proof of Theorem 1 below, we set the total probability of deviations from equilibrium at the action stage from the part of player $(i,t)$ to be equal to $\varepsilon^{\frac{1}{(n+1)^{2.m+1}}}$. 
vectors that are feasible in $G$, which gives strictly positive weight to $u(a^*)$. Clearly, depending on the position of $u(a^*)$ within $V$ this may include vectors that are on the boundary of $V$.

**Remark 3.** Generalized Prisoners’ Dilemma: Whenever the stage game $G$ is a version of the $n$-player Prisoners Dilemma (with $n \geq 3$), Theorem 2 guarantees that all interior feasible payoff vectors can be sustained as an SE of the dynastic repeated game, provided that $\delta$ is near 1.

To see this, observe that if we label $C_i$ the “cooperate” action and $D_i$ the “defect” action for player $i$, in an $n$-person Prisoners’ Dilemma we obviously have that $u_i(C) > u_i(D) > u_i(C_i, D_{-i})$ for every $i \in I$. Hence identifying $a^*$ and $a'$ of the statement of Theorem 2 with the profiles $C$ and $D$ respectively, immediately yields the result.

Moreover, notice that Theorem 2 applies equally immediately to any stage game that is “locally” like an $n$-person Prisoners’ Dilemma. In other words, the result applies to any stage game $G$ in which we can identify any pair of profiles $C$ and $D$ as above, regardless of how many other actions may be available to any player, and of which payoff vectors they may yield.

Before outlining the argument behind Theorem 2, it is necessary to clear-up a preliminary issue that concerns both the proof of Theorem 2 and the proof of Theorem 3 below.

For simplicity, in both cases, we work with message spaces that are smaller than the applicable set of finite histories $H^t$. As Lemma T.2.1 demonstrates, enlarging message spaces from the ones we use in these two proofs back to $H^t$ will not shrink the set of SE payoffs. This is because we can “replicate” any SE of the dynastic repeated game with restricted message spaces as an SE of the dynastic repeated game with larger message spaces by mapping each message in the smaller set to a finite set of messages in the larger message space. A choice of message in the smaller message space corresponds to a (uniform) randomization over the entire corresponding set in the larger message space. A player receiving one of the randomized messages in the larger message space acts just like the corresponding player who receives the corresponding message in the smaller message set. It is then straightforward to check that the new strategies constitute an SE of the dynastic repeated game with larger message spaces, provided of course that we started off with an SE of the dynastic game with restricted message spaces in the first place.

In the SE that we construct to prove Theorems 2 and 3, player $\langle i, t - 1 \rangle$ may want to communicate to player $\langle i, t \rangle$ that dynasty $i$ is being punished for having deviated, but will be unable to do so in an effective way. Given that in both cases we work with message spaces that are smaller than $H^t$, one may be tempted to conclude that this is due to the fact that player $\langle i, t - 1 \rangle$ “lacks the message” to communicate effectively to his successor $\langle i, t \rangle$ that he should respond in an appropriate way. This is misleading. When the message spaces coincide with $H^t$, it clearly cannot be that this inability to communicate is due to a shortage
of possible messages. Given the argument (the proof of Lemma T.2.1) that we sketch out above the correct interpretation is that in equilibrium there is no message (in $H^t$) that player $\langle i, t \rangle$ might possibly interpret in the way that $\langle i, t - 1 \rangle$ would like.\textsuperscript{20}

We are now ready for an actual outline of the proof of Theorem 2. The intuition can be divided into two parts. First, we will argue that if $\delta$ is close to one, it is possible to sustain the payoff vector $u(a^*)$ as an SE of the dynastic repeated game using $u(a')$ as “punishment” payoffs. Notice that $u(a^*)$ could well already be below the minmax payoff for one or more players. We call this the “local” part of the argument. Second, we will argue that via a judicious use of the action-stage correlation device it is possible to go from the local argument to a “global” one and sustain every feasible payoff vector as required by the statement of the theorem.

Our construction relies on every player $\langle i, t \rangle$ having a message space with three elements. So, set $M^t_i = \{m^*, m^A, m^B\}$ for every $i$ and $t$. Notice also that the assumptions made in Theorem 2 obviously guarantee that $a^*_i \neq a'_i$ for every $i \in I$.

We begin with the local part of the argument. So, suppose that $\delta$ is close to one and that we want to implement the payoff $v = u(a^*)$ as an SE of the dynastic repeated game. The strategy of every player $\langle i, 0 \rangle$ at the action stage is to play $a^*_i$. The strategy of each player $\langle i, t \rangle$ with $t \geq 1$ at the action stage is to play $a^*_i$ after receiving message $m^*$ and to play action $a'_i$ after receiving message $m^A$ or message $m^B$. If no player deviates from the prescriptions of his equilibrium strategy at the action and message stages, then play follows a path that involves the message profile $(m^*, \ldots, m^*)$ and the action profile $a^*$ in every period. Moreover, even after deviations, if the message profile ever returns to being equal to $(m^*, \ldots, m^*)$, then the path of play returns to being as above. We call this the equilibrium phase.

Play starts in the equilibrium phase. Suppose now that some player deviates to playing action $a'_i$ (deviations by two or more players are ignored). This triggers what we call the temporary punishment phase. During the temporary punishment phase all players $\langle i \in I, t \rangle$ condition the messages they send to their successors on the realization $y^t$ of the message-stage correlation device. The message-stage correlation device takes value $y(1)$ with probability $\gamma(\delta)$ and value $y(0)$ with probability $1 - \gamma(\delta)$. If $y^t = y(1)$ then all players send message $m^*$, and hence play returns to the equilibrium phase. If on the other hand $y^t = y(0)$ then all players send message $m^A$, and play remains in the temporary punishment phase. That is all players now play $a'_i$, and continue to coordinate their messages as we just described. Play remains in the temporary punishment phase until the realization of the message-stage coordination device is $y(1)$, at which point play goes back to the equilibrium phase.

Any other deviation, taking place in either the equilibrium phase or in the temporary

\textsuperscript{20}At this point it is legitimate of course to wonder whether the concept of “neologism-proof” equilibrium (Farrell 1993) has any impact on what we are saying here. The answer is that it simply does not apply. We return to this point at some length in Section 10 below.
punishment phase, triggers the permanent punishment phase. Suppose that any player \(\langle i, t \rangle\) deviates to playing any action \(a_i\) different from both \(a_i^*\) and \(a'_i\) during the equilibrium phase or that he deviates from playing \(a'_i\) during the temporary punishment phase (deviations by two or more players are again ignored). Then all players \(\langle i \in I, t \rangle\) send message \(m^B\) to their successors. From this point on the prescriptions of the equilibrium strategies are that all subsequent players should send message \(m^B\), and that they should play action \(a'_i\). Figure 1 is a schematic depiction of the three phases of play we have outlined.

To check that the strategies we have described constitute an SE of the dynastic repeated game, the players’ beliefs need to be specified. All players in the \(t = 0\) cohort of course have correct beliefs. All other players, after receiving message \(m^*\) believe that all other players have also received message \(m^*\) with probability one. Similarly, after receiving message \(m^A\) all time-\(t \geq 1\) players believe that all other players have received message \(m^A\) with probability one. So, when play is either in the equilibrium phase all players believe that this is indeed the case, and the same is true for the temporary punishment phase. Not surprisingly, it is possible to sustain these beliefs via a sequence of completely mixed strategies using Bayes’ rule, as required for an SE.

After receiving message \(m^B\), all time-\(t \geq 1\) players have non-degenerate beliefs as follows.
With probability $\beta(\delta)$ they believe that play is in the equilibrium phase so that all other players have in fact received message $m^*$. With probability $1 - \beta(\delta)$ they believe that play is in the permanent punishment phase, so that all other players have received message $m^B$.

So, when play is in the permanent punishment phase the players’ beliefs assign positive probability to something that is in fact not taking place. Why is this possible in an SE? The intuition is not hard to outline. Upon receiving message $m^B$, player $\langle i, t \rangle$ needs to compare two possibilities: am I receiving message $m^B$ because some action-deviations have occurred that triggered the permanent punishment path, or is it simply the case that play is in the equilibrium phase and my immediate predecessor has sent message $m^B$ when he should have sent $m^*$? Clearly the answer depends on the probability of a message-deviation relative to the probability of action-deviations that could have triggered the permanent punishment phase. By a careful use of infinitesimals of different orders in the sequence of completely mixed strategies, it is then possible to sustain the beliefs we have described via Bayes’ rule.

Given the beliefs we have specified, we can now argue that no player has an incentive to deviate, either at the action or at the message stage, from the strategies we have described. We distinguish again between the three phases of play identified above.

If play is in the equilibrium phase it is not hard to see that no player $\langle i, t \rangle$ has an incentive to deviate. If he adheres to the equilibrium strategy, player $\langle i, t \rangle$ gets a payoff of $u_i(a^*)$. Since $\delta$ is close to 1, and player $\langle i, t \rangle$ takes the strategies of all other players (including his successors) as given, any deviation may produce an instantaneous gain, but will decrease the overall payoff of player $\langle i, t \rangle$.

If play is in the temporary punishment phase and player $\langle i, t \rangle$ does not deviate then play will eventually go back to the equilibrium phase. Hence, since $\delta$ is close to one, the dynastic payoff to player $\langle i, t \rangle$ in this case is close to (but below) $u_i(a^*)$. Therefore, the same argument as for the equilibrium phase applies to show that he will not want to deviate during the temporary punishment phase.

Suppose now that play is in the permanent punishment phase. Begin with the action stage. Recall that at this point, that is after receiving message $m^B$, player $\langle i, t \rangle$ has the non-degenerate beliefs we described above giving positive probability to both the event that play is the equilibrium phase and to the event that play is in the permanent punishment phase.

If play were in the equilibrium phase, since $\delta$ is close to one, for an appropriate choice of $\gamma(\delta)$, player $\langle i, t \rangle$ would prefer taking action $a^*_i$ to taking action $a'_i$, and would prefer the latter to taking any other action $a_i$ different from both $a^*_i$ and $a'_i$. This is because his continuation payoff (from the beginning of period $t + 1$ onwards) is higher in the equilibrium phase than in the temporary punishment phase, and lowest in the permanent punishment phase. Notice that if $\gamma(\delta)$ were too close to one, then the continuation payoff in the equilibrium phase and in the temporary punishment phase could be so close as to reverse (via an instantaneous gain
in period $t$ from playing $a'_i$) the preference of player $\langle i, t \rangle$ between $a^*_i$ and $a'_i$. On the other hand if $\gamma(\delta)$ were too low, then the continuation payoff in the temporary punishment phase and in the permanent punishment phase could be so close as to reverse (via an instantaneous gain in period $t$ from playing some action $a_i$ different from both $a^*_i$ and $a'_i$) the preference of player $\langle i, t \rangle$ between $a'_i$ and some other action $a_j$.

If play were in the permanent punishment phase then clearly player $\langle i, t \rangle$ would prefer taking action $a'_i$ to taking action $a^*_i$. This is simply because by assumption $u_i(a') > u_i(a^*_i, a'_{-i})$. Of course, it is possible that some other action(s) $a_i$ would be preferable to $a'_i$.

So, in one of two cases that player $\langle i, t \rangle$ entertains with positive probability after receiving message $m^B$ action $a^*_i$ is preferable to action $a'_i$ which in turn is preferable to any other action, while in the other case action $a'_i$ is preferable to $a^*_i$, but some other action(s) may be preferable to both. What is critical here is that there is no action ($a^*_i$ or any other one) that is preferable to $a'_i$ with probability one. As it turns out, this is sufficient to show that for some value of $\beta(\delta)$ action $a'_i$ is in fact optimal for player $\langle i, t \rangle$. Therefore he does not want to deviate from the equilibrium strategy we have described at the action stage during the permanent punishment phase.

Finally, consider the choice of player $\langle i, t \rangle$ at the message stage when play is in the permanent punishment phase. Notice that after receiving message $m^B$, player $\langle i, t \rangle$ will discover that play is in the permanent punishment phase during the action stage of period $t$. This is so even if some other time-$t$ player were to deviate from his prescribed action in period $t$. The reason is our assumption that $n \geq 3$. The fact that $n - 2$ or more players $\langle j, t \rangle$ play $a'_j$ is sufficient to tell player $\langle i, t \rangle$ that play is in fact in the permanent punishment phase. Clearly, it is at this point that player $\langle i, t \rangle$ would like to “communicate effectively” to player $\langle i, t + 1 \rangle$ that play is in the permanent punishment phase but is unable to do so, as in our discussion above concerning message spaces. In fact $a'_i$ could be very far from being a best response from $a'_{-i}$ in the stage game. Yet, given the strategies of all other players, inducing all his successors to play a myopic best-response to $a'_{-i}$ is simply not an option that is open to player $\langle i, t \rangle$ even after he discovers that play is in the permanent punishment phase. Given that, by assumption, $u_i(a') > u_i(a^*_i, a'_{-i})$ the best that player $\langle i, t \rangle$ can do at this point is to send message $m^B$ as required by his equilibrium strategy.

The argument we have just outlined suffices to show that the payoff vector $u(a^*)$ can be sustained as an SE of the dynastic repeated game. We now argue that this fact can be used

---

21 If we had $n = 2$ players the following problem would arise with our construction. Suppose that each player were to monitor the action of the other to decide whether play is in the equilibrium phase or in the permanent punishment phase. Then, after receiving $m^B$, say for instance player $\langle 1, 1 \rangle$, could find it advantageous to play $a^*_i$ instead of $a'_i$. This is because this deviation, together with player $\langle 1, 1 \rangle$ sending message $m^*$, would put the path of play back in the equilibrium phase with probability one.

22 Note that this would necessarily be the case if, for instance, $u_i(a^*)$ were below $i$’s minmax payoff in the stage game.
as a “local anchor” for our global argument that shows that any interior payoff vector can be sustained as an SE. Fix any \( v^* \in \text{int} V \) to be sustained in equilibrium. Since \( v^* \) is interior it is obvious that it can be expressed as \( v^* = qu(a^*) + (1 - q)v \) for some \( q \in (0, 1) \) and some \( v \in V \).

The construction we put forward uses the payoffs \( v' = qu(a') + (1 - q)v \) as “punishments.” Clearly, \( v^*_i > v'_i \) for every \( i \in I \).

We use the action stage correlation device to combine the local part of the argument with the global one. The possible realizations of \( \tilde{x}_t \) are \( (x(1), \ldots, x(|A|)) \), with the probability that \( \tilde{x}_t = x(1) \) equal to \( q \).

Whenever the realization of \( \tilde{x}_t \) is \( x(1) \), the action and message strategies of all players are just as we described above. The action strategies of all players do not depend on the message received whenever \( x_t \neq x(1) \). Moreover, the probability law governing \( \tilde{x}_t \) and the action strategies of all players for the case \( x_t \neq x(1) \) are such that the per-period expected payoff (conditional on \( x_t \neq x(1) \)) is \( v_i \) for every \( i \in I \). Whenever \( x_t \neq x(1) \), any deviation from the prescription of the equilibrium strategies triggers the permanent punishment phase, as described above. It is not hard to verify that the latter is enough to keep all players from deviating at any point.

One point that is worth emphasizing here is that, in contrast to what happens in the local argument above, if play is in the permanent punishment phase, so that player \( (i, t) \) receives message \( m^B \), and \( x_t \neq x(1) \), then player \( (i, t) \) does not discover from \( a_t \) whether play is in the equilibrium phase or in the permanent punishment phase. Both at the beginning and at the end of period \( t \) he believes with probability \( \beta(\delta) \) that play in is the equilibrium phase, and with probability \( 1 - \beta(\delta) \) that play is in the permanent punishment phase. However, he does know that any deviation to an action different from the prescribed one will trigger the permanent punishment phase for sure. Given that \( \delta \) is close to one, he is then better-off not deviating. This ensures that the first of his successors who observes a realization of the action correlation device equal to \( x(1) \) will play action \( a'_i \), which is in fact optimal given the beliefs that player \( (i, t) \) has.

7. Four Dynasties or More

We now turn to the case in which the stage game \( G \) has four or more players. We need to introduce some further notation to work towards our next main result. We begin by constructing what we will refer to as the restricted correlated minmax.

Given a stage game \( G = (A, u, I) \), we indicate by \( \hat{A} \subseteq A \) a typical set of pure action profiles with a product structure. In other words, we require that there exist an array \( (\hat{A}_1, \ldots, \hat{A}_n) \) with \( \hat{A} = \times_{i \in I} \hat{A}_i \). Given a product set \( \hat{A} \), we let \( V(\hat{A}) \) be the convex hull of the set of payoff

\[ |\cdot| \] denotes the cardinality of a set.

\[ \text{int} \] denotes the interior of a set.

\[ \beta(\delta) \] denotes a probability function.

\[ \times \] denotes the Cartesian product.
vectors that can be achieved in $G$ using pure action profiles in $\hat{A}$. As before, $\text{int} V(\hat{A})$ will denote the (relative) interior of $V(\hat{A})$.

**Definition 3.** Restricted Correlated Minmax: Let a product set $\hat{A} \subseteq A$ be given. Now let

$$\omega_i(\hat{A}) = \min_{z_i \in \Delta(A_{-i})} \max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} z_i(a_i) u_i(a_i, a_{-i})$$

(6)

where $z_i$ is any probability distribution over the finite set $\hat{A}_{-i}$ (not necessarily the product of independent marginals), and $z_i(a_{-i})$ denotes the probability that $z_i$ assigns to the profile $a_{-i}$.

We then say that $\omega_i(\hat{A})$ is the restricted (to $\hat{A}$) correlated minmax for $i$ in $G$.

Roughly speaking, the restricted (to $\hat{A}$) correlated minmax payoff for $i$ is the best payoff that $i$ can achieve when he is restricted to choosing an element of $\hat{A}_i$, while all other players are choosing a profile of correlated mixed strategies with support at most $\hat{A}_{-i}$.

We are now ready to state our next result. The proof of Theorem 3 begins in Appendix B, and continues in the technical addendum to the paper (Sections T.9 through T.12).

**Theorem 3.** Dynastic Folk Theorem: Four Dynasties or More: Let a stage game $G$ with four or more players be given. Assume that $G$ is such that we can find a product set $\hat{A} \subseteq A$ and an array of $n + 1$ payoffs vectors $\hat{v}, \hat{v}^1, \ldots, \hat{v}^n$ for which the following conditions hold.

1. For every $i \in I$, the set $\hat{A}_i$ contains at least two elements.
2. $\hat{v} \in \text{int} V(\hat{A})$, and $\hat{v}^i \in V(\hat{A})$ for every $i \in I$.
3. $\omega_i(\hat{A}) < \hat{v}_i$ and $\hat{v}_i < \hat{v}_j$ for every $i \in I$ and every $j \neq i$.

Then for every $v \in \text{int} V$ there exists a $\delta \in (0, 1)$ such that $\delta > \delta_i$ implies $v \in E^D(\delta)$.

Before any discussion of the argument behind Theorem 3 we proceed with a remark concerning its scope.

---

25 Recall that the “standard minmax” for player $i$ is defined as $\underline{v}_i = \min_{z_i \in \Pi_{j \neq i} \Delta(A_j)} \max_{a_i \in A_i} u_i(a_i, \sigma_{-i})$, where $\Pi_{j \neq i} \Delta(A_j)$ is the set of (independent) mixed strategy profiles for $-i$.

It is then straightforward to see that the restricted correlated minmax of Definition 3 is a “stronger” concept than the standard minmax in the following sense. Given any $G = (A, u, I)$, if we let the standard minmax payoff for $i$ be denoted by $\underline{v}_i$, we obviously have that $\omega_i(\hat{A}) \leq \underline{v}_i$ for every $i \in I$.

Of course, the relationship between the restricted correlated minmax and the standard minmax also depends on the “size” of product set $\hat{A}$ relative to $A$. In particular, let two product sets $\hat{A}$ and $A'$ be given. It is then straightforward to check that if $\hat{A}_{-i} \subseteq A_{-i}'$ and $A_i' \subseteq A_i$, then $\omega_i(A') \leq \omega_i(\hat{A})$. Therefore, in general, depending on $\hat{A}$, the value of $\omega_i(\hat{A})$ could be below, equal or above $\underline{v}_i$. 
Remark 4. **Theorem 3 and the “Standard” Folk Theorem:** Suppose that the stage game $G$ has four or more players. Suppose further that $G$ is such that the standard Folk Theorem (Fudenberg and Maskin 1986, Theorems 2 and 5) guarantees that, as $\delta$ approaches one, the standard repeated game has at least one SPE payoff vector that strictly dominates the standard minmax.\(^{26}\) Then Theorem 3 guarantees that, as $\delta$ approaches one, all (interior) feasible payoff vectors can be sustained as an SE of the dynastic repeated game.

To see this, assume that $V$ contains a payoff vector that yields all players more than their (standard) minmax $v_i$ and that $V$ has full dimension. Next, observe that (see footnote 25) if we set $\tilde{A} = A$ we know that $\omega_i(A) \leq v_i$ for every $i$. Therefore, using the full-dimensionality of $V$, it is straightforward to see that setting $\tilde{A} = A$ all the hypotheses required by Theorem 3 are in fact satisfied.

It is important to emphasize at this point that Theorem 3 implies that all (interior) payoff vectors may be sustained as SE of the dynastic repeated game as $\delta$ goes to one, even when the standard Folk Theorem of Fudenberg and Maskin (1986) does not imply any multiplicity of payoffs. This is because of the two critical features that distinguish the Restricted Correlated Minmax of Definition 3 from the standard minmax: the correlation that it allows, and the restriction to the product set $\tilde{A}$.

To appreciate the role of correlation, consider a stage game $G$ with four or more players, each with at least two pure actions, and with $V$ of full dimension. Recall that (see footnote 25) we know that it is possible that $\omega_i(A) < v_i$. If the latter condition is satisfied for all players, it is clearly possible that no payoff vector in $V$ gives all players more than $v_i$, but at the same time all the hypotheses of Theorem 3 are in fact satisfied.\(^{27}\)

To see the role of the restriction to $\tilde{A}$, notice that, as in Theorem 2, the necessary conditions listed in the statement of Theorem 3 are “local.” In other words (see footnote 25), it is clearly possible that no payoff vector in $V$ gives all players more than $v_i$, but that by excluding one or more pure strategies for one or more players we obtain a product set $\tilde{A}$ for which conditions (i), (ii) and (iii) of Theorem 3 are in fact satisfied.

\(^{26}\)Notice that there is an open set of stage games for which the standard Folk Theorem does not imply any multiplicity of equilibrium payoffs. To see this, consider, for instance, a game $G$ in which every player $i$ has a strictly dominant strategy $a_i^*$ and such that the standard minmax value is $u_i(a^*)$ for every $i \in I$. Clearly, these conditions can be satisfied even when $u(a^*)$ is not (weakly) Pareto-dominated by any other payoff vector in $V$. Moreover, if these conditions are satisfied, they are also satisfied for an entire open set of games around $G$.

\(^{27}\)The example we provided in footnote 26 may prompt the following question. Is it the case that whenever the standard Folk Theorem does not yield any payoff multiplicity it must necessarily be the case that every player has a dominant strategy in $G$? The answer is no. This observation is relevant to the role of correlation we are discussing here. If all players have a dominant strategy in $G$, then the correlated minmax $\omega_i(A)$ is in fact the same as the standard minmax $v_i$ for every $i$. Clearly, what we are saying here has real content only when $G$ is such that the standard Folk Theorem does not yield any payoff multiplicity and it is not the case that all players have a dominant strategy.
Similarly to the case of Theorem 2, the construction we use to prove Theorem 3 relies on message spaces that are smaller than the appropriate set of finite histories \( H^T \). Also similarly to Theorem 2, it is convenient to divide the argument behind Theorem 3 into a local one and a global one. As before, we begin with the local part of the argument.

Consider a product set \( \tilde{A} \) satisfying the hypotheses of Theorem 3. Let \( V(\tilde{A}) \) be the convex hull of payoff vectors generated with action profiles in \( \tilde{A} \). For simplicity, assume that the payoff vectors \( \tilde{\omega}^1, \ldots, \tilde{\omega}^n \) can all be obtained from pure profiles of actions in \( \tilde{A} \). Also for simplicity, assume that each of the payoffs \( \omega_i(\tilde{A}) \) can be obtained from some (pure) profile of actions in \( \tilde{A} \).

The local part of the argument behind Theorem 3 shows that, for \( \delta \) close enough to one the payoff vector \( \hat{v} \in V(\tilde{A}) \) of the statement of the Theorem can be sustained as an SE of the dynastic repeated game.

The equilibrium path generated by the strategies we construct consists of \( n + 1 \) phases. We call the first one the standard equilibrium phase, the second one the diversionary-1 equilibrium phase, through to the diversionary-\( n \) equilibrium phase.

If all players \( \langle i \in I, t \rangle \) receive message \( m^* \) then play is in the standard equilibrium phase. For simplicity again we proceed with our outline of the construction assuming that the equilibrium prescribes that the players \( \langle i \in I, t \rangle \) play a pure action profile during the standard equilibrium phase, denoted by \( a' \). The associated payoff vector is \( \hat{v} \).

If all players \( j \neq i \) in the \( t \)-th cohort receive message \( \tilde{m}^i \), and player \( \langle i, t \rangle \) receives any message \( m^{i^T} \) in a finite set \( M(i, t) = \{m^{i^1}, \ldots, m^{i^T} \} \subset M_i^T \), then play is in the diversionary-\( i \) phase. Let \( a_i^j \) be the vector of actions (pure, for simplicity) for which \( i \) receives his restricted correlated minmax payoff \( \omega_i(\tilde{A}) \). During the diversionary-\( i \) equilibrium phase player \( \langle i, t \rangle \) plays action \( a_i^j \). For all players \( j \neq i \), let \( \tilde{a}_i^j \) be any action in \( \tilde{A}_j \) that is not equal to \( a_i^j \). Such action can always be found since, by assumption, each \( \tilde{A}_j \) contains at least two actions. During the diversionary-\( i \) equilibrium phase, any player \( j \neq i \) plays action \( \tilde{a}_i^j \). The (per-period) payoff vector associated with the diversionary-\( i \) equilibrium phase is denoted by \( \tilde{u}_i^j \).

If in period \( t \) play is in any of the equilibrium phases we have just described, and no deviation occurs at the action stage, at the end of period \( t \) all players use the realization \( y^j \) of the message-stage correlation device to select the message to send to their successors. The

---

\(^{28}\)Clearly, when \( \omega_i(\tilde{A}) \) is achieved via a pure action profile in \( \tilde{A} \) the restricted correlated minmax is the same as the standard minmax restricted to \( \tilde{A} \). It is still obviously the case that if \( \tilde{A} \) is a strict subset of \( A \), then it is possible that \( \omega_i(\tilde{A}) \) would be below the standard (unrestricted) minmax \( \omega_i \).

\(^{29}\)The formal proof of Theorem 3 (see Appendix B and the technical addendum to the paper) actually treats period 0 differently from all other periods. In period 0, in equilibrium, the players use the action-stage correlation device to play actions that yield exactly \( \hat{v} \). This payoff vector is achieved in expected terms (across different equilibrium phases) on the equilibrium path by the construction we outline here.

\(^{30}\)In the formal proof of Theorem 3, presented in Appendix B and the technical addendum, the set of messages \( M(i, t) \) actually does depend on the time index \( t \) because not all messages are available for the first \( T \) periods of play. This is so in order to avoid any message space \( M_i^T \) having a cardinality that exceeds that of \( H^T \).
possible realizations of \( \tilde{y}^t \) are \((y(0), y(1), \ldots, y(n))\). The probability that \( y^t = y(0) \) is \( 1 - \eta \) and the probability that \( y^t = y(i) \) is \( \eta/n \) for every \( i \in I \). Consider now the end of any period \( t \) in any equilibrium phase, and assume that no deviation has occurred. If \( y^t = y(0) \) then all players \( (i \in I, t) \) send message \( m^* \) to their successors, so play in period \( t+1 \) is in the standard equilibrium phase. If \( y^t = y(i) \), then all players \( j \neq i \) send message \( \tilde{m}^i \) to their successors and player \( (i, t) \) sends a (randomly selected) message \( m^{i,T} \in M(i,t) \) to player \( (i, t+1) \). So, in this case in period \( t+1 \) play is in the diversionary-\( i \) equilibrium phase.

The profiles to be played in each diversionary-\( i \) equilibrium phase may of course be entirely determined by the (need to differ from the) correlated minmax action profiles, so we have no degrees of freedom there. However, we are free to choose the profile \( a' \) in constructing the standard equilibrium phase. Recall that we take \( a' \) to be pure solely for expositional simplicity. Using the action-stage correlation device, clearly we could select correlated actions for the standard equilibrium phase that yield any payoff vector \( v' \) in \( V(\hat{A}) \). Since \( \hat{v} \in \text{int}V(\hat{A}) \) we can be sure that for some \( v' \in V(\hat{A}) \) and some \( \eta \in (0,1) \)

\[
\hat{v} = (1 - \eta)v' + \frac{\eta}{n} \sum_{i=1}^{n} \hat{u}_i
\]  

(7)

So that (modulo our expositional assumption that \( v' = u(a') \)) when play is in any equilibrium phase the expected (across all possible realizations of \( \tilde{y}^t \)) continuation payoff to any player \( (i, t) \) from the beginning of period \( t+1 \) onward is \( \hat{v}_i \).

The strategies we construct also define (off the equilibrium path) \( n \) punishment phases, one for each \( i \in I \) and \( n \) terminal phases, again one for each dynasty \( i \). In the punishment-\( i \) phase, in every period player \( i \) receives his restricted correlated minmax payoff \( \omega_{i}(\bar{a}) \) payoff, and the phase lasts \( T \) periods. In the terminal-\( i \) phase in every period the players receive the vector of payoffs \( \pi' \). The transition between any of the equilibrium phases and any of the punishment or terminal phases is akin to the benchmark construction in Fudenberg and Maskin (1986). In other words, a deviation by dynasty \( i \) during any of the equilibrium, any of the punishment phases or any of the terminal phases triggers the start (or re-start) of the punishment-\( i \) phase (deviations by two players or more are ignored). The terminal-\( i \) phase begins after play has been, without subsequent deviations, in the punishment-\( i \) phase for \( T \) periods. For an appropriately chosen (large) \( T \), as \( \delta \) approaches 1, with one critical exception, the inequalities in (iii) of the statement of Theorem 3 are used in much the same way as in Fudenberg and Maskin (1986) to guarantee that no player deviates from the prescriptions of the equilibrium strategies.

In Fudenberg and Maskin (1986), during the punishment-\( i \) phase player \( i \) plays a myopic best response to the actions of other players. Critically, this is not the case here. During the punishment-\( i \) phase dynasty \( i \) plays a best response to the (correlated) strategy of others
restricted to the set of pure actions in $\bar{A}_i$. Clearly this could be very far away (in per-period payoff terms) from an unconstrained best reply chosen at will within $A_i$. To understand how this can happen in an SE we need to specify what message profiles mark the beginning of the punishment-$i$ phase and what the players’ beliefs are.

Suppose that player $\langle i, t \rangle$ deviates from the prescriptions of the equilibrium strategy and triggers the start of the punishment-$i$ phase as of the beginning of period $t + 1$. Then all players $\langle j \in I, t \rangle$ send message $m^{i,T}$ to their successors. These messages are interpreted as telling all players $\langle j \neq i, t + 1 \rangle$ that the punishment-$i$ phase has begun, and that there are $T$ periods remaining. We return to the beliefs of player $\langle i, t + 1 \rangle$ shortly. In the following period of the punishment-$i$ phase all players $\langle j \in I, t + 1 \rangle$ send message $m^{i,T-1}$, then $m^{i,T-2}$ and so on, so that the the index $\tau$ in a message $m^{i,\tau}$ is effectively interpreted (by dynasties $j \neq i$) as a “punishment clock,” counting down how many periods remain in the punishment-$i$ phase.

The players’ beliefs in the SE we construct are “correct” in all phases of play except for the beliefs of player $\langle i, t \rangle$ whenever play is in the punishment-$i$ phase at time $t$. Upon receiving any message $m^{i,\tau} \in M(i, t)$, player $\langle i, t \rangle$ believes that play is in the punishment-$i$ phase with probability zero. Instead he believes that play is in the $i$-diversionary equilibrium phase with probability one. This is possible in an SE because player $\langle i, t \rangle$ receives the same message in both cases and play is in the diversionary-$i$ phase with positive probability on the equilibrium path, while the punishment-$i$ phase is entirely off the equilibrium path. It follows easily that player $\langle i, t \rangle$ does not want to deviate from the equilibrium strategies we have described when play is in the punishment-$i$ phase.

Notice moreover that if at time $t$ play is in the punishment-$i$ phase, after the profile $a^t$ is observed, player $\langle i, t \rangle$ will discover that play is in fact in the punishment-$i$ phase at time $t$. This is because, by construction, all players $\langle j \neq i, t \rangle$ play an action (namely $\bar{a}_j$) in the diversionary-$i$ phase that is different from what they play in the punishment-$i$ phase (namely $a^i_j$). This, coupled with the assumption assumption that $n \geq 4$ will ensure that $\langle i, t \rangle$ will discover the truth, and the identity of any deviator.$^{31}$ Therefore any player $\langle i, t \rangle$ who knows that in period $t + 1$ play will be in the punishment-$i$ phase would like would like to “communicate effectively” to player $\langle i, t + 1 \rangle$ that play is in the punishment-$i$ phase but is unable to do so, as in our discussion above concerning message spaces. After receiving $m^{i,\tau}$ and discovering that play is in the punishment-$i$ phase, sending any message to player $\langle i, t + 1 \rangle$ that is not $m^{i,\tau-1}$ may cause him, or some of his successors, to deviate and hence to re-start the punishment-$i$ phase.$^{32}$ Sending $m^{i,\tau-1}$ will cause player

---

$^{31}$Clearly, if $\langle i, t \rangle$ could not be guaranteed to discover that play is in the punishment-$i$ phase, or the identity of any deviator at time $t$, then we could not construct strategies that guarantee that if $\langle j \neq i, t \rangle$ deviates during the punishment-$i$ phase then play switches to the punishment-$j$ phase, as required.

$^{32}$It is worth emphasizing here that checking sequential rationality at the message stage is somewhat more
⟨i, t + 1⟩ and his successors to play a best response among those that can be induced by any available message. This is because \( g_i \) is in fact a best response to the the correlated minmax of the other players within the set \( \tilde{A}_i \). Therefore, given the inequalities in (iii) of the statement of Theorem 3, given that \( T \) is sufficiently large, and that \( \delta \) is close enough to one, no profitable deviation is available to player \( ⟨i, t⟩ \).

The argument we have just outlined suffices to show that the payoff vector \( \hat{v} \) of the statement of the theorem can be sustained as an SE of the dynastic repeated game. Similarly to Theorem 2, we now argue that this fact can be used as a “local anchor” for an argument that shows that any interior payoff vector can be sustained as an SE.

Fix any \( v^* \in \text{int}V \) to be sustained in equilibrium. Since \( v^* \) is interior it is obvious that it can be expressed as \( v^* = q\hat{v} + (1 - q)z \) for some \( q \in (0, 1) \) and some \( z \in V \). The “global” argument then consists of using the action-stage correlation device so that in each period with probability \( q \) play proceeds as in the construction above, while with probability \( 1 - q \) the (expected) payoff vector is \( z \). The latter is achieved with action-stage strategies that do not depend on the messages received. A deviation by \( i \) from the (correlated) actions needed to implement \( z \) triggers the punishment-\( i \) phase. With one proviso to be discussed shortly, it is not hard to then verify that this is sufficient to keep all players from deviating at any point, and hence that \( v^* \) can be sustained as an SE payoff vector of the dynastic repeated game for \( \delta \) sufficiently close to one.

The difficulty with the global argument we have outlined that needs some attention is easy to point out. The periods in which the action-stage correlation device tells the players to implement the payoff vector \( z \) cannot be counted as real punishment periods. They in fact stochastically interlace all phases of play, including any punishment-\( i \) phase. However, the length of effective punishment \( T \) has to be sufficiently large to deter deviations. The solution we adopt is to ensure that the punishment clock does not decrease in any period in which (with probability \( 1 - q \) the payoff vector \( z \) is implemented at the action stage. In effect, this makes the length of any punishment-\( i \) phase stochastic, governed by a punishment clock that counts down only with probability \( q \) in every period.

8. Inter-Generational (Dis)Agreement

Some of the SE of the dynastic repeated game we have identified in Theorems 2 and 3 above clearly do not correspond in any meaningful sense to any SPE of the standard repeated game. This is obvious if we consider an SE of the dynastic repeated game in which one or more players receive a payoff below their standard minmax value.

involved than may appear from our intuitive outline of the argument given here. This is because a deviation at the message stage may trigger multiple deviations; that is deviations at the action and/or message stage by more than one successor of any given player. The core of the argument dealing with this case is Lemma T.10.4.
An obvious question to raise at this point is then the following. What is it that makes these SE viable? To put it another way, can we identify any properties of an SE of the dynastic repeated game which ensure that it must correspond in a meaningful sense to an SPE of the standard repeated game? The answer is yes, and this is what this section of the paper is devoted to.

The critical properties of an SE that we identify concern the players’ beliefs. The first is that a player’s (revised) beliefs at the end of the period over the messages received by other players at the beginning of the period should be the same as at the beginning of the period. This is equation (8) below. The second is that the end-of-period beliefs of a player (over messages sent by his opponents) should be the same as the beginning-of-period beliefs of his successor (over messages received by his opponents). This is equation (9) below. In fact we will be able to show that if this property holds for all players (and all information sets) in an SE of the dynastic repeated game, then this SE must be payoff-equivalent to some SPE of the standard repeated game. For want of a better term, when an SE of the dynastic repeated game has the two properties (of beliefs) that we just described informally, we will say that it displays Inter-Generational Agreement.

**Definition 4. Inter-Generational Agreement:** Let an SE triple \((g, \mu, \Phi)\) of the dynastic repeated game be given.

We say that this SE displays Inter-Generational Agreement if and only if for every \(i \in I, \ t \geq 0, \ m_i^t \in H^t, \ x^t \in X, \ a^t \in A \) and \(y^t \in Y\) we have that

\[
\Phi^{tR}_i(m_i^t, x^t, a^t, y^t) = \Phi^{tB}_i(m_i^t) \tag{8}
\]

and for every \(m_{i}^{t+1}\) in the support of \(\mu_{i}^{t}(m_i^t, x^t, a^t, y^t)\)

\[
\Phi^{tE}_i(m_i^t, x^t, a^t, y^t) = \Phi^{t+1B}_i(m_i^{t+1}) \tag{9}
\]

We are now ready to state our last result.

**Theorem 4. SE of the Dynastic and SPE of the Standard Repeated Game:** Fix a stage game \(G\), any \(\delta \in (0, 1)\), and any \(\bar{x}\) and \(\bar{y}\). Let \((g, \mu, \Phi)\) be an SE of the dynastic repeated game. Assume that this SE displays Inter-Generational Agreement as in Definition 4. Let \(v^*\) be the vector of payoffs for this SE.

Then \(v^* \in \mathcal{E}^S(\delta)\). 

\textsuperscript{33}As will be clear from the proof of Theorem 4 we are able to show that there is an, appropriately defined, strategic equivalence between such SE of the dynastic repeated game and the SPE of the standard repeated game.
The proof of Theorem 4 is in Appendix C. Before proceeding with an intuitive outline of the proof, we state a remark on the implications of the theorem.

Clearly, Theorem 4 implies that if a payoff vector \( v \in E^D(\delta) \) is not sustainable as an SPE, so that \( v \not\in E^S(\delta) \), then it must be the case that no SE which sustains \( v \) in the dyadic repeated game displays Inter-Generational Agreement.

**Remark 5.** *SE and SPE Payoffs:* Observe that the SE that we construct in the proof of Theorem 1 all in fact satisfy Inter-Generational Agreement. In other words, we know that if a payoff vector \( v \) is in \( E^S(\delta) \) then there is an SE of the dyadic repeated game that displays Inter-Generational Agreement which sustains \( v \).

Together with Theorem 4, this gives us a complete characterization in payoff terms of the relationship between the SE of the dyadic and the SPE of the standard repeated game as follows.

Fix any stage game \( G \) and any \( \delta \in (0, 1) \). Then a payoff vector \( v \in E^D(\delta) \) is in \( E^D(\delta)/E^S(\delta) \) if and only if no SE which sustains \( v \) in the dyadic repeated game displays Inter-Generational Agreement.

To streamline the exposition of the outline of the argument behind Theorem 4, make the following two simplifying assumptions. First, assume that at the message stage the players do not condition their behavior at the message-stage on the correlation device. The simplest way to fix ideas here is to consider a message-stage correlation device with a singleton \( Y \) (the set of possible realizations). Second, assume that all message-strategies \( \mu^i \) are pure. In other words, even though they may randomize at the action stage, all players at the message stage send a single message, denoted \( \mu^i_t(m^i_t, x^i, a^j) \), with probability one.

Now consider an SE \((g, \mu, \Phi)\) of the dyadic repeated game that satisfies Inter-Generational Agreement as in Definition 4. Fix any history of play \( h^t = (x^0, a^0, \ldots, x^{t-1}, a^{t-1}) \). For each dynasty \( i \), using the message strategies of players \( i, 0 \) through to \( i, t-1 \), we can now determine the message \( m^i_t \) that player \( i, t-1 \) will send to his successor, player \( i, t \). Denote this message by \( m^i_t(h^t) \). Notice that \( m^i_t(h^t) \) can be determined simply by recursing forward from period 0. Recall that at the beginning of period 0 all players \( i \in I, 0 \) receive message \( m^i_0 = \emptyset \). Therefore, given \( h^1 = (a^0, a^0) \), using \( \mu^0_0 \), we know \( m^1_1(h^1) \). Now using \( \mu^1_1 \) we can compute \( m^2_1(h^2) = \mu^1_1(m^1_1(h^1), x^1, a^1) \), and so recursing forward the value of \( m^i_t(h^t) \) can be worked out.

---

Note that we are somewhat abusing notation here. Recall that for every player \( i, t \), both \( g^i_t(m^i_t, x^t) \) and \( \mu^i_t(m^i_t, x^t, a^j, y^j) \) denote probability distributions (over actions and messages respectively). Here and throughout the rest of the paper and the technical addendum, we will sometimes construct strategies in which a single action is played with probability one and/or a single message is sent with probability one. Abusing notation slightly, we will write \( g^i_t(m^i_t, x^t) = a^i \) to mean that the distribution \( g^i_t(m^i_t, x^t) \) assigns probability one to \( a^i \). Similarly, we will write \( \mu^i_t(m^i_t, x^t, a^j, y^j) = m^i_{t+1} \) to mean that the distribution \( \mu^i_t(m^i_t, x^t, a^j, y^j) \) assigns probability one to \( m^i_{t+1} \).
Because the SE \((g, \mu, \Phi)\) satisfies Inter-Generational Agreement it must be the case that, after any actual history of play (on or off the equilibrium path) \(h^t\), and therefore after receiving message \(m^1_t(h^t)\), player \(\langle i, t \rangle\) believes that his opponents have received messages \((m^1_1(h^t), \ldots, m^1_{t-1}(h^t), m^1_{t+1}(h^t), \ldots, m^1_n(h^t))\) with probability one.

To see why this is the case, we can recurse forward from period 0 again. Consider the end of period 0. Since all players in the \(t = 0\) cohort receive message \(\emptyset\), after observing \((x^0, a^0)\), player \(\langle i, 0 \rangle\) knows that any player \(\langle j \neq i, 0 \rangle\) is sending message \(m^1_j(x^0, a^0) = \mu^0_j(\emptyset, x^0, a^0)\) to his successor player \(\langle j, 1 \rangle\).

Equation (9) of Definition 4 guarantees that the beginning-of-period beliefs of player \(\langle i, 1 \rangle\) must be the same as the end-of-period beliefs of player \(\langle i, 0 \rangle\). So, at the beginning of period 1, player \(\langle i, 1 \rangle\) believes with probability one that every player \(\langle j \neq i, 1 \rangle\) has received message \(m^1_j(x^0, a^0)\) as above.

Equation (8) of Definition 4 guarantees that player \(\langle i, 1 \rangle\) will not revise his beginning-of-period beliefs during period 1. Therefore, after observing any \((x^1, a^1)\), player \(\langle i, 1 \rangle\) still believes that every player \(\langle j \neq i, 1 \rangle\) has received message \(m^1_j(x^0, a^0)\) as above. But this, via the message strategies \(\mu^1_j\) implies that player \(\langle i, 1 \rangle\) must believe with probability one that every player \(\langle j, 1 \rangle\) sends message \(m^2_j(h^2) = m^2_j(x^0, a^0, x^1, a^1) = \mu^1_j(m^1_j(h^1), x^1, a^1)\) to his successor \(\langle j, 2 \rangle\). Continuing forward in this way until period \(t\) we can then see that the beginning-of-period beliefs of player \(\langle i, t \rangle\) are as we claimed above.

Before we proceed to close the argument for Theorem 4, notice that both conditions of Definition 4 are necessary for our argument so far to be valid. Intuitively, the forward recursion argument we have outlined essentially ensures that the “correct” (because all its members receive a given message \(m^0_I = \emptyset\)) beliefs of the first cohort “propagate forward” as follows. At the beginning of period \(t = 1\) each player \(\langle i \in I, 1 \rangle\) must have correct beliefs since the end-of-period beliefs of all players in period 0 are trivially correct, and equation (9) of Definition 4 tells us that the end-of-period beliefs must be the same as the beginning-of-periods beliefs of the next cohort. Now some pair \((x^1, a^1)\) is observed by all players \(\langle i \in I, 1 \rangle\). If this pair is consistent with their beginning-of-period beliefs, then clearly no player \(\langle i \in I, 1 \rangle\) can possibly revise his beliefs on the messages received by others at the beginning of period 1. However, if \((x^1, a^1)\) is not consistent with the beliefs of players \(\langle i \in I, 1 \rangle\) and their action strategies, some players in the \(t = 1\) cohort may be “tempted” to revise their beginning-of-period beliefs. This is because an observed “deviation” from what they expect to observe in period 1 can always be attributed to two distinct sources. It could be generated by an actual deviation at action stage of period 1, or it could be the result of one (or more) players in the \(t = 0\) cohort having deviated at the message stage of period 0. Equation (8) of Definition 4 essentially requires that the \(t = 1\) players should always interpret an “unexpected” pair \((x^1, a^1)\) as an actual deviation at the action stage. The same applies to all subsequent periods. So, while equation (9) of Definition 4 ensures that the initially correct
beliefs are passed on from one generation to the next, equation (8) of Definition 4 guarantees that actual deviations will be treated as such in the beliefs of players who observe them. The beliefs of players \(\langle i \in I, 0 \rangle\) are correct and the end-of-period beliefs of any cohort are guaranteed to be the same as the beginning-of-period beliefs of the next cohort by equation (9). However, without equation (8) following an action deviation from the equilibrium path the end-of-periods beliefs of some players \(\langle i, t \rangle\) could be incorrect, and be passed on to the next cohort intact.

Now recall that the punch-line of the forward recursion argument we have outlined is that if the SE \((g, \mu, \Phi)\) satisfies Inter-Generational Agreement then we know that after any actual history of play \(h^t\), player \(\langle i, t \rangle\) believes that his opponents have received messages \((m_i^1(h^t), \ldots, m_i^{t-1}(h^t), m_{i+1}^t(h^t), \ldots, m_n^t(h^t))\) with probability one. To see how we can construct an SPE of the standard repeated game that is equivalent to the given SE, consider the strategies \(g_i^t\) for the standard repeated game defined as \(g_i^t(h^t, x^t) = g_i^t(m_i(h^t), x^t)\). Clearly, these strategies implement the same payoff vector that is obtained in the given SE of the dynastic repeated game. Now suppose that the strategy profile \(g^*\) we have just constructed is not an SPE of the standard repeated game. Then, by the one-shot deviation principle (Remark 1 above) we know that some player \(i\) in the standard repeated game would have an incentive do deviate in a single period \(t\) after some history of play \(h^t\). However, given the property of beliefs in the SE \((g, \mu, \Phi)\) with Inter-Generational Agreement that we have shown above, this implies that player \(\langle i, t \rangle\) would have an incentive to deviate at the action stage in the dynastic repeated game. This of course contradicts the fact that \((g, \mu, \Phi)\) is an SE of the dynastic repeated game. Hence the argument is complete. The proof of Theorem 4 that appears in Appendix C of course does not rely on the two simplifying assumptions we made here. However, modulo some additional notation and technical issues, the argument presented there runs along the same lines as the sketch we have given here.

9. Relation to the Literature

The infinitely repeated game model is a widely used construct in economics. Fudenberg and Maskin (1986), Aumann (1981), and Pearce (1992) are all standard references for the benchmark model.\(^{35}\) The prevailing view is that this benchmark model, which consists of a repeated stage game played by infinitely lived agents with perfect monitoring, is outcome-equivalent to a dynastic model in which short-lived individuals are replaced by their heirs in the game.

In one formal sense, this view is correct: when it is assumed that all individuals have full and direct knowledge of the past history of play, the two models are outcome-equivalent in subgame perfect equilibria. However, some recent work challenges the underlying “full memory” assumption. This literature argues that any literal interpretation of dynastic players

\(^{35}\)Pearce (1992) is also an excellent source for further references.
entails that new entrants do not typically have direct memory of play prior to their entry. The question posed then is: to what extent can intergenerational communication substitute for memory? Recent papers by Anderlini and Lagunoff (2005), Kobayashi (2003), and Lagunoff and Matsui (2004) all posit dynastic game models to address this question.

Among these, Anderlini and Lagunoff (2005) is the closest and, in many ways, the most direct predecessor of the current paper. Anderlini and Lagunoff (2005) examines this dynastic model when each player \((i, t)\) receives a public messages from the player \((j \in I, t − 1)\) about the previous history of play. If the public messages from all player in the previous cohort are simultaneous, then a Folk Theorem in the sense of Fudenberg and Maskin (1986) can be obtained. Namely, if there are three or more players, all individually rational feasible payoffs can be sustained as an SE. Intuitively, this is because a version of a well known “cross-checking” argument that goes back to Maskin (1999) can be applied in this case.\(^{36}\) By contrast, the present paper studies the model in which private communication (within each dynasty) may occur. We show that the difference between purely public and possible private communication is potentially large. Equilibria that sustain payoffs below some dynasty’s minmax exist, but they require inter-generational “disagreement.”

Kobayashi (2003) and Lagunoff and Matsui (2004) examine dynastic OLG games. As in Anderlini and Lagunoff (2005), these models assume entrants have no prior memory, and they also allow for communication across generations. Though substantive differences exist between each of the models, they both prove standard (for OLG games) Folk Theorems.\(^{37}\) Interestingly, both Folk Theorems make use of intra-generational disagreement of beliefs in the equilibrium continuations following deviations. Nevertheless, the constructed equilibria in both papers leave no room for inter-generational disagreement at the message stage.

The role of public messages has been studied in other repeated game contexts such as in games with private monitoring. Models of Ben-Porath and Kahneman (1996), Compte (1998), and Kandori and Matsushima (1998) all examine communication in repeated games when players receive private signals of others’ past behavior. As in Anderlini and Lagunoff (2005), these papers exploit cross-checking arguments to sustain the truthful revelation of one’s private signal in each stage of the repeated game.

Recent works by Schotter and Sopher (2001a), Schotter and Sopher (2001b), and Chaudhuri, Schotter, and Sopher (2001) examine the role of communication in an experimental dynastic environment. These papers set up laboratory experiments designed to mimic the dynastic game. The general conclusion seems to be that the presence of private communica-

\(^{36}\)Baliga, Corchon, and Sjostrom (1997) use a similar type of mechanism in another model of communication when there are three or more players.

\(^{37}\)Bhaskar (1998) examines a related OLG model with very little, albeit some, direct memory by entrants. He shows that very limited memory is enough to sustain optimal transfers in a 2-period consumption-loan smoothing OLG game.
tion has a significant (if puzzling) effect, even in the full memory game.

It is also worth noting the similarity between the present model and games with imperfect recall. Each dynastic player could be viewed as an infinitely lived player with imperfect recall (e.g., the “absent-minded driver” with “multiple selves” in Piccione and Rubinstein (1993)) who can write messages to his future self at the end of each period.

By contrast, the present model is distinguishable from dynamic models that create memory from a tangible “piece” of history. For instance, Anderlini and Lagunoff (2005) extend the analysis to the case where history may leave “footprint,” i.e., hard evidence of the past history of play. In another instance, memory may be created from a tangible but intrinsically worthless asset such as fiat money. A number of contributions in monetary theory (e.g., Kocherlakota (1998), Kocherlakota and Wallace (1998), Wallace (2001), and Corbae, Temzelides, and Wright (2001)) have all shown, to varying degrees, the substitutability of money for memory.

10. Concluding Remarks

We began by arguing that the literal interpretation of an infinitely repeated game stretches the limits of plausibility, and hence that a closer investigation of the dynastic “equivalent” is worth pursuing. However, once we remove the assumption that players in a dynastic repeated game observe the history of play that takes place before they are born, the equivalence between these two models ceases to hold.

We posit a dynastic repeated game populated by one-period-lived individuals who rely on private messages from their predecessors to fathom the past. The set of equilibrium payoffs expands dramatically relative to the corresponding standard repeated game. Under extremely mild conditions, as the dynastic players care more and more about the payoffs of their successors, all interior payoff vectors that are feasible in the stage game are sustainable in an SE of the dynastic repeated game.

We are able to characterize entirely, via a property of the players’ beliefs, when an SE of the dynastic repeated game can yield a payoff vector not sustainable as an SPE of the standard repeated game: the SE in question must display a failure of what we have termed Inter-Generational Agreement.

It is natural to ask why we focus on SE as opposed to other solution concepts. The answer is two-fold. First the concept of SE is an extremely widely accepted and used benchmark in the literature. Hence it seems an appropriate point to start. Second, the surprising features of some of the equilibria of our dynastic repeated game arise from a failure of Inter-Generational Agreement. Broadly speaking, failures of Inter-Generational Agreement can be traced back to the fact that, upon receiving an unexpected private message from his predecessor, a player

\[38\text{See the Special Issue of Games and Economic Behavior (1997) on Games with Imperfect Recall for extensive references.}\]
always has to weigh two sets of possibilities. The message he observes could have been generated by a deviation at the action stage, or it could have been generated by a deviation at the message stage. The concept of SE requires all players to have a complete (and common across players) theory of the mistakes that might have caused deviations from equilibrium. Therefore, it seems like a prime candidate to require that the players’ beliefs should be plausible in an intuitive sense in the dynastic repeated game we analyze.

We leave the analysis of the impact of further equilibrium refinements on the set of equilibria of the dynastic repeated game for possible future research. However, it is important to comment here on the relationship between our results and the idea of “neologism-proofness” that has been put forth in the literature (Farrell 1993, Mattheus, Okuno-Fujiwara, and Postlewaite 1991, among others). As we mentioned above, at least in its current form, neologism-proofness simply does not apply to our framework. The reason is simple. Roughly speaking, neologism-proofness builds into the solution concept the idea that in a sender-receiver game, provided the appropriate incentive-compatibility constraints are satisfied, a player’s *exogenous type* (in the standard sense of a “payoff type”) will be able to create a “neologism” (use an hitherto unused message) to distinguish himself from other types. The point is that in our dynastic game there are *no payoff types* for any of the players. It would therefore be impossible to satisfy any form of incentive-compatibility constraints. The different “types” of each player in our dynastic repeated game are only distinguished by their beliefs, which in turn are determined by equilibrium strategies together with a complete theory of mistakes as in any SE. To see how slippery the logic of neologism-proofness can become in our context, consider for instance the construction we use to prove Theorem 2 above. Suppose that some player \( \langle i, t \rangle \) deviates so as to trigger the permanent punishment phase against dynasty \( i \). At the end of period \( t \) player \( \langle i, t \rangle \) would like to communicate to player \( \langle i, t + 1 \rangle \) that play is in the permanent punishment phase so that dynasty \( i \) can play a myopic best response to the actions of others from \( t + 1 \) onwards. For a “neologism” to work at this point player \( \langle i, t + 1 \rangle \) would have to believe it. He would have to believe what player \( \langle i, t \rangle \) is saying: I have deviated, therefore respond appropriately; you should believe me because I am rational (there are no exogenous types to which \( \langle i, t \rangle \) can appeal in his “speech”). However, the very fact that \( \langle i, t \rangle \) has deviated is an indication that he is *not* rational given the original SE strategies. In the SE we construct above, player \( \langle i, t + 1 \rangle \) trusts his complete theory of mistakes more than the “speech” of player \( \langle i, t \rangle \). This, as we described above, tells him to believe with positive probability that play is in fact in the equilibrium phase.

While our results apply only to the actual formal model we have set forth, it is natural to ask which ones are essential and which ones are not. We have several remarks to make.

As we noted already, the absence of public messages alongside private ones is completely inessential to what we do here. Public messages could be added to our model without altering
our results. It is always possible to replicate the SE of this model in another model with public messages as well; the public messages would be ignored by the players’ equilibrium strategies and beliefs.

We make explicit use of correlation devices both at the action and at the message stage of the dynastic repeated game. While the use of two separate devices is not essential for our results (see footnote 13 above), the use of some correlation device is. This is contrast with the arguments in Fudenberg and Maskin (1986). In the standard repeated game, all individually rational payoffs can be *approximately* sustained as SPE even without a correlation device. The equilibrium path can be constructed so that, as the discount factor approaches one, “time averaging” across different pure action profiles in each period can (approximately) substitute for a correlation device. By contrast, in our constructions the correlation device is essential to merge the “local” part of the argument with the “global” one. We have examples showing that even without any correlation devices it is possible to display SE that push one or more dynasties below their minmax value. A full characterization of the set of SE payoffs in this case is an open question at this point.

We have stipulated a very specific set of “demographics” for our dynastic repeated game: all players live one period and are replaced by their successor at the end of their lives. We believe that this is not essential to the qualitative nature of our results, although some features of the demographic structure of the model play an important role. We conjecture that results similar to ours would hold in a dynastic repeated game in which all players are simultaneously replaced by their successors in the same dynasty with positive probability, thus ensuring that looking forward from any point in time total replacement happens with probability one at some future date.

The actual history of play leaves no visible trace in our model. To what extent would our results change if a (noisy or incomplete, or both) “footprint” of the past history of play were available to the players? Intuitively this would make failures of Inter-Generational Agreement harder to generate in equilibrium. Does the set of equilibria of the dynastic repeated game shrink as the direct information about the past history of play that is available to the players becomes more and more precise? While it seems plausible to conjecture that our results would change across this range of possibilities, we have no option but to say that the topic is worthy of future research.

Our Folk Theorems for the dynastic repeated game assume three or more dynasties in one case, and four or more in another. Is this essential to our results? (It clearly is essential to our construction in both cases.) It is not hard to construct examples of dynastic repeated games with two dynasties that admit SE in which the players’ payoffs are below their minmax in the stage game. Thus, it seems that there is no definite need to have more than two dynasties to generate SE payoffs in the dynastic repeated game that are not sustainable as SPE of the corresponding standard repeated game. Whether and under what conditions a Folk Theorem
like the ones presented here is available for the case of two dynasties is an open question. We leave the characterization of the equilibrium set in this case for future work.

Lastly, our Folk Theorems for the dynastic repeated game show that, as \( \delta \) approaches one, the set of SE payoffs includes “worse” vectors that push some (or even all) players below their minmax payoffs in the stage game. We do not have a full characterization of the SE payoffs for the dynastic repeated game when \( \delta \) is bounded away from one. However, it is possible to construct examples showing that the set of SE payoffs includes vectors that Pareto-dominate those on the Pareto-frontier of the set of SPE payoffs in the standard repeated game when \( \delta \) is bounded away from one. Intuitively, this is because some “bad” payoff vectors (pushing some players below the minmax) are sustainable in an SE when \( \delta \) is bounded away from one. Thus, “harsher” punishments are available as continuation payoffs in the dynastic repeated game than in the standard repeated game. Using these punishments, higher payoffs are sustainable in equilibrium in the dynastic repeated game.

Appendix A: The Basics of the Proof of Theorem 2

A.1. Outline

Our proof is constructive. It runs along the following lines. Given a \( v^* \in \text{int}(V) \) and a \( \delta \in (0, 1) \) we construct a correlation device \( \tilde{x} \), a correlation device \( \tilde{y}(\delta) \), and an assessment \( (g, \mu, \Phi^\delta) \), which implements \( v^* \), and which for \( \delta \) sufficiently large constitutes an SE of the dynastic repeated game.

Notice that the profile \( (g, \mu) \) is defined independently of \( \delta \), as is \( \tilde{x} \). On the other hand, the probability distribution of the message-stage correlation device \( \tilde{y}(\delta) \) and the system of beliefs \( \Phi^\delta \) are defined using \( \delta \) as a parameter. This is simply a property of our construction. Given that our task is to show that there exists a \( \delta \in (0, 1) \) such that \( \delta > \delta^* \implies v^* \in E_D(\delta) = \bigcup_{\tilde{x}, \tilde{y}} E_D(\delta, \tilde{x}, \tilde{y}) \), if \( (g, \mu) \) and \( \tilde{x} \) were also dependent on \( \delta \) this would not matter at all. Sometimes, when it does not cause any ambiguity, \( \delta \) will be dropped from the notation for the message-stage correlation device and/or the system of beliefs.

Throughout the argument, we assume that the message space for any player \( \langle i, t \rangle \) consists of three elements only. Formally, we let \( M_t^i = \{ m^*, m^A, m^B \} \). Since we assume that \( n \geq 3 \) and that \( |A_i| \geq 2 \) for every \( i \), it is obvious that \( |H^t| > 3 \) for every \( t \geq 1 \). Therefore we can think of each \( M_t^i \) as a “restricted message space” in the sense of Definition T.2.1. It then follows from Lemma T.2.1 that proving the result for these restricted message spaces is sufficient to prove our claim for the case \( M_t^i = H^t \), as required by the statement of the theorem.

In Section A.2 we define formally the correlation devices \( \tilde{x} \) and \( \tilde{y}(\delta) \) and the profile \( (g, \mu) \). In Section A.3 we define the completely mixed strategies that generate the SE beliefs \( \Phi^\delta \). In Section T.5 we define formally the system of beliefs \( \Phi^\delta \). In Section T.6 we check that the assessment \( (g, \mu, \Phi^\delta) \) satisfies sequential rationality when \( \delta \) is close to one. Finally in Section T.7 we verify the consistency of the equilibrium beliefs.

Throughout the argument, \( v^* \in \text{int}(V) \) is the vector of payoffs to be sustained as an SE, and \( a^* \) and \( a' \) are two action profiles as in the statement of the theorem.

A.2. Strategies and Correlation Devices

Definition A.1: Let \( (a(1), \ldots, a(\ell), \ldots, a(|A|)) \) be a list of all possible outcomes in \( G \). Without loss of generality assume that the first element of this list is \( a^* \) so that \( a(1) = a^* \). We can then define a set of
weights \( \{ p(a(\ell)) \}_{\ell=1}^{\|A\|} \) adding up to one and such that

\[
v^* = \sum_{\ell=1}^{\|A\|} p(a(\ell))u(a(\ell)) \tag{A.1}
\]

Notice that since \( v^* \in \text{int}(V) \), we can assume that \( p(a(\ell)) > 0 \) for all \( \ell = 1, \ldots, \|A\| \).

Since it will play a special role, we assign a separate symbol to \( p(a(1)) \) (the weight of \( a^* \) in equation (A.1)). In what follows we set \( p(a(1)) = q \).

**Definition A.2.** *Action-Stage Correlation Device:* Given \( v^* \), the action-stage correlation device \( \tilde{x} \) is defined as follows. The set \( X \) consists of \( \|A\| \) elements denoted by \((x(1), \ldots, x(\ell), \ldots, x(\|A\|))\). We then set

\[
\Pr(\tilde{x} = x(\ell)) = p(a(\ell)) \tag{A.2}
\]

where each \( p(a(\ell)) \) is as in Definition A.1.

**Definition A.3:** Let \( \underline{u}_i = \min_{a_i \in A} u_i(a) \) and \( \overline{u}_i = \max_{a_i \in A} u_i(a) \).\(^{30}\) Let also \( \alpha \) be a real number in \((0,1)\) such that

\[
\alpha < \frac{u_i(a') - u_i(a^*_{\ell-1})}{u_i(a') - u_i(a^*_\ell) + \overline{u}_i - \underline{u}_i} \quad \forall i \in I \tag{A.3}
\]

**Definition A.4.** *Message-Stage Correlation Device:* Given \( \delta \) the message-stage correlation device \( \tilde{y}(\delta) \) is defined as follows. The set \( Y \) consists of two elements, \( y(0) \) and \( y(1) \).

Let \( \alpha \) be as in Definition A.3. Now define

\[
\gamma(\delta) = \frac{(1 - \delta)(1 - \alpha)}{\alpha \delta} \tag{A.4}
\]

Notice that, given \( \alpha \), as \( \delta \) increases towards one, clearly \( \gamma(\delta) \) is between zero and one. We then take it to be the case that

\[
\Pr[\tilde{y}(\delta) = y(0)] = 1 - \gamma(\delta)
\]

\[
\Pr[\tilde{y}(\delta) = y(1)] = \gamma(\delta) \tag{A.5}
\]

**Definition A.5.** *Action-Stage Strategies:* Let \( \{a(\ell)\}_{\ell=1}^{\|A\|} \) be an enumeration of the strategy profiles in \( G \) as in Definition A.1, and let \( \tilde{x} \) be as in Definition A.2. Recall that at the beginning of period 0 all players receive the null message \( m_0^0 \). For all players \( \langle i, t \rangle \) define

\[
g^0_i(m_0^0, x^0) = a_i(\ell) \quad \text{if} \ x^0 = x(\ell) \tag{A.6}
\]

and for any player \( \langle i, t \rangle \) with \( t \geq 1 \) define

\[
g^t_i(m_t^i, x^t) = \begin{cases} 
a_i^* & \text{if} \ m_t^i = m^* \text{ and } x^t = x(1) \\
a'_i & \text{if} \ m_t^i \in \{m^A, m^B\} \text{ and } x^t = x(1) \\
a_i(\ell) & \text{if} \ x^t = x(\ell) \text{ with } \ell \geq 2 \end{cases} \tag{A.7}
\]

\(^{30}\)Throughout the rest of the paper and the technical addendum, we will use the symbols \( \underline{u}_i \) and \( \overline{u}_i \) with the meaning we have just defined.
Definition A.6. Message-Stage Strategies: Let \( \{a(\ell)\}_{\ell=1}^{1|A|} \) be an enumeration of the strategy profiles in \( G \) as in Definition A.1. Let \( x \) be as in Definition A.2. Let \( y \) be as in Definition A.4, where the dependence on \( \delta \) has been suppressed for notational convenience.

To describe the message-stage strategies it is convenient to distinguish between two cases.\(^{40}\)

For any player \( \langle i, t \rangle \), whenever \( x^t = x(1) \) let\(^{41}\)

\[
\mu_i^t(m^t_i, x(1), a^t, y^t) = \begin{cases} 
  m^A & \text{if } y^t = y(0) \text{ and } a^t = (a_{j}^*, a_{-j}^*) \text{ for some } j \in I \\
  m^A & \text{if } m^t_i = m^A, \ y^t = y(0) \text{ and } a^t = a' \\
  m^A & \text{if } m^t_i = m^A, \ y^t = y(0) \text{ and } a^t = (a_{j}^*, a_{-j}^*) \text{ for some } j \in I \\
  m^B & \text{if } m^t_i = m^B \text{ and } a^t = a' \\
  m^B & \text{if } m^t_i = m^B, \text{ and } a^t = (a_{j}^*, a_{-j}^*) \text{ for some } j \in I \\
  m^* & \text{otherwise} 
\end{cases}
\] (A.8)

For any player \( \langle i, t \rangle \), whenever \( x^t = x(\ell) \) with \( \ell \geq 2 \) let

\[
\mu_i^t(m^t_i, x(\ell), a^t, y^t) = \begin{cases} 
  m^* & \text{if } m^t_i = m^*, \text{ and } a^t = a(\ell) \\
  m^* & \text{if } m^t_i = m^A, \ y^t = y(1) \text{ and } a^t = a(\ell) \\
  m^A & \text{if } m^t_i = m^A, \ y^t = y(0) \text{ and } a^t = a(\ell) \\
  m^B & \text{otherwise} 
\end{cases}
\] (A.9)

A.3. Completely Mixed Strategies

Definition A.7: Throughout this section we let \( \varepsilon \) denote a small positive number, which parameterizes the completely mixed strategies that we construct. It should be understood that our construction of beliefs involves the limit \( \varepsilon \to 0 \).

For every \( t \geq 0 \), given \( \varepsilon \), define

\[
\varepsilon_t = \frac{1}{\max_{i \in I} \|A_i\}} \varepsilon^\frac{1}{n^{2T}} 
\] (A.10)

Notice also that in Definition A.9 below we make use of the quantity \( \psi_\delta(\delta) \) that is defined in equation (T.7.1). What needs to be specified at this point is that, for \( \delta \) sufficiently close to one, \( \psi_\delta(\delta) \) is a real number in \( (0, 1) \) for every \( i \in I \). (See Remark T.7.1.)

Definition A.8. Completely Mixed Action Strategies: Given \( \varepsilon \), the completely mixed strategies for all players \( \langle i, t \rangle \) at the action stage are denoted by \( g_{i, \varepsilon}^t \) and are defined as follows.

After receiving message \( m_i^0 \) and observing the realization \( x^t \) of the action-stage correlation device any player \( \langle i, 0 \rangle \) plays the action prescribed by the proposed equilibrium strategy described in (A.6) with probability \( 1 - \varepsilon_t(\|A_i\| - 1) \) and plays all other actions in \( A_i \) with probability \( \varepsilon_t \) each.

\(^{40}\)In the interest of brevity, we avoid writing down the message-stage strategies for players \( \langle i \in I, 0 \rangle \) separately. Equations (A.8) and (A.9) that follow can be interpreted as defining the profile \( \mu^\ell \) by re-defining \( m^t_i \) to be equal to \( m^* \) for all players \( \langle i \in I, 0 \rangle \).

\(^{41}\)Notice that to distinguish between the first and the fifth case, and between the first and the last case, on the right-hand side of (A.8), player \( \langle i, t \rangle \) must be able to distinguish between an action profile of the type \( (a_{j}^*, a_{-j}^*) \) and an action profile of the type \( (a_{j}^*, a_{-j}^*) \). This is always possible because of our assumption that \( n \geq 3 \), and because our assumptions about \( a^* \) and \( a' \) obviously imply that \( a_{j}^* \neq a_{j}^* \) for every \( j \in I \).
After receiving message $m^*_t$ and observing the realization $x^t$ of the action-stage correlation device any player $(i,t)$ plays the action prescribed by the proposed equilibrium strategy described in (A.7) with probability $1 - \varepsilon_t(|A_i| - 1)$ and plays all other actions in $A_i$ with probability $\varepsilon_t$ each.

After receiving any message $m^t_i \neq m^*_t$ and observing the realization $x^t$ of the action-stage correlation device any player $(i,t)$ plays the action prescribed by the proposed equilibrium strategy described in (A.7) with probability $1 - \sqrt{\varepsilon_t(|A_i| - 1)}$ and plays all other actions in $A_i$ with probability $\sqrt{\varepsilon_t}$ each.

**Definition A.9.** Completely Mixed Message Strategies: Given $\varepsilon$, the completely mixed strategies for all players $(i,t)$ at the message stage are denoted by $\mu^t_i$, and are defined as follows.\(^{42}\)

Suppose that player $(i,t)$ receives message $m^*_t$ and that after observing $(x^t, a^t, y^t)$ the proposed equilibrium strategy described in Definition A.6 prescribes that he should send message $m^*$. Then player $(i,t)$ sends message $m^*$ with probability $1 - \varepsilon^2_t - \psi_i(\delta)\varepsilon_t$, sends message $m^A$ with probability $\varepsilon^2_t$, and sends message $m^B$ with probability $\psi_i(\delta)\varepsilon_t$.

Suppose that player $(i,t)$ receives message $m^*_t$ and that after observing $(x^t, a^t, y^t)$ the proposed equilibrium strategy described in Definition A.6 prescribes that he should send message $m^A$. Then player $(i,t)$ sends message $m^A$ with probability $1 - \varepsilon_t - \varepsilon^2_t$, sends message $m^*$ with probability $\varepsilon_t$, and sends message $m^B$ with probability $\varepsilon^2_t$.

Suppose that player $(i,t)$ receives message $m^*_t$ and that after observing $(x^t, a^t, y^t)$ the proposed equilibrium strategy described in Definition A.6 prescribes that he should send message $m^B$. Then player $(i,t)$ sends message $m^A$ with probability $1 - \varepsilon_t - \varepsilon^2_t$, sends message $m^*$ with probability $\varepsilon_t$, and sends message $m^A$ with probability $\varepsilon^2_t$.

Finally, after receiving any message $m^t_i \neq m^*$ and observing $(x^t, a^t, y^t)$, player $(i,t)$ sends the message prescribed by the equilibrium strategy described in Definition A.6 with probability $1 - 2\varepsilon_t$, and sends each of the other two messages with probability $\varepsilon_t$.

**Appendix B: The Basics of the Proof of Theorem 3**

**B.1. Outline**

Once again, our proof is constructive. It runs along the following lines. Given a $v^* \in \text{int}(V)$, we construct a correlation device $\tilde{x}$, a correlation device $\tilde{y}$, and an assessment $(g, \mu, \Phi)$, which implements the vector of payoffs $v^*$, and which for $\delta$ sufficiently large constitutes an SE of the dynastic repeated game.

Notice that all the elements of our construction are defined independently of $\delta$. The sequential rationality of the strategy profile given the postulated beliefs holds when $\delta$ is sufficiently close to one.

Throughout the argument, we assume that the message space for any player $(i, t-1)$ consists of a set smaller than the set $H^t$. Therefore we can think of each $M^t_i$ as a “restricted message space” in the sense of Definition T.2.1. It then follows from Lemma T.2.1 that proving the result for these restricted message spaces is sufficient to prove our claim for the case $M^t_i = H^t$, as required by the statement of the theorem.

In Section B.2 we define formally the correlation devices $\tilde{x}$ and $\tilde{y}$, the players’ message spaces $M^t_i$ and the profile $(g, \mu)$. In Section B.3 we define the completely mixed strategies that generate the SE beliefs $\Phi$. In Section T.9 we define formally the system of beliefs $\Phi$. In Section T.10 we check that the assessment $(g, \mu, \Phi)$ satisfies sequential rationality when $\delta$ is close to one. Finally in Section T.11 we verify the consistency of the equilibrium beliefs.

In what follows, $v^* \in \text{int}(V)$ is the vector of payoffs to be sustained as an SE as in the statement of the theorem. Throughout the argument, $A$ is a product set and $\bar{v}$ and $\bar{v}^*$ through $\bar{v}^n$ are vectors of payoffs as in the statement of the theorem. Of course, these are fixed throughout the proof.

\(^{42}\)In the interest of brevity, we avoid an explicit distinction between the $t = 0$ players and all others. What follows can be interpreted as applying to all players re-defining $m^0_i$ to be equal to $m^*$ for players $(i \in I, 0)$.
B.2. Strategies, Correlation Devices and Message Spaces

Definition B.1: As in Definition A.1, let \((a(1), \ldots, a(\ell), \ldots, a(|A|))\) be a list of all possible outcomes in G. Without loss of generality, assume that the first \(|\tilde{A}| \leq |A|\) elements in this enumeration are the strategy profiles in the product set \(\tilde{A}\). This enumeration will be taken as fixed throughout the rest of the argument.

Remark B.1: From Definition 3, for each \(i \in I\) we can find a set of of profiles of actions \(\tilde{A}^i \subset \tilde{A}\) corresponding to a set of indices \((i_1, \ldots, i_\ell, \ldots, i_{|\tilde{A}^i|})\) in the enumeration of Definition B.1, and a set of positive weights \(\{p^i(a(i_\ell))\}_{\ell=1}^{|	ilde{A}^i|}\) adding up to one and such that

\[
\omega_i(\tilde{A}) = \max_{a_i \in A_i} \sum_{\ell=1}^{|	ilde{A}^i|} p^i(a(i_\ell))u_i(a_i, a_{-i}(i_\ell)) = \sum_{\ell=1}^{|	ilde{A}^i|} p^i(a(i_\ell))u_i(a(i_\ell)) \tag{B.1}
\]

Without loss of generality, we can take it to be the case that for every \(i_\ell, a_i(i_\ell)\) is the same action in \(\tilde{A}_i\). We denote this by \(a_i^i\) so that \(a_i^i(i_\ell) = a_i(i_\ell)\) for \(\ell = 1, \ldots, |\tilde{A}^i|\).

For convenience, since \(\tilde{A}\) is fixed throughout the argument, in what follows we will use the following notation for the payoffs of each \(i\) corresponding to the weights \(\{p^i(a(j_\ell))\}_{\ell=1}^{|	ilde{A}^i|}\).

\[
\omega_i^i = \sum_{\ell=1}^{|	ilde{A}^i|} p^i(a(j_\ell))u_i(a(j_\ell)) \tag{B.2}
\]

Of course, we have that \(\omega_i^i = \omega_i(\tilde{A})\).

Definition B.2: Let \(\tilde{A}^i\) be as in Remark B.1. For each \(i \in I\) and for each element \(a(i_\ell)\) of \(\tilde{A}^i\), construct a new action profile \(\tilde{a}^i(i_\ell)\) as follows. For all \(j \neq i\), set \(\tilde{a}_j^i(i_\ell)\) to satisfy \(\tilde{a}_j^i(i_\ell) \neq a_j(i_\ell)\) and \(\tilde{a}_j^i(i_\ell) \in \tilde{A}_j\). Notice that this is always possible since, by assumption, \(\tilde{A}_j\) contains at least two elements for every \(j \in I\). Finally, set \(\tilde{a}_i^i(i_\ell) = a_i(i_\ell) = a_i^i\).

In what follows, for every \(i\) and \(j\) in \(I\) we will let

\[
\tilde{a}_i^j = \sum_{\ell=1}^{|	ilde{A}^i|} p^i(a(j_\ell))u_i(\tilde{a}_i^j(j_\ell)) \tag{B.3}
\]

Remark B.2: Since each of the payoff vectors \(\mathbf{v}^j\) must only satisfy strong inequalities (see (iii) of the statement of the theorem), without loss of generality we can take it to be the case that \(\mathbf{v} \in \text{int}(V(\tilde{A}))\), for each \(j \in I\). It then follows that for every \(j \in I\) we can find a set of positive weights \(\{\mathbf{v}^j(a(\ell))\}_{\ell=1}^{|	ilde{A}^i|}\) adding up to one and such that for every \(i \in I\)

\[
\mathbf{v}_i^j = \sum_{\ell=1}^{|	ilde{A}^i|} \mathbf{v}^j(a(\ell))u_i(a(\ell)) \tag{B.4}
\]

Remark B.3: Since the payoff vector \(\mathbf{v}\) is in \(\text{int}(V(\tilde{A}))\), we can find an \(\eta \in (0, 1)\) and a set of positive weights \(\{\mathbf{v}^j(a(\ell))\}_{\ell=1}^{|	ilde{A}^i|}\) adding up to one and such that for every \(i \in I\)

\[
\hat{v}_i = (1 - \eta)\sum_{\ell=1}^{|	ilde{A}^i|} \mathbf{v}^j(a(\ell))u_i(a(\ell)) + \frac{\eta}{n} \sum_{j=1}^{n} \tilde{a}_i^j \tag{B.5}
\]
Definition B.3: Throughout the rest of the argument we let
\[ \hat{v}_i = \sum_{\ell=1}^{|A|} \hat{p}(a(\ell))u_i(a(\ell)) \]  
(B.6)
where the weights \( \{\hat{p}(a(\ell))\}_{\ell=1}^{|A|} \) are as in Remark B.3.

Remark B.4: Since the payoff vector \( v^* \) is in \( \text{int}(V) \), we can find a \( q \in (0,1) \) and a set of positive weights \( \{p^*(a(\ell))\}_{\ell=1}^{|A|} \) adding up to one and such that for every \( i \in I \)
\[ v^*_i = (1-q)\sum_{\ell=1}^{|A|} p^*(a(\ell))u_i(a(\ell)) + q\hat{v}_i \]  
(B.7)
In what follows we will let \( z_i = \sum_{\ell=1}^{|A|} p^*(a(\ell))u_i(a(\ell)) \).

Remark B.5: Recall that by assumption the payoff vector \( v^* \) is in \( \text{int}(V) \). Therefore, we can find a set of positive weights \( \{p^0(a(\ell))\}_{\ell=1}^{|A|} \) adding up to one and such that for every \( i \in I \)
\[ v^*_i = \sum_{\ell=1}^{|A|} p^0(a(\ell))u_i(a(\ell)) \]  
(B.8)

Definition B.4. Action-stage Correlation Device: The action stage correlation device \( \hat{x} \) is defined as follows. The set \( X \) consists of \( |A| \cdot |\hat{A}|^{n+1} \prod_{i \in I} |\hat{A}|^4 \) elements.

Let \( x(1), \ldots, x(\kappa), \ldots, x(|X|) \) be an enumeration of the elements of \( X \), and let \( \pi = |A| \cdot |\hat{A}|^{n+1} \prod_{i \in I} |\hat{A}|^4 \).

Each of the first \( \pi \) elements of \( X \) can be identified by a string of \( 1+(n+1)+n = 2n+2 \) indices as follows. With a slight abuse of notation, for \( \kappa \leq \pi \), we will write \( x(\kappa) = x(\ell_0, \ell, \ell_1, \ldots, \ell_n, \ell_{\pi}, \ldots, \ell_{\pi+n}) \) with \( \ell_0 \) running from 1 to \( |A| \), \( \ell \) and each of the indices \( \ell_1 \) through \( \ell_n \) running from 1 to \( |\hat{A}| \), and each of the \( n \) indices \( \ell_i \), each with \( \ell \) running from 1 to \( |\hat{A}| \).

We are now ready to define the probability distribution governing the realization of \( \hat{x} \). For \( \kappa \leq \pi \) let
\[ \Pr(\hat{x} = x(\ell_0, \ell, \ell_1, \ldots, \ell_n, \ell_{\pi}, \ldots, \ell_{\pi+n})) = q \left[ p^0(a(\ell_0)) \hat{p}(a(\ell)) \hat{p}^1(a(\ell_1)) \ldots \hat{p}^n(a(\ell_n)) p^1(a(1\ell)) \ldots p^n(a(n\ell)) \right] \]  
(B.9)
and for \( \kappa = \pi + 1, \ldots, |X| \) let
\[ \Pr(\hat{x} = x(\ell_0, \ell^*)) = (1-q) \left[ p^0(a(\ell_0))p^*(a(\ell^*)) \right] \]  
(B.10)

Definition B.5. Message-Stage Correlation Device: The message stage correlation device \( \tilde{y} \) is defined as follows. The set \( Y \) consists of \( n+1 \) elements, which we denote \( \{y(0), y(1), \ldots, y(n)\} \). The random variable \( \tilde{y} \) takes value \( y(0) \) with probability \( 1-\eta \), and each of the other possible values with probability \( \eta/n \).

Definition B.6: Throughout the rest of the argument, we let \( T \) be an integer sufficiently large so as to guarantee that the following inequality is satisfied for all \( i \in I \).
\[ T (\pi_i^* - \omega_i) > \pi_i - u_i \]  
(B.11)
Definition B.7. Message Spaces: For each \( j \in I \) and each \( t = 1, \ldots, T - 1 \) let
\[
M(j, t) = \{m^{j,T-t+1}, m^{j,T-t+2}, \ldots, m^{j,T}\}
\]
and for every \( t \geq T \) let
\[
M(j, t) = M(j, T) = \{m^{j,1}, \ldots, m^{j,T}\}
\]
Define also \( \bar{M} = \{\bar{m}^1, \ldots, \bar{m}^n\} \), and \( \bar{M} = \{\bar{m}^1, \ldots, \bar{m}^n\} \), and recall that according to our notational convention when we write \( M_n \) we mean \( \{\bar{m}^1, \ldots, \bar{m}^{n-1}, \bar{m}^{n+1}, \bar{m}^n\} \).

Recall that \( M_t^i \) denotes the set of messages that a player \((i, t-1)\) can send to player \((i, t)\). For any \( t = 1, \ldots, T \) let
\[
M_t^i = \{m^{\ast}\} \cup \bar{M}_{-i} \cup M(1, t) \cup \ldots \cup M(n, t)
\]
For any \( t \geq T + 1 \) let
\[
M_t^i = \{m^{\ast}\} \cup \bar{M}_{-i} \cup M(1, t) \cup \ldots \cup M(n, t) \cup \bar{M}
\]

Definition B.8. Action-Stage Strategies: Let \( k \) be an element of \( I \), and \( j \) be an element of \( I \) not equal to \( i \).

Let \((a(1), \ldots, a(|A|))\) be the enumeration of the elements of \( A \) of Definition B.1 and consider the indexation of the elements of \( X \) in Definition B.4, according to whether \( x(\kappa) \) has \( \kappa \leq \pi \) or not.

Recall that at the beginning of period 0 all players \((i, 0)\) receive message \( m_i^0 = \emptyset \). For all players \((i, 0)\) then define
\[
g_i^0(m_i^0, x^0) = \begin{cases} 
  a_i(\ell_0) & \text{if } x^0 = x(\ell_0, \ldots) \\
  a_i(\ell_{00}) & \text{if } x^0 = x(\ell_{00}, \cdot) 
\end{cases}
\]

Now consider any player \((i, t)\) with \( t \geq 1 \). It is convenient to distinguish between the two cases \( x^t = x(\kappa) \) with \( \kappa \leq \pi \) and with \( \kappa > \pi \).

For any \( i \in I \) and \( t \geq 1 \) whenever \( x^t = x(\kappa) \) with \( \kappa \leq \pi \) define\(^{43}\)
\[
g_i^t(m_i^t, x^t) = \begin{cases} 
  a_i(\ell) & \text{if } m_i^t = m^t \quad \text{and } x^t = x(\ell, \ell, \ldots) \\
  a_i(\ell_j) & \text{if } m_i^t = \hat{m}_j \quad \text{and } x^t = x(\ldots, j, \ell, \ell, \ldots) \\
  a_i(\ell_k) & \text{if } m_i^t = \underline{m}_k \quad \text{and } x^t = x(\ell, \ell, \ell, \ldots) \\
  a_i(k) & \text{if } m_i^t \in M(k, t) \quad \text{and } x^t = x(\ell, \ell, k, \ell, \ldots) 
\end{cases}
\]

For any \( i \in I \), \( t \geq 1 \) and \( m_i^t \), whenever \( x^t = x(\kappa) \) with \( \kappa > \pi \) define
\[
g_i^t(m_i^t, x^t) = a_i(\ell^*) \quad \text{if } x^t = x(\cdot, \ell^*)
\]

Definition B.9. Message-Stage Strategies: Let \( k \) be any element of \( I \), and \( j \) be any element of \( I \) not equal to \( i \).

We begin with period \( t = 0 \). Recall that \( m_i^0 = \emptyset \) for all \( i \in I \). Let also \( g(0)(m_0^0, x^0) = (g_1^0(m_1^0, x^0), \ldots, g_n^0(m_n^0, x^0)), \)
and define \( g_{t+1}(m_0^0, x^0) \) in the obvious way.

\(^{43}\) Notice that the third case in (B.17) can only possibly apply when \( t \geq T + 1 \).
We let

\[
\rho^{k}_{0}(m_{i,0}^{0}, x_{0}, a_{0}, y_{0}) = \begin{cases} 
\hat{m}^{j} & \text{if } a^{0} = g^{0}(m_{0}, x_{0}) \text{ and } y^{0} = y(j) \\
m_{i,T}^{k} & \text{if } a^{0} = g^{0}(m_{0}, x_{0}) \text{ and } y^{0} = y(i) \\
m_{k,T}^{k} & \text{if } a_{-k}^{0} = g_{-k}^{0}(m_{0}, x_{0}) \text{ and } a_{k}^{0} \neq g_{k}^{0}(m_{0}, x_{0}) \\
m^{*} & \text{otherwise}
\end{cases} \tag{B.19}
\]

For the periods \( t \geq 1 \) it is convenient to distinguish between several cases. Assume first that \( x^{t} = x^{t}(\kappa) \) with \( \kappa > \pi \). Let

\[
\rho^{i}_{t}(m_{i}^{t}, x^{t}, a^{t}, y^{t}) = \begin{cases} 
\hat{m}^{j} & \text{if } x^{t} = x(\cdot , \ell^{*}) \text{ and } a^{t} = a(\ell^{*}) \text{ and } y^{t} = y(j) \\
m_{i}^{k,T} & \text{if } x^{t} = x(\cdot , \ell^{*}), a_{-k}^{t} = a_{-k}(\ell^{*}) \text{ and } a_{k}^{t} \neq a_{k}(\ell^{*}) \text{ and } y^{t} = y(i) \\
m^{*} & \text{otherwise}
\end{cases} \tag{B.20}
\]

Now consider the case \( x^{t} = x^{t}(\kappa) \) with \( \kappa \leq \pi \). We divide this case into several subcases, according to which message player \((i, t)\) has received. We begin with \( m_{i}^{t} = m^{*} \). Let\(^{44}\)

\[
\rho^{i}_{t}(m^{*}, x^{t}, a^{t}, y^{t}) = \begin{cases} 
\hat{m}^{j} & \text{if } x^{t} = x(\cdot , \ell, \cdots), a^{t} = a(\ell) \text{ and } y^{t} = y(j) \\
\nu(M(i, t + 1)) & \text{if } x^{t} = x(\cdot , \ell, \cdots), a^{t} = a(\ell) \text{ and } y^{t} = y(i) \\
m_{k,T}^{k} & \text{if } x^{t} = x(\cdot , \ell, \cdots), a_{-k}^{t} = a_{-k}(\ell) \text{ and } a_{k}^{t} \neq a_{k}(\ell) \\
m^{*} & \text{otherwise}
\end{cases} \tag{B.21}
\]

Our next subcase of \( \kappa \leq \pi \) is that of \( m_{i}^{t} = \hat{m}^{j} \). With the understanding that \( j^{*} \) is any element of \( I \) not equal to \( i \), we let

\[
\rho^{i}_{t}(\hat{m}^{j}, x^{t}, a^{t}, y^{t}) = \begin{cases} 
\hat{m}^{j'} & \text{if } x^{t} = x(\cdots , j, \ell, \cdots), a^{t} = a(\ell) \text{ and } y^{t} = y(j^{*}) \\
\nu(M(i, t + 1)) & \text{if } x^{t} = x(\cdots , j, \ell, \cdots), a^{t} = a(\ell) \text{ and } y^{t} = y(i) \\
m_{k,T}^{k} & \text{if } x^{t} = x(\cdots , j, \ell, \cdots), a_{-k}^{t} = a_{-k}(\ell) \text{ and } a_{k}^{t} \neq a_{k}(\ell) \\
m^{*} & \text{otherwise}
\end{cases} \tag{B.22}
\]

Still assuming \( \kappa \leq \pi \) we now deal with the subcase \( m_{i}^{t} \in M(i, t) \). For any \( m^{i,\tau} \in M(i, t) \), we let

\[
\rho^{i}_{t}(m^{i,\tau}, x^{t}, a^{t}, y^{t}) = \begin{cases} 
\hat{m}^{j} & \text{if } x^{t} = x(\cdots , j, \ell, \cdots), a^{t} = a(\ell) \text{ and } y^{t} = y(j) \\
\nu(M(i, t + 1)) & \text{if } x^{t} = x(\cdots , j, \ell, \cdots), a^{t} = a(\ell) \text{ and } y^{t} = y(i) \\
m_{k,T}^{k} & \text{if } x^{t} = x(\cdots , j, \ell, \cdots), a_{-k}^{t} = a_{-k}(\ell) \text{ and } a_{k}^{t} \neq a_{k}(\ell) \\
m^{*} & \text{otherwise}
\end{cases} \tag{B.23}
\]

where we set \( m^{i,0} = m^{*} \). Notice that player \((i, t)\) may need to distinguish between the third and fourth cases of (B.23) since clearly they may be generated by different values of the index \( k \in I \). To verify that this distinction is always feasible, recall that, by construction (see Definition B.2), \( a_{-k}(\ell) \) differs from \( a_{-k}(\ell) \) in every component, and that of course \( n \geq 4 \).

\(^{44}\)Throughout the paper we adopt the following notational convention. Given any finite set, we denote by \( \nu(\cdot) \) the uniform probability distribution over the set. So, if \( Z \) is a finite set, \( \nu(Z) \) assigns probability \( 1/|Z| \) to every element of \( Z \).
The next subcase of \( \kappa \leq \pi \) we consider is that of \( m^t_i \in M(j,t) \). For any \( m^{i,\tau} \in M(j,t) \), we let
\[
\mu^t_i(m^{i,\tau}, x^t, a^t, y^t) = \begin{cases} 
\frac{m^{i,\tau-1}}{m^*} & \text{if } x^t = x(\cdots, j_t, \cdots) \\
\frac{m^{i,T}}{m^*} & \text{if } x^t = x(\cdots, j_k, \cdots), a^t_k = a_k(j_k) \text{ and } a^t_k \neq a_k(j_k) \\
m^* & \text{otherwise}
\end{cases}
\] (B.24)

where we set \( m^{i,0} = \bar{m}^i \).

Finally, still assuming that \( \kappa \leq \pi \), we consider the case in which \( m^t_i = \bar{m}^{k'} \) for some \( k' \in I \). We let
\[
\mu^t_i(\bar{m}^{k'}, x^t, a^t, y^t) = \begin{cases} 
\bar{m}^{k'} & \text{if } x^t = x(\cdots, \ell_{k'}, \cdots) \\
\frac{m^{k,T}}{m^*} & \text{if } x^t = x(\cdots, \ell_{k'}, \cdots), a^t_k = a_k(\ell_{k'}) \text{ and } a^t_k \neq a_k(\ell_{k'}) \\
m^* & \text{otherwise}
\end{cases}
\] (B.25)

B.3. Completely Mixed Strategies

Definition B.10: Throughout this section we let \( \varepsilon \) denote a small positive number, which parameterizes the completely mixed strategies that we construct. It should be understood that our construction of beliefs involves the limit \( \varepsilon \to 0 \).

Definition B.11. Completely Mixed Action Strategies: Given \( \varepsilon \), the completely mixed strategies for all players \( (i,t) \) at the action stage are denoted by \( g^i_{t,\varepsilon} \) and are defined as follows.\(^{45}\)

After receiving a message \( m \in \{m^*\} \cup \hat{M}_{-i} \cup M(i,t) \) and observing the realization \( x^t \) of the action-stage correlation device, any player \( (i,t) \) plays the action prescribed by the action-stage strategy described in Definition B.8 with probability \( 1 - \varepsilon^2 \|A_i\| - 1 \) and plays all other actions in \( A_i \) with probability \( \varepsilon^2 \) each.

After receiving any message \( m \not\in \{m^*\} \cup \hat{M}_{-i} \cup M(i,t) \) and observing the realization \( x^t \) of the action-stage correlation device, any player \( (i,t) \) plays the action prescribed by the action-stage strategy described in Definition B.8 with probability \( 1 - \varepsilon \|A_i\| - 1 \) and plays all other actions in \( A_i \) with probability \( \varepsilon \) each.

Definition B.12. Completely Mixed Message Strategies: Given \( \varepsilon \), the completely mixed strategies for all players \( (i,t) \) at the message stage are denoted by \( \mu^i_{t,\varepsilon} \) and are defined as follows.

Player \( (i,t) \) sends the message prescribed by the message-stage strategy described in Definition B.9 with probability \( 1 - \varepsilon^{2n+1} \|M_i^{t+1}\| - 1 \) and sends all other messages in \( M_i^{t+1} \) with probability \( \varepsilon^{2n+1} \) each.

Appendix C: The Proof of Theorem 4

Definition C.1: Let a profile of message strategies \( \mu \) be given. Fix an “augmented history” \( \kappa^t = (x^0, a^0, y^0, \ldots, x^{t-1}, a^{t-1}, y^{t-1}) \). In other words, fix a history \( h^t \), together with a sequence of realizations of the message-stage correlation device \( (y^0, \ldots, y^{t-1}) \). In what follows, \( \kappa^0 \) will denote the null history \( \emptyset \), and for any \( \tau \leq t \), \( \kappa^\tau \) will denote the appropriate subset of \( \kappa^t \).

For every \( i \in I \) let \( \mathcal{M}^0_i(m^0_i|\kappa^0, \mu_i) = 1 \). Then, recursively forward, define
\[
\mathcal{M}^t_i(m^t_i|\kappa^t, \mu_i) = \sum_{m^{i,\tau-1} \in \hat{M}^{t-1}} \mu^{i-1}_{t-1}(m^{i,\tau-1}_i, x^{t-1}, a^{t-1}, y^{t-1}) \mathcal{M}^{t-1}_i(m^{i,t-1}_i|\kappa^{t-1}, \mu_i) \tag{C.1}
\]

So that \( \mathcal{M}^t_i(m^t_i|\kappa^t, \mu_i) \) is the probability that player \( (i,t-1) \) sends message \( m^t_i \) given \( \kappa^t \) and the profile \( \mu_i \).

We also let \( \mathcal{M}^t_{-i}(m^t_{-i}|\kappa^t, \mu_{-i}) = \mathcal{M}^t_{-i}(m^t_{-i}|\kappa^t, \mu_{-i}) = \prod_{j \neq i} \mathcal{M}^t_j(m^t_j|\kappa^t, \mu_j) \).

\(^{45}\)In the interest of brevity, we avoid an explicit distinction between the \( t = 0 \) players and all others. What follows can be interpreted as applying to any players re-defining \( m^0_i \) to be equal to \( m^* \) for players \( i \in I, 0 \).
Our working hypothesis can now be used to assert that (C.7) is in turn equal to
\[ \Phi^B_i(m_{-i}^t | m_i^t) = \mathcal{M}^t_{-i}(m_{-i}^t | \kappa^t, \mu_{-i}) \]  
(C.2)

Proof: We proceed by induction. Given the fixed \( \kappa^t \), let \( \kappa^0 = 0 \) and \( \kappa^\tau \) with \( \tau = 1, \ldots, t \) be the augmented histories comprising the first three components \((x^0, a^0, y^0)\) of \( \kappa^t \), the first six components \((x^0, a^0, y^0, x^1, a^1, y^1)\) of \( \kappa^t \) and so on. First of all notice that setting \( \tau = 0 \) yields
\[ \Phi^B_i(m_{-i}^0 | m_i^0) = \mathcal{M}^0_{-i}(m_{-i}^0 | \kappa^0, \mu_{-i}) = 1 \]  
(C.3)
which is trivially true given that all players \( i \in I, 0 \) receive the null message by construction.

Our working hypothesis is now that the claim is true for an arbitrary \( \tau - 1 < t - 1 \), and our task is to show that it holds for \( \tau \).

Consider any message \( m_i^\tau \) in \( \text{Supp} \left( \mathcal{M}^\tau_{-i}(\cdot | \kappa^t, \mu_i) \right) \). Then there must exist a message \( m_i^{\tau-1} \) such that
\[ \mu^{\tau-1}_i(m_i^{\tau-1}, x^{\tau-1}, a^{\tau-1}, y^{\tau-1}) \mathcal{M}^{\tau-1}_{-i}(m_i^{\tau-1} | \kappa^{\tau-1}, \mu_i) > 0 \]  
(C.4)
Therefore, using (9) we can write
\[ \Phi^B_i(m_{-i}^\tau | m_i^\tau) = \Phi^{\tau-1}_i(m_{-i}^{\tau-1}, x^{\tau-1}, a^{\tau-1}, y^{\tau-1}) \]  
(C.5)
Notice that in any SE it must be the case that the right-hand side of (C.5) is equal to
\[ \sum_{m_{-i}^{\tau-1}} \Phi^{\tau-1}_i(m_{-i}^{\tau-1}, m_i^{\tau-1}, x^{\tau-1}, a^{\tau-1}, y^{\tau-1}) \prod_{j \neq i} \mu^{\tau-1}_j(m_j^{\tau-1}, x^{\tau-1}, a^{\tau-1}, y^{\tau-1}) \]  
(C.6)
Using (8), we know that (C.6) is equal to
\[ \sum_{m_{-i}^{\tau-1}} \Phi^{\tau-1}_i(m_{-i}^{\tau-1}, m_i^{\tau-1}) \prod_{j \neq i} \mu^{\tau-1}_j(m_j^{\tau-1}, x^{\tau-1}, a^{\tau-1}, y^{\tau-1}) \]  
(C.7)
Our working hypothesis can now be used to assert that (C.7) is in turn equal to
\[ \sum_{m_{-i}^{\tau-1}} \mathcal{M}^{\tau-1}_{-i}(m_{-i}^{\tau-1} | \kappa^{\tau-1}, \mu_{-i}) \prod_{j \neq i} \mu^{\tau-1}_j(m_j^{\tau-1}, x^{\tau-1}, a^{\tau-1}, y^{\tau-1}) \]  
(C.8)
Rearranging terms (C.8) we find that it can also be written as
\[ \prod_{j \neq i} \left[ \sum_{m_j^{\tau-1}} \mathcal{M}^{\tau-1}_j(m_j^{\tau-1} | \kappa^{\tau-1}, \mu_j) \mu^{\tau-1}_j(m_j^{\tau-1}, x^{\tau-1}, a^{\tau-1}, y^{\tau-1}) \right] \]  
(C.9)

\[ ^{46}\text{Supp} (\cdot) \text{ denotes the support of a probability distribution.} \]
Using now (C.1), it is immediate that (C.9) is equal to
\[
\prod_{j \neq i} \mathcal{M}_{ij}(m_j^\tau | \kappa^\tau, \mu_j) = \mathcal{M}_{i-i}(m_{i-i}^\tau | \kappa^\tau, \mu_{-i})
\]  
(C.10)
and hence the claim is proved. ■

**Definition C.2:** Fix any \( \delta \in (0, 1) \), any \( \tilde{x} \) and any \( \tilde{y} \). Fix any strategy profile, \((g, \mu)\), for the dynastic repeated game.

Consider the standard repeated game with the same common discount factor \( \delta \), and with the following action-stage correlation device \( \hat{x} \). The random variable \( \hat{x} \) takes values in the finite set \( Y \times X \) (the sets in which \( \tilde{y} \) and \( \tilde{x} \) take values respectively), and the probability that \( \hat{x} \) is equal to \( \hat{x} = (y, x) \) is \( \Pr(\tilde{y} = y) \Pr(\tilde{x} = x) \).

For notational convenience we will denote the realization action-stage correlation device \( \hat{x} \) from the dynastic repeated game profile \((\hat{x}, \tilde{x})\), as above if and only if it is defined as follows.

\[
g_i^{\text{st}}(h^t, \hat{x}^t) = g_i^{\text{st}}(y^{-1}, \kappa^t, x^t) = \sum_{m_i^t} \mathcal{M}_{i}^t(m_i^t | \kappa^t, \mu_i) g_i^t(m_i^t, x^t)
\]  
(C.11)

**Lemma C.2:** Fix any \( \delta \in (0, 1) \), any \( \tilde{x} \) and any \( \tilde{y} \). Consider any SE, \((g, \mu)\), of the dynastic repeated game that displays Inter-Generational Agreement as in Definition 4.

Now consider the strategy profile \( g^* \) for the standard repeated game with correlation device \( \hat{x} \) that is derived from \((g, \mu)\) as in Definition C.2.

Given \( g^* \), fix any pair \((h^t, \hat{x}^t)\) representing a history and realized correlation device for the standard repeated game. For any \( a_{-i}^t \in A_{-i} \), let \( \mathcal{P}_{g^* | h^t, \hat{x}^t}(a_{-i}^t) \) be the probability that the realized action profile for all players but \( i \) at time \( t \) is \( a_{-i}^t \).

Given the pair \((h^t, \hat{x}^t)\), consider the corresponding triple \((y^{-1}, \kappa^t, x^t)\) as in Definition C.2. Then

\[
\mathcal{P}_{g^* | h^t, \hat{x}^t}(a_{-i}^t) = \prod_{j \neq i} \left\{ \sum_{m_j^t} \mathcal{M}_{j}^t(m_j^t | \kappa^t, \mu_j) g_j^t(a_j^t | m_j^t, x^t) \right\}
\]  
(C.12)

**Proof:** The claim is a direct consequence of (C.11) of Definition C.2. ■

**Lemma C.3:** Fix any \( \delta \in (0, 1) \), any \( \tilde{x} \) and any \( \tilde{y} \). Consider any SE \((g, \mu, \Phi)\) of the dynastic repeated game that displays Inter-Generational Agreement as in Definition 4.

Fix any pair \((h^t, \hat{x}^t)\) representing a history and realized correlation device for the standard repeated game. Given the pair \((h^t, \hat{x}^t)\), consider the corresponding triple \((y^{-1}, \kappa^t, x^t)\) as in Definition C.2. Given the last two elements of this triple \((\kappa^t, x^t)\), now use (C.1) to find a message \( \overline{m}^i \) such that \( \mathcal{M}_{i}^t(\overline{m}^i | \kappa^t, \mu_i) > 0 \).

Finally, consider the following alternative action-stage and message-stage strategies \((\overline{g}_i, \overline{\mu}_i)\) for player \((i, t)\). Whenever \( m_i^t \neq \overline{m}^i \), set \( \overline{g}_i^t = g_i^t \) and \( \overline{\mu}_i^t = \mu_i^t \). Then define

\[
\overline{g}_i^t(\overline{m}_i^t, x^t) = \sum_{m_i^t} \mathcal{M}_{i}^t(m_i^t | \kappa^t, \mu_i) g_i^t(m_i^t, x^t)
\]  
(C.13)
and

$$\pi_i^t(\tilde{m}_i^t, x', a^t, y') = \sum_{m_i^t} M_i^t(m_i^t|x^t, \mu_i) \mu_i^t(m_i^t, x^t, a^t, y')$$  \hfill (C.14)

Then

$$v_i^t(g, \mu|\tilde{m}_i^t, x', \Phi_i^{tB}) = v_i^t(\tilde{g}_i^t, g_i^{-t}, g_i^{-t} \tilde{m}_i^t, \mu_i^{-t}, \mu_i^{-t} \tilde{m}_i^t, x', \Phi_i^{tB})$$  \hfill (C.15)

**Proof:** The claim is a direct consequence of Lemma C.1 and of (8) of Definition 4. The details are omitted for the sake of brevity.

### C.1. Proof of the Theorem

Fix any $\delta \in (0, 1)$, any $\tilde{x}$ and any $\tilde{g}$. Consider any SE triple $(g, \mu, \Phi)$ for the dyadic repeated game. Assume that this SE displays Inter-Generational Agreement as in Definition 4.

Now consider the strategy profile $g^*$ for the standard repeated game with common discount $\delta$ and correlation device $\tilde{x}$ that is derived from $(g, \mu)$ as in Definition C.2.

Since $(g, \mu)$ and $g^*$ obviously give rise to the same payoff vector, to prove the claim it is enough to show that $g^* \in G^S(\delta, \tilde{x})$. By way of contradiction, suppose that $g^* \not\in G^S(\delta, \tilde{x})$.

By Remark 1 (One-Shot Deviation Principle) this implies that there exist an $i$, an $h^t$, an $\tilde{h}^t$ and a $\sigma_i$ such that

$$v_i(\sigma_i, g_i^{-t}, g_i^*|h^t, \tilde{h}^t) > v_i(g^*|h^t, \tilde{h}^t)$$  \hfill (C.16)

Given the pair $(h^t, \tilde{h}^t)$, consider the corresponding triple $(y^{-1}, k^t, x^t)$ as in Definition C.2. Given the last two elements of this triple $(k^t, x^t)$, now use (C.1) to find a message $\tilde{m}_i^t$ such that $M_i^t(\tilde{m}_i^t|x^t, \mu_i) > 0$.

Using Lemmas C.1 and C.2 we now conclude that (C.16) implies that

$$v_i^t(\sigma_i, g_i^{-t}, g_i^{-t} \tilde{m}_i^t, \mu_i^{-t}, \mu_i^{-t} \tilde{m}_i^t, x', \Phi_i^{tB}) > v_i^t(\tilde{g}_i^t, g_i^{-t}, g_i^{-t} \tilde{m}_i^t, \mu_i^{-t}, \mu_i^{-t} \tilde{m}_i^t, x', \Phi_i^{tB})$$  \hfill (C.17)

where $\sigma_i$ is the profitable deviation identified in (C.16) and $\tilde{g}_i^t$ and $\tilde{m}_i^t$ are the alternative action-stage and message-stage strategies of Lemma C.3.

However, using (C.15) of Lemma C.3, the inequality in (C.17) clearly implies that

$$v_i^t(\sigma_i, g_i^{-t}, g_i^{-t} \tilde{m}_i^t, \mu_i^{-t}, \mu_i^{-t} \tilde{m}_i^t, x', \Phi_i^{tB}) > v_i^t(g, \mu|\tilde{m}_i^t, x', \Phi_i^{tB})$$  \hfill (C.18)

But since (C.18) contradicts the fact that $(g, \mu, \Phi)$ is an SE of the dyadic repeated game, the proof is now complete.

### References


---

47 Here and throughout the rest of the paper and the technical addendum, we denote by $v_i^t(g, \mu|m_i^t, x', \Phi_i^{tB})$ the continuation payoff to player $(i,t)$ given the profile $(g, \mu)$, after he has received message $m_i^t$ has observed the realization $x'$, and given that his beliefs over the $n-1$-tuple $m_{i-1}$ are $\Phi_i^{tB}$. See also our Point of Notation T.1.1.


