Supplement to "Communication and Learning:" Omitted Proofs

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June 2011

Proof of Lemma A.2: Let $\hat{U}_t(m_t,\omega)$ be the same type of continuation payoff as $U_t(m_{t-1},\omega)$ defined in (A.2), but this time assuming that t *sends* message m_t. Clearly, $\hat{U}_t(m_t,\omega)$ and $U_t(m_{t-1},\omega)$ are related so that

$$U_t(m_{t-1},\omega) = \int\limits_{\gamma_t\in[0,1]} [\Pr(s_t=0|\omega)\hat{U}_t(\sigma_t(m_{t-1},s_t=0,\gamma_t),\omega) + \Pr(s_t=1|\omega)\hat{U}_t(\sigma_t(m_{t-1},s_t=1,\gamma_t),\omega)]\,\mathrm{d}\,\gamma_t$$

Fix two messages m_{t-1} and m'_{t-1} with $x_t(m_{t-1}) > x_t(m'_{t-1})$. Consider any $\tau > 0$ and any sequence $s^{t,t+\tau-1}$. We define the set

$$Z^{t,t+\tau-1}(m_{t-1},m'_{t-1},s^{t,t+\tau-1}) = \Big\{ \gamma^{t,t+\tau-1}\in[0,1]^\tau \text{ such that } \mathrm{a}_{t+\tau'}(m_{t-1},s^{t,t+\tau'-1},\gamma^{t,t+\tau'-1}) > \mathrm{a}_{t+\tau'}(m'_{t-1},s^{t,t+\tau'-1},\gamma^{t,t+\tau'-1}) \ \ \forall\, \tau'=1,...,\tau \Big\}$$

For convenience, the rest of the argument is divided into three separate steps.

Step 1: *Fix an M and σ^* as in Remark A.1. Let $m_{\bar{t}-1}$ and $m'_{\bar{t}-1}$ be two messages such that $x_{\bar{t}}(m_{\bar{t}-1}) > x_{\bar{t}}\left(m'_{\bar{t}-1}\right)$. Then for every $\omega\in\{0,1\}$ it must be that*

$$\begin{aligned} &U_{\bar{t}}(m_{\bar{t}-1},\omega) - U_{\bar{t}}(m'_{\bar{t}-1},\omega) = \\ &\quad (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\sum_{s^{\bar{t},\bar{t}+t-1}\in\{0,1\}^t}\Pr(s^{\bar{t},\bar{t}+t-1}\,|\,\omega) \\ &\quad\int\limits_{Z^{\bar{t},\bar{t}+t-1}(m_{t-1},m'_{t-1},s^{\bar{t},\bar{t}+t-1})} [(\omega-\mathrm{a}_{\bar{t}+t}(m_{t-1},s^{\bar{t},\bar{t}+t-1},\gamma^{\bar{t},\bar{t}+t-1}))^2 - (\omega-\mathrm{a}_{\bar{t}+t}(m_{t-1},' s^{\bar{t},\bar{t}+t-1},\gamma^{\bar{t},\bar{t}+t-1}))^2]\,\mathrm{d}\,\gamma^{\bar{t},\bar{t}+t-1} \end{aligned} \tag{S.1}$$

To keep notation usage down, during the proof of this step we let $m_{\bar{t}-1}=m$ and $m'_{\bar{t}-1}=m'$.

Suppose that for some $t\geq 0$ there is a pair of sequences $s^{\bar{t},\bar{t}+t}=(s_{\bar{t}},\ldots,s_{\bar{t}+t})$ and $\gamma^{\bar{t},\bar{t}+t}=(\gamma_{\bar{t}},\ldots,\gamma_{\bar{t}+t})$ such that: (a) $\mathrm{a}_{\bar{t}+\tau}(m,s_{\bar{t}},\ldots,s_{\bar{t}+\tau-1},\gamma_{\bar{t}},\ldots,\gamma_{\bar{t}+\tau-1}) > \mathrm{a}_{\bar{t}+\tau}(m',s_{\bar{t}},\ldots,s_{\bar{t}+\tau-1},\gamma_{\bar{t}},\ldots,\gamma_{\bar{t}+\tau-1})$ for every $\tau\leq t$, and (b) $\mathrm{a}_{\bar{t}+t+1}(m,s_{\bar{t}},\ldots,s_{\bar{t}+t},\gamma_{\bar{t}},\ldots,\gamma_{\bar{t}+t}) \leq \mathrm{a}_{\bar{t}+t+1}(m',s_{\bar{t}},\ldots,s_{\bar{t}+t},\gamma_{\bar{t}},\ldots,\gamma_{\bar{t}+t})$. (If such a pair of sequences cannot be found then there is nothing to prove.) We need to consider two, mutually exclusive and exhaustive, cases.

Case 1: After the sequences $s^{\bar{t},\bar{t}+t}$ and $\gamma^{\bar{t},\bar{t}+t}$, player $\bar{t}+t$ sends the same message (say $\tilde{m}$) both when player $\bar{t}$ behaves according to m and when player $\bar{t}$ behaves according to m'.

Clearly, after these sequences, the messages m and m' will induce the same action in every period $\bar{t}+t+\tau$ for any $\tau\geq 1$. Hence in Case 1 there is nothing further to prove.

Case 2: After the sequences $s^{\bar{t},\bar{t}+t}$ and $\gamma^{\bar{t},\bar{t}+t}$, player $\bar{t}+t$ sends message $\tilde{m}$ if player $\bar{t}$ behaves according to m and message $\tilde{m}'\neq\tilde{m}$ if player $\bar{t}$ behaves according to m'.

Let $\tilde{y}$ (resp. $\tilde{y}'$) denote the beginning-of-period belief of player $\bar{t}+t+1$ when he receives $\tilde{m}$ (resp. $\tilde{m}'$). Of course, $\tilde{y}' \geq \tilde{y}$ since

$$\tilde{y}-\beta \;=\; \mathrm{a}_{\bar{t}+t+1}(m, s^{\bar{t},\bar{t}+t}, \gamma^{\bar{t},\bar{t}+t}) \;\leq\; \mathrm{a}_{\bar{t}+t+1}(m', s^{\bar{t},\bar{t}+t}, \gamma^{\bar{t},\bar{t}+t}) \;=\; \tilde{y}'-\beta$$

Let $\tilde{x}$ denote the end-of-period belief of player $\bar{t}+t$ when player $\bar{t}$ behaves according to m and the sequences of realized signals and randomization devices are $s^{\bar{t},\bar{t}+t}$ and $\gamma^{\bar{t},\bar{t}+t}$, respectively. Finally, let $\tilde{x}'$ denote the end-of-period belief of player $\bar{t}+t$ when player $\bar{t}$ behaves according to m' and the sequences are $s^{\bar{t},\bar{t}+t}$ and $\gamma^{\bar{t},\bar{t}+t}$. Notice that $\tilde{x} > \tilde{x}'$ since by assumption $\mathrm{a}_{\bar{t}+t}(m, s_{\bar{t}+1}, \ldots, s_{\bar{t}+t-1}, \gamma_{\bar{t}+1}, \ldots, \gamma_{\bar{t}+t-1}) > \mathrm{a}_{\bar{t}+t}(m', s_{\bar{t}+1}, \ldots, s_{\bar{t}+t-1}, \gamma_{\bar{t}+1}, \ldots, \gamma_{\bar{t}+t-1})$.

Let $\mathcal{T}_{\bar{t}+t} \subset M_{\bar{t}+t-1} \times \{0,1\} \times [0,1]$ denote the set of types (a type consisting of the message received and the two random variables observed) of player $\bar{t}+t$ who send message $\tilde{m}$. Also, let $\tilde{X}^E_{\bar{t}+t}$ denote the set of corresponding beliefs, so that $\tilde{X}^E_{\bar{t}+t} = \bigcup_{(m_{\bar{t}+t-1}, s_{\bar{t}+t}, \gamma_{\bar{t}+t}) \in \mathcal{T}_{\bar{t}+t}} x^E_{\bar{t}+t}(m_{\bar{t}+t-1}, s_{\bar{t}+t}, \gamma_{\bar{t}+t})$. Define $\mathcal{T}'_{\bar{t}+t}$ and $\tilde{X}^{E'}_{\bar{t}+t}$ in a similar way (replace $\tilde{m}$ with $\tilde{m}'$).

Since each player uses Bayes' rule to compute his beginning-of-period beliefs, $\tilde{y}$ belongs to the convex hull of $\tilde{X}^E_{\bar{t}+t}$ and $\tilde{y}'$ belongs to the convex hull of $\tilde{X}^{E'}_{\bar{t}+t}$. Notice that $\tilde{x} \in \tilde{X}^E_{\bar{t}+t}$ and $\tilde{x}' \in \tilde{X}^{E'}_{\bar{t}+t}$, and recall that $\tilde{x} > \tilde{x}'$. This, together with $\tilde{y}' \geq \tilde{y}$, imply that one of the following two mutually exclusive subcases, (a) and (b), must be true.

(a) We can find three types $(m^1_{\bar{t}+t-1}, s^1_{\bar{t}+t}, \gamma^1_{\bar{t}+t})$, $(m^2_{\bar{t}+t-1}, s^2_{\bar{t}+t}, \gamma^2_{\bar{t}+t})$ and $(m^3_{\bar{t}+t-1}, s^3_{\bar{t}+t}, \gamma^3_{\bar{t}+t})$ such that

$$x^{(1)} = x^E_{\bar{t}+t}(m^1_{\bar{t}+t-1}, s^1_{\bar{t}+t}, \gamma^1_{\bar{t}+t}) < x^{(2)} = x^E_{\bar{t}+t}(m^2_{\bar{t}+t-1}, s^2_{\bar{t}+t}, \gamma^2_{\bar{t}+t}) < x^{(3)} = x^E_{\bar{t}+t}(m^3_{\bar{t}+t-1}, s^3_{\bar{t}+t}, \gamma^3_{\bar{t}+t}) \qquad \text{(S.2)}$$

and, the two extreme types $(m^1_{\bar{t}+t-1}, s^1_{\bar{t}+t}, \gamma^1_{\bar{t}+t})$ and $(m^3_{\bar{t}+t-1}, s^3_{\bar{t}+t}, \gamma^3_{\bar{t}+t})$ send the same message, equal to either $\tilde{m}$ or $\tilde{m}'$, while the intermediate type sends the other. Suppose that the extreme types send $\tilde{m}$ and $(m^2_{\bar{t}+t-1}, s^2_{\bar{t}+t}, \gamma^2_{\bar{t}+t})$ sends $\tilde{m}'$ (mutatis mutandis, the reverse case is identical). Since in equilibrium no type can have a profitable deviation when selecting which message to send, it must be that for $k = 1, 3$

$$x^{(k)}\hat{U}_{\bar{t}+t}(\tilde{m}, 1) + (1-x^{(k)})\hat{U}_{\bar{t}+t}(\tilde{m}, 0) \geq x^{(k)}\hat{U}_{\bar{t}+t}(\tilde{m}', 1) + (1-x^{(k)})\hat{U}_{\bar{t}+t}(\tilde{m}', 0) \qquad \text{(S.3)}$$

and

$$x^{(2)}\hat{U}_{\bar{t}+t}(\tilde{m}, 1) + (1-x^{(2)})\hat{U}_{\bar{t}+t}(\tilde{m}, 0) \leq x^{(2)}\hat{U}_{\bar{t}+t}(\tilde{m}', 1) + (1-x^{(2)})\hat{U}_{\bar{t}+t}(\tilde{m}', 0) \qquad \text{(S.4)}$$

Recall that by (S.2) we have $x^{(1)} < x^{(2)} < x^{(3)}$. Thus inequalities (S.3) and (S.4) can only be satisfied if $\hat{U}_{\bar{t}+t}(\tilde{m}, \omega) = \hat{U}_{\bar{t}+t}(\tilde{m}', \omega)$, $\forall\, \omega \in \{0,1\}$. Therefore, after the sequences $s^{\bar{t},\bar{t}+t}$ and $\gamma^{\bar{t},\bar{t}+t}$, player $\bar{t}$ receives the same expected continuation payoff regardless of ω and regardless of whether he behaves according to m or behaves according to m'. This concludes the argument in subcase (a).

(b) There are four types, $(m^1_{\bar{t}+t-1}, s^1_{\bar{t}+t}, \gamma^1_{\bar{t}+t})$, $(m^2_{\bar{t}+t-1}, s^2_{\bar{t}+t}, \gamma^2_{\bar{t}+t})$, $(m^3_{\bar{t}+t-1}, s^3_{\bar{t}+t}, \gamma^3_{\bar{t}+t})$ and $(m^4_{\bar{t}+t-1}, s^4_{\bar{t}+t}, \gamma^4_{\bar{t}+t})$ such that

$$x^E_{\bar{t}+t}(m^1_{\bar{t}+t-1}, s^1_{\bar{t}+t}, \gamma^1_{\bar{t}+t}) = x^E_{\bar{t}+t}(m^2_{\bar{t}+t-1}, s^2_{\bar{t}+t}, \gamma^2_{\bar{t}+t}) = \tilde{x}$$

and

$$x^E_{\bar{t}+t}(m^3_{\bar{t}+t-1}, s^3_{\bar{t}+t}, \gamma^3_{\bar{t}+t}) = x^E_{\bar{t}+t}(m^4_{\bar{t}+t-1}, s^4_{\bar{t}+t}, \gamma^3_{\bar{t}+t}) = \tilde{x}'$$

and the two types $(m^1_{\bar{t}+t-1}, s^1_{\bar{t}+t}, \gamma^1_{\bar{t}+t})$ and $(m^3_{\bar{t}+t-1}, s^3_{\bar{t}+t}, \gamma^3_{\bar{t}+t})$ send message $\tilde{m}$ while the two types $(m^2_{\bar{t}+t-1},$

$s^2_{\bar{t}+t}$, $\gamma^2_{\bar{t}+t}$) and ($m^4_{\bar{t}+t-1}, s^4_{\bar{t}+t}, \gamma^3_{\bar{t}+t}$) send message $\tilde{m}'$.

Again, in equilibrium no type can have a profitable deviation when selecting which message to send. Recalling that $\tilde{x} > \tilde{x}'$, it is then immediate to see that this implies that $\hat{U}_{\bar{t}+t}(\tilde{m}, \omega) = \hat{U}_{\bar{t}+t}(\tilde{m}', \omega)$, $\forall\ \omega \in \{0, 1\}$. Therefore, after the sequence $s^{\bar{t},\bar{t}+t}$ player $\bar{t}$ receives the same expected continuation payoff regardless of ω and regardless of whether he behaves according to m or behaves according to m'. This closes the argument in case (b) and hence concludes the proof of Step 1.

Step 2: *Fix an M and $\boldsymbol{\sigma}^*$ as in Remark A.1. For any $\eta > 0$ there exists an $\varepsilon > 0$ such that the following is true for every t. Suppose that m_{t-1} and m'_{t-1} are two messages in M_{t-1} with $\min\{x_t(m_{t-1}), x_t(m'_{t-1})\} > 1-\varepsilon$ and $x_t(m_{t-1}) \neq x_t(m'_{t-1})$. Then*

$$\left|U_t\left(m_{t-1}, 1\right) - U_t\left(m'_{t-1}, 1\right)\right| < \eta$$

Fix η and choose $\varepsilon > 0$ such that $\varepsilon(1-\beta)^2/(1-\varepsilon) < \eta$. By hypothesis, we can find two messages m_{t-1} and m'_{t-1} in M_{t-1} with $x_t(m_{t-1}) > 1-\varepsilon$ and $x_t(m'_{t-1}) > 1-\varepsilon$. To keep notation usage down, during the proof of this step we let $m_{t-1} = m$, $m'_{t-1} = m'$, $x_t(m_{t-1}) = x$ and $x_t(m'_{t-1}) = x'$. Without loss of generality, assume $x > x'$.

Since player t must have no incentives to deviate from his equilibrium strategy after observing s_t and γ_t, taking averages, the following two inequalities must be satisfied

$$x\, U_t(m, 1) + (1-x)U_t(m, 0) \geq x\, U_t(m', 1) + (1-x)U_t(m', 0)$$

and

$$x' U_t(m, 1) + (1-x')U_t(m, 0) \leq x' U_t(m', 1) + (1-x')U_t(m', 0)$$

Therefore, for some $\bar{x} \in [x', x]$ it must be that

$$\bar{x}\, U_t(m, 1) + (1-\bar{x})U_t(m, 0) = \bar{x}\, U_t(m', 1) + (1-\bar{x})U_t(m', 0)$$

Hence, using the fact that $\bar{x} > 1-\varepsilon$, we conclude that

$$|U_t(m, 1) - U_t(m', 1)| = \frac{(1-\bar{x})}{\bar{x}}|U_t(m', 0) - U_t(m, 0)| < \frac{\varepsilon}{1-\varepsilon}|U_t(m', 0) - U_t(m, 0)| \tag{S.5}$$

Notice now that, using (3), it is trivial that no player will ever choose an action outside $[-\beta, 1-\beta]$. From (A.2) it then follows directly that the continuation payoff $U_t(\cdot, 0)$ is bounded above by 0 and below by $-(1-\beta)^2$. It is then obvious that $|U_t(m', 0) - U_t(m, 0)| < (1-\beta)^2$. Hence, because of the way ε was chosen, (S.5) is enough to prove the claim in Step 2.

Step 3: *Fix an M and $\boldsymbol{\sigma}^*$ as in Remark A.1. For any $\eta > 0$ there exists an $\varepsilon > 0$ such that the following is true for every t. Suppose that m_{t-1} and m'_{t-1} are two messages in M_{t-1} with $\min\{x_t(m_{t-1}), x_t(m'_{t-1})\} > 1-\varepsilon$ and $x_t(m_{t-1}) \neq x_t(m'_{t-1})$. Then*

$$|U_t(m_{t-1}, 0) - U_t(m'_{t-1}, 0)| < \eta$$

We proceed by contradiction. Then by hypothesis there must exist an η such that for every ε we can find messages m_{t-1} and m'_{t-1} (for some t) such that $\min\{x_t(m_{t-1}), x_t(m'_{t-1})\} > 1-\varepsilon$, $x_t(m_{t-1}) \neq x_t(m'_{t-1})$ and

$$|U_t(m_{t-1}, 0) - U_t(m'_{t-1}, 0)| > \eta \tag{S.6}$$

To keep notation usage down, during the proof of this step we let $m_{t-1} = m$, $m'_{t-1} = m'$, $x_t(m_{t-1}) = x$ and $x_t(m'_{t-1}) = x'$. Without loss of generality, assume $x > x'$. From Step 1 we know that

$$U_t(m,0) - U_t(m',0) =$$

$$(1-\delta)\sum_{\tau=1}^{\infty}\delta^{\tau-1}\sum_{s^{t,t+\tau-1}\in\{0,1\}^t}\Pr(s^{t,t+\tau-1}\,|\,\omega)$$
$$\int_{Z^{t,t+\tau-1}(m,m',s^{t,t+\tau-1})}\left[\mathrm{a}_{t+\tau}(m',s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2 - \mathrm{a}_{t+\tau}(m,s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2\right]\ \mathrm{d}\gamma^{t,t+\tau-1} \tag{S.7}$$

Just as in the proof of Step 2, the difference between continuation payoffs (appropriately normalized) conditional on $\omega = 0$ is bounded above by $(1-\beta)^2$. Therefore, from (S.7) we conclude that for any $T \geq 1$ it must be that

$$|U_t(m,0) - U_t(m',0)| \leq$$

$$(1-\delta)\sum_{\tau=1}^{T}\delta^{\tau-1}\sum_{s^{t,t+\tau-1}\in\{0,1t\}^t}\Pr(s^{t,t+\tau-1}|\omega)$$
$$\int_{Z^{t,t+\tau-1}(m,m',s^{t,t+\tau-1})}\left|\mathrm{a}_{t+\tau}(m',s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2 - \mathrm{a}_{t+\tau}(m,s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2\right|\ \mathrm{d}\,\gamma^{t,t+\tau-1}\ +\ \delta^T(1-\beta)^2$$

Since T can always be chosen so that $\delta^T(1-\beta)^2 < \eta/2$, we conclude that there exists a T such that

$$|U_t(m,0) - U_t(m',0)| <$$

$$(1-\delta)\sum_{\tau=1}^{T}\delta^{\tau-1}\sum_{s^{t,t+\tau-1}\in\{0,1\}^t}\Pr(s^{t,t+\tau-1}\,|\,\omega)$$
$$\int_{Z^{t,t+\tau-1}(m,m',s^{t,t+\tau-1})}\left|\mathrm{a}_{t+\tau}(m',s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2 - \mathrm{a}_{t+\tau}(m,s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2\right|\ \mathrm{d}\gamma^{t,t+\tau-1}\ +\ \frac{\eta}{2} \tag{S.8}$$

Inequalities (S.6) and (S.8) directly imply that

$$(1-\delta)\sum_{\tau=1}^{T}\delta^{\tau-1}\sum_{s^{t,t+\tau-1}\in\{0,1\}^t}\Pr(s^{t,t+\tau-1}\,|\,\omega)$$
$$\int_{Z^{t,t+\tau-1}(m,m',s^{t,t+\tau-1})}\left|\mathrm{a}_{t+\tau}(m',s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2 - \mathrm{a}_{t+\tau}(m,s^{t,t+\tau-1},\gamma^{t,t+\tau-1})^2\right|\ \mathrm{d}\,\gamma^{t,t+\tau-1}\ >\ \frac{\eta}{2} \tag{S.9}$$

However, inequality (S.9) implies that there exist a $\bar{\tau} = 1,\ldots,T$ and some sequence of signals $s^{t,t+\bar{\tau}-1}$

such that

$$\int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} |\, \mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})^2 - \mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})^2 \,|\, \mathrm{d}\,\gamma^{t,t+\bar\tau-1} \;>\; \frac{\eta}{2(1-\delta^T)}$$

By definition, for any $\gamma^{t,t+\bar\tau-1} \in Z^{t,t+\bar\tau-1}(m, m', s^{t,t+\bar\tau-1})$, we have that $\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) > \mathrm{a}_{t+\bar\tau}(m',\, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})$. Since all actions are weakly smaller than $1-\beta$, we have

$$\frac{\eta}{2(1-\delta^T)} < \int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} |\mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})^2 - \mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})^2| \mathrm{d}\,\gamma^{t,t+\bar\tau-1} =$$

$$\int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} |\mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) + \mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})|$$

$$[\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) - \mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]\, \mathrm{d}\gamma^{t,t+\bar\tau-1} <$$

$$\int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} [2(1-\beta)][\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) - \mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]\, \mathrm{d}\,\gamma^{t,t+\bar\tau-1}$$

which implies

$$\int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} [\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) - \mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]\, \mathrm{d}\,\gamma^{t,t+\bar\tau-1} > \frac{\eta}{4(1-\delta^T)(1-\beta)}$$

Consider now the payoffs in state $\omega = 1$ We have

$$\int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} |[1-\mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]^2 - [1-\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]^2|\;\; d\,\gamma^{t,t+\bar\tau-1} =$$

$$\int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} [1-\mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]^2 - [1-\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]^2\;\; \mathrm{d}\,\gamma^{t,t+\bar\tau-1} =$$

$$\int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} [\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) - \mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]$$

$$[2-\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) - \mathrm{a}_{t+\bar\tau}(m', s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]\, \mathrm{d}\;\gamma^{t,t+\bar\tau-1} \geq$$

$$2\beta \int_{Z^{t,t+\bar\tau-1}(m,m',s^{t,t+\bar\tau-1})} [\mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1}) - \mathrm{a}_{t+\bar\tau}(m, s^{t,t+\bar\tau-1}, \gamma^{t,t+\bar\tau-1})]\;\; \mathrm{d}\,\gamma^{t,t+\bar\tau-1} > \frac{2\beta\eta}{4(1-\delta^T)(1-\beta)}$$

The strict inequality between the first and the last expression above, together with Step 1, implies that the difference $U_t(m,1) - U_t(m',1)$ is bounded below by

$$\frac{\beta\eta(1-\delta)\delta^T(1-p)^T}{2(1-\delta^T)(1-\beta)}$$

which contradicts Step 2. Hence the proof of Lemma A.2 is now complete.