# ECON 459 — Game Theory

# Lecture Notes

# Nash Bargaining


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These notes have been used and commented on before. If you can still spot any errors or have any suggestions for improvement, please let me know.

# 1 Bargaining Theory – Preamble

• Economic transactions generate **surplus**.

• Bargaining Theory addresses the question of how the surplus will be **divided** among the participants.

• In virtually all cases it is restricted to the case of **two agents**.

• Suppose one agent has an object to sell, and another h as the opportunity to buy it.

• Suppose the seller values the object less than the potential buyer. The difference between the valuation of the seller and that of the buyer is the potential surplus.

• Since there are mutual gains from trade, it is reasonable to suppose that the transaction will take place.

• The objective of Bargaining Theory is to say something (if possible to pin down completely) about the price at which the transaction will take place.

• Another way to put it is that Bargaining Theory addresses the question: **How will the surplus be split between the two?**

• Once we frame the question in terms of surplus, we have a framework that applies much more generally than the single-object transaction. The two agents could, for instance, be bargaining over the terms of a complex contractual arrangement.

• Lastly, notice that price-taking models are no use whatsoever in answering the question at hand.

• Bargaining Theory studies situation in which no-one can reasonably be assumed to be taking the price as given.

• The question in Bargaining Theory is precisely: **Where do the prices come from** in a situation of **bilateral monopoly?**

• Two types of Bargaining Theory have been developed.

• The first one we look at is also the first one to have been developed historically. It is known as "Nash Bargaining."

• This way of proceeding amounts to stating a number of "desirable properties" that the "solution" to a bargaining problem should have, and then showing that the properties in fact do pin down the solution uniquely.

• Nash Bargaining belongs to a body of work called "Cooperative Game Theory."

• The second approach to Bargaining Theory belongs firmly to Non-Cooperative Game Theory (the stuff we have been concerned with so far).

• In this approach, we write down an extensive form game that we think captures the essence of how the bargaining will in fact proceed. We then apply the tools of Non-Cooperative Game Theory to solve the extensive form game and – hopefully – find a unique prediction about the outcome.

• One of the surprises we find along the way is that the *solutions* to the bargaining problem we find using these two – seemingly unrelated – approaches are in fact closely related. Under appropriate circumstances the answer is the same.

# 2 Nash Bargaining

## 2.1 Ingredients: The Set-Up

• There are two participants, $i = 1, 2$.

• A **bargaining problem** is a pair $\mathcal{B} = (\mathcal{U}, d)$.

• In $(\mathcal{U}, d)$, $\mathcal{U}$ is a **set** of possible **agreements** in terms of **utilities** that they yield to 1 and 2. An element of $\mathcal{U}$ is a *pair* $u = (u_1, u_2) \in \mathcal{U}$.

• The interpretation is that if agreement $u = (u_1, u_2) \in \mathcal{U}$ is reached, then 1 gets utility $u_1$ and 2 gets utility $u_2$.

• Throughout, we are going to take $\mathcal{U}$ to be a *convex* set. (More on this later.)

• In $(\mathcal{U}, d)$, $d$ is a *pair* $(d_1, d_2)$ called the **disagreement** point.

• The interpretation is that if *no agreement* is reached then 1 gets utility $d_1$ and 2 gets utility $d_2$.

## 2.2 Ingredients: The Solution Function

• What sort of "solution" are we after?

• We seek a "solution function" $f$ of the following kind.

• The function $f$ takes as input any bargaining problem $(\mathcal{U}, d)$, and returns a pair of utilities $u = (u_1, u_2) \in \mathcal{U}$.

• So, we write $u = f(\mathcal{B})$ or alternatively $u = f(\mathcal{U}, d)$. When we need to refer to the "components" of $f$ we write $u_1 = f_1(\mathcal{B})$ and $u_2 = f_2(\mathcal{B})$ or alternatively $u_1 = f_1(\mathcal{U}, d)$ and $u_2 = f_2(\mathcal{U}, d)$.

• The **interpretation** is that, given any bargaining problem $\mathcal{B} = (\mathcal{U}, d)$, the solution function tells us that the agreement $u = f(\mathcal{U}, d)$ will be reached.

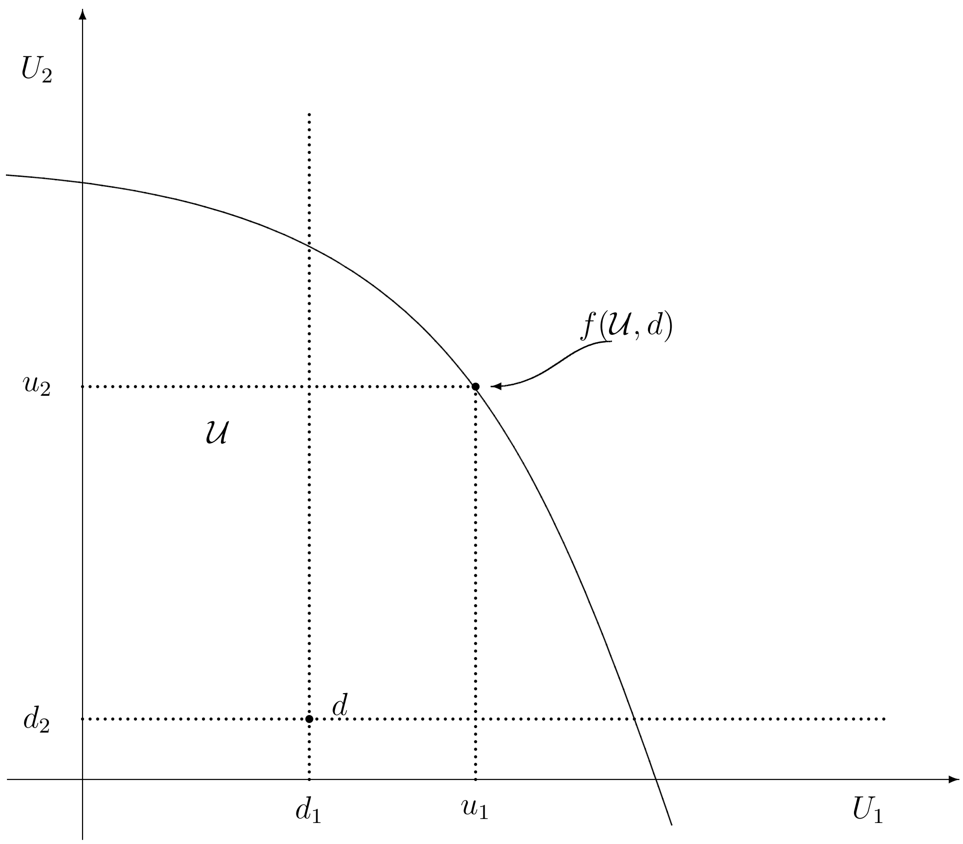



Figure 1: Bargaining Problem and Solution Function

## 2.3 A Canonical Interpretation

- There are a buyer and a seller.
- The seller has an object potentially for sale that costs him $c$.
- The buyer places a value of $v$ on the object.
- To make this interesting we take it to be the case that $v > c$.
- At what price $p$ will the object be sold?
- If it is sold at $p$, then the seller's utility is $U_S(p - c)$ and the buyer's utility is $U_B(v - p)$.
- If no transaction takes place, then both buyer and seller get a utility of 0.
- This situation gives rise to a bargaining problem of the type we described in the abstract before.
- Take $\mathcal{U}$ to be the set of utility pairs that can be obtained as $p$ varies between $c$ and $v$. (So, notice if both $U_B$ and $U_S$ are concave, we get a convex $\mathcal{U}$.)
- Take $d$ to be $(0, 0)$.
- A Solution function would tell us what utility the buyer and the seller get, and hence the *price* at which the object is traded.

## 2.4 Question

• Suppose we list a bunch of "appealing" properties that $f$ should satisfy.

• Can we "pin down" $f$ completely?

• Answer: YES.

## 2.5 The Axioms

### 2.5.1 Pareto (PAR)

• This axiom imposes that the point that $f$ picks out must be Pareto-efficient.

• Formally, $f(\mathcal{U}, d)$ has the property that there does *not* exist a point $(u_1, u_2) \in \mathcal{U}$ such that

$$u_1 \geq f_1(\mathcal{U}, d),\ u_2 \geq f_2(\mathcal{U}, d),\ (u_1, u_2) \neq f(\mathcal{U}, d) \quad (2.1)$$

• In other words there are no points in $\mathcal{U}$ that are "North-East" of $f(\mathcal{U}, d)$. (See Figure 2.)

### 2.5.2 Symmetry (SYM)

• This axiom imposes that *if* everything is symmetric in $\mathcal{B} = (\mathcal{U}, d)$, then the solution function should pick out a symmetric solution.

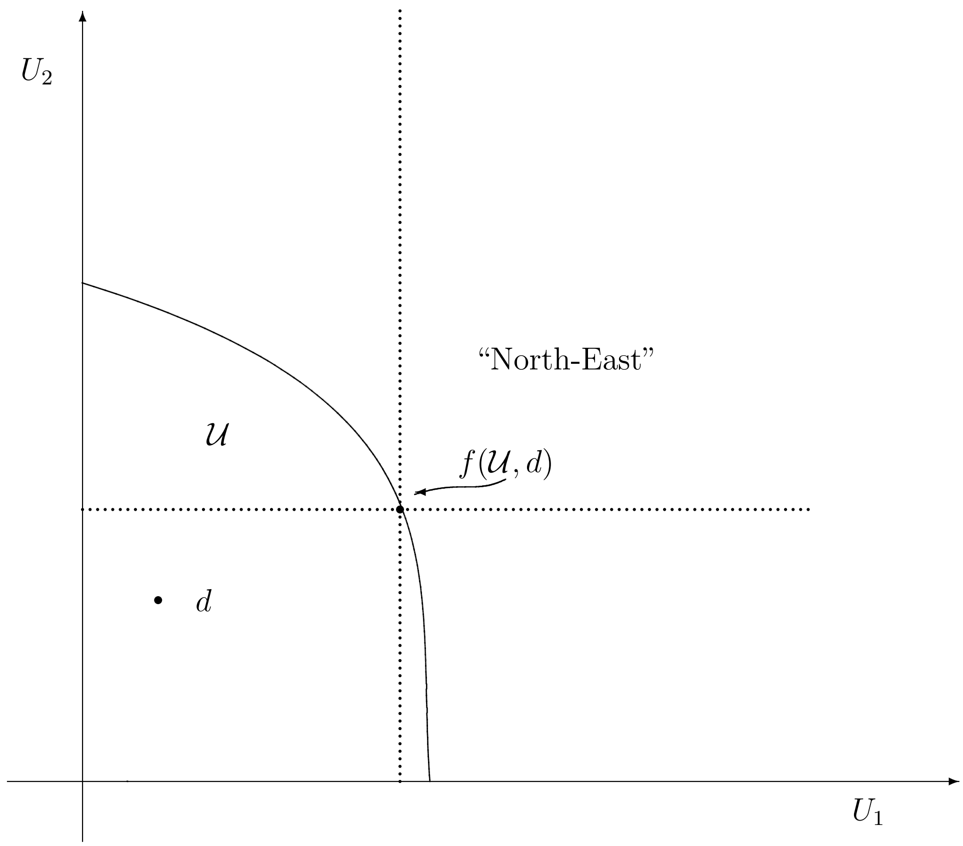



Figure 2: Pareto

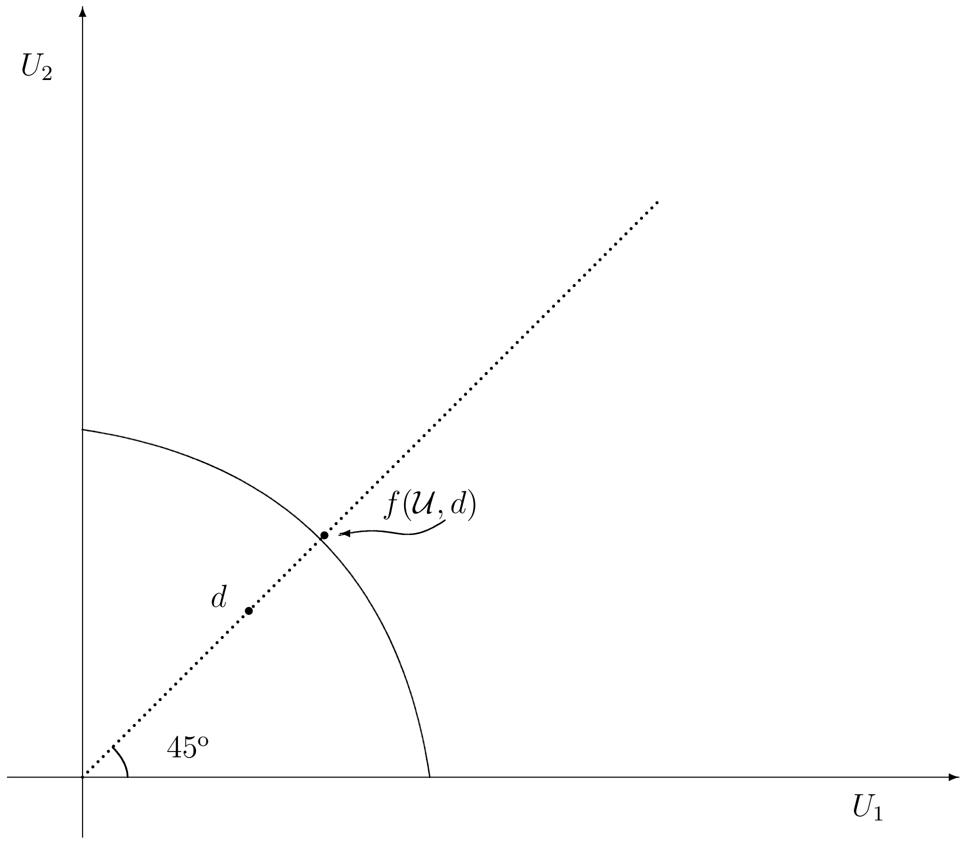



Figure 3: Symmetry

• Formally, suppose that $(\mathcal{U}, d)$ is such that $\mathcal{U}$ is symmetric around the $45^o$ line and $d_1 = d_2$, *then*

$$f_1(\mathcal{U}, d) = f_2(\mathcal{U}, d) \tag{2.2}$$

• In other words, when everything in $\mathcal{B}$ is symmetric, the point $f(\mathcal{U}, d)$ is itself on the $45^o$ line. (See Figure 3.)

### 2.5.3 Independence of Utility Origins (IUO)

• As always, the origin of any utility function can be changed arbitrarily.

• This axiom imposes that if we add or subtract a constant from the utility of either 1 or 2 or both, the solution should not be affected.

• Suppose we have two bargaining problems $\mathcal{B} = (\mathcal{U}, d)$ and $\mathcal{B}' = (\mathcal{U}', d')$ with the following property.

• For some **vector** $b = (b_1, b_2)$

$$d' = d + b \tag{2.3}$$

and

$$\mathcal{U}' = \mathcal{U} + b \tag{2.4}$$

where (2.3) means that a point $(u'_1, u'_2)$ is in $\mathcal{U}'$ if and only if for some $(u_1, u_2) \in \mathcal{U}$ we have that

$$(u'_1, u'_2) = (u_1, u_2) + (b_1, b_2) \tag{2.5}$$

• Then the IUO axiom imposes that (See Figure 4.)

$$f(\mathcal{U}', d') = f(\mathcal{U}, d) + b \tag{2.6}$$

• Notice, using IUO there is **no loss of generality** in considering **only** bargaining problems with $d = (0, 0)$.

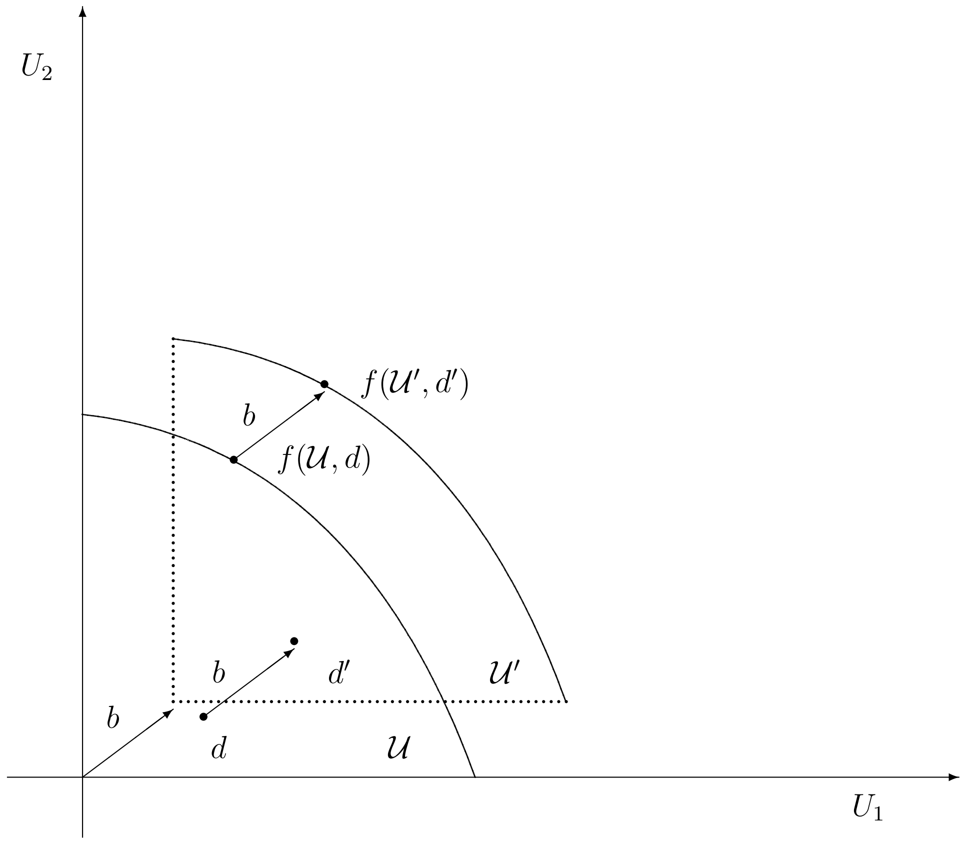



Figure 4: IUO

- To see this just set $b = -d$ and use IUO.
- From here on, whenever it is convenient we will assume $d = (0, 0)$.

### 2.5.4 Independence of Utility Units (IUU)

• As always, the units of any utility function can be changed arbitrarily.

• This axiom imposes that if we multiply by a positive constant the utility of either 1 or 2 or both, the solution should not be affected.

• This axiom is stated more conveniently for problems with $d = (0, 0)$. This is how we proceed (but see IUO).

• Suppose we have two bargaining problems $\mathcal{B} = (\mathcal{U}, d)$ and $\mathcal{B}' = (\mathcal{U}', d)$ with $d = (0, 0)$ and the following property.

$$U_1' = k_1 U_1 \text{ and } U_2' = k_2 U_2 \tag{2.7}$$

where (2.7) means that a point $(u_1', u_2')$ is in $\mathcal{U}'$ if and only if for some $(u_1, u_2) \in \mathcal{U}$ we have that

$$u_1' = k_1 u_1 \text{ and } u_2' = k_2 u_2 \tag{2.8}$$

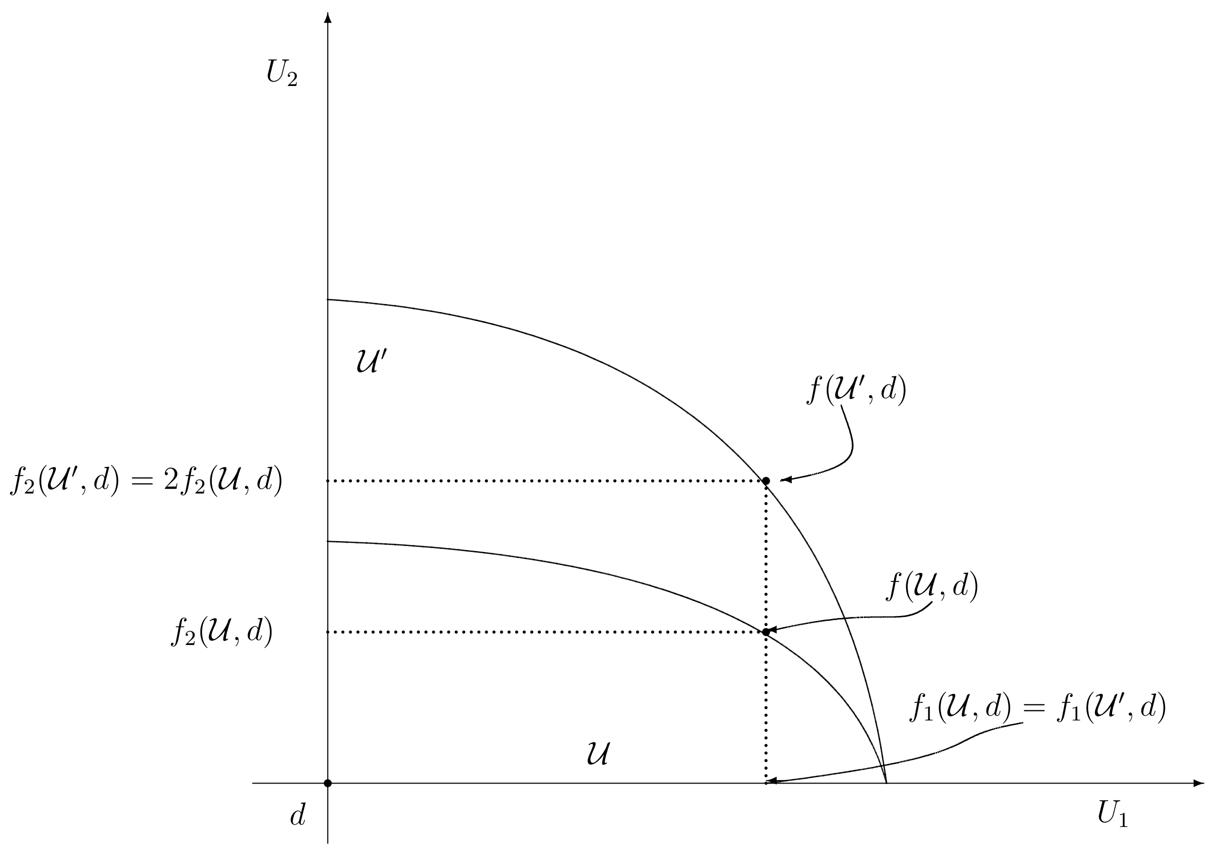
• Then the IUU axiom imposes that

$$f_1(\mathcal{U}', d) = k_1 f_1(\mathcal{U}, d) \tag{2.9}$$

and

$$f_2(\mathcal{U}', d) = k_2 f_2(\mathcal{U}, d) \tag{2.10}$$

• In Figure 5 we depict a change for $U_2$ only with $k_2 = 2$.



$U_2$
$\mathcal{U}'$
$f(\mathcal{U}', d)$
$f_2(\mathcal{U}', d) = 2f_2(\mathcal{U}, d)$
$f(\mathcal{U}, d)$
$f_2(\mathcal{U}, d)$
$f_1(\mathcal{U}, d) = f_1(\mathcal{U}', d)$
$\mathcal{U}$
$d$
$U_1$


Figure 5: IUU

### 2.5.5 Independence of Irrelevant Alternatives (IIA)

• The last axiom has the same flavor of a property widely used in "social choice" problems.

• Intuitively, the property embodied by IIA is simple to state.

• Suppose that we render infeasible some agreements. Suppose also that none the agreements we render infeasible is the chosen one. Then the chosen agreement should not change.

• Eliminating some "Irrelevant Alternatives" should not change the point picked out by the solution function.

• Formally, suppose we have two bargaining problems $\mathcal{B} = (\mathcal{U}, d)$ and $\mathcal{B}' = (\mathcal{U}', d')$ with $d = d'$ and $\mathcal{U}' \subset \mathcal{U}$. Suppose also that $f(\mathcal{U}, d) \in \mathcal{U}'$.

• Then the IIA axiom imposes that (See Figure 6.)

$$f(\mathcal{U}, d) = f(\mathcal{U}', d') \tag{2.11}$$

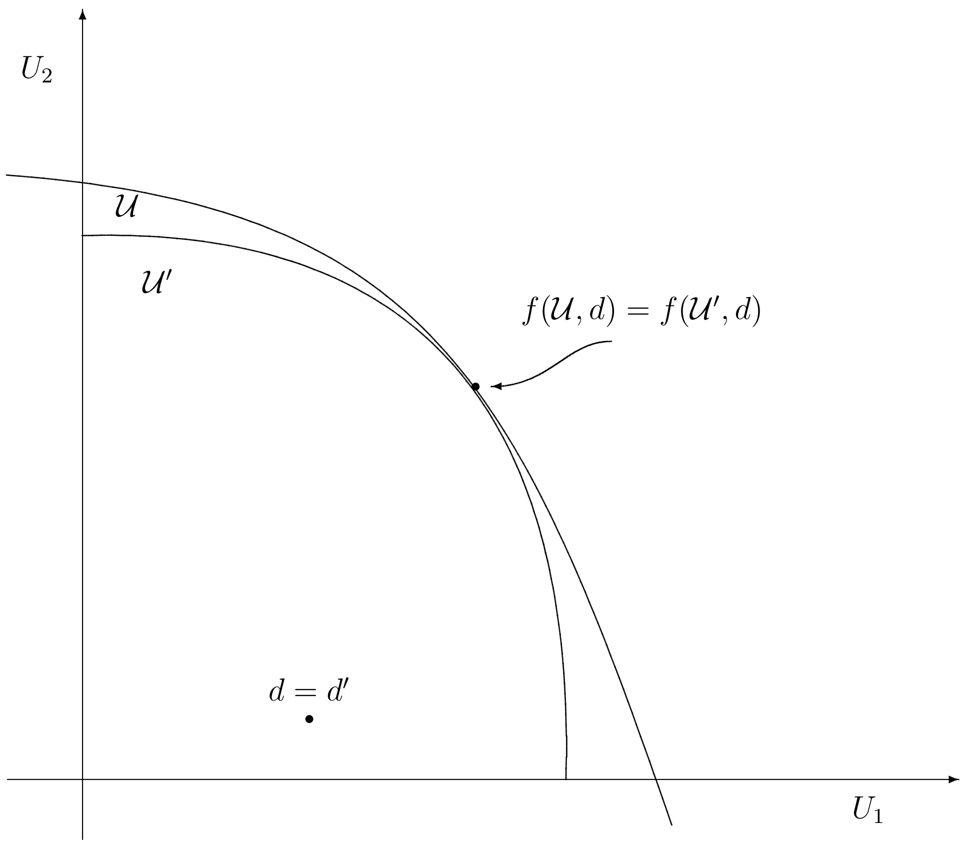



Figure 6: IIA

## 2.6 Pinning Down $f$

### 2.6.1 Symmetric Problems

• Our first observation is that because of PAR and SYM, we know everything about $f(\mathcal{B}) = f(\mathcal{U}, d)$ if $\mathcal{B}$ ia symmetric problem ($d_1 = d_2$ and $\mathcal{U}$ symmetric around the $45^o$ line).

• In this case $f(\mathcal{U}, d)$ must be on the upper boundary of $\mathcal{U}$ on the $45^o$ line.

• These two requirements together pin down $f$ uniquely, just as in Figure 3.

### 2.6.2 Linear Frontier Problems

• Our second observation concerns any $\mathcal{B}$ with a *linear frontier*.

• We say that a bargaining problem $\mathcal{B}$ has a linear frontier if and only if the upper boundary of $\mathcal{U}$ is a (downward sloping) straight line.

• To argue what $f$ has to be like in the case of a linear frontier $\mathcal{B}$ we proceed in two steps.

• We do this setting $d_1 = d_2 = 0$ for simplicity. (We know this can always be done.)

• From our observation about symmetric problems above, we know that if $\mathcal{B}$ has a linear frontier and is also symmetric, then

$$f_1(\mathcal{U}, d) = f_2(\mathcal{U}, d) \tag{2.12}$$

• Now consider a symmetric $\mathcal{B}$ with a linear frontier and notice that in this case the slope of the upper boundary of $\mathcal{U}$ must be $-1$.

• It follows that the segment (up) on the left of $f(\mathcal{U}, d)$ on the frontier of $\mathcal{U}$ must be of the **same length** as the segment (down) on the right of $f(\mathcal{U}, d)$ on the frontier of $\mathcal{U}$. (See Figure 7 – the two segments $A$ and $B$ have the same length.)

• Now consider a *new* linear frontier bargaining problem $\mathcal{B}' = (\mathcal{U}', d')$ with $d' = d$, and $\mathcal{U}'$ with a boundary that cuts the horizontal axis in the same place as $\mathcal{U}$, but cuts the *vertical* axis twice as high as $\mathcal{U}$. (See Figure 7.)

• Notice $\mathcal{B}'$ is **not** a **symmetric** problem.

• However, IUU tells us what the solution $f(\mathcal{U}', d'$ should be.

• Since we have kept $U_1$ the same and we have multiplied $U_2$ by 2 (see Figure 7), we should have

$$f_1(\mathcal{U}', d') = f_1(\mathcal{U}, d) \tag{2.13}$$

and

$$f_2(\mathcal{U}', d') = 2\, f_2(\mathcal{U}, d) \tag{2.14}$$

• But this (see Figure 7) tells us something **general** about bargaining problems with a linear frontier — symmetric or not.

• Geometrically, in Figure 7, it is clear that the triangle above the dotted line has the same shape and dimensions as the triangle to the right of the dotted line.

• Hence, it follows that in Figure 7 the segment (up) on the left of $f(\mathcal{U}', d')$ on the frontier of $\mathcal{U}'$ must be of the **same length** as the segment (down) on the right of $f(\mathcal{U}', d')$ on the frontier of $\mathcal{U}'$ — the two segments $C$ and $D$ have the same length.

• We have done this diagrammatically scaling $U_2$ by a factor of 2. But clearly the geometric argument generalizes to any re-scaling of a symmetric problem with a linear frontier.

• Hence, we have reached the following key conclusion.

• In any bargaining problem $\mathcal{B} = (\mathcal{U}, d)$ with $d = (0, 0)$ and with a **linear frontier** (whether symmetric or not), $f(\mathcal{U}, d)$ picks out the point on the frontier of $\mathcal{U}$ that divides the frontier into two **segments of equal length**.

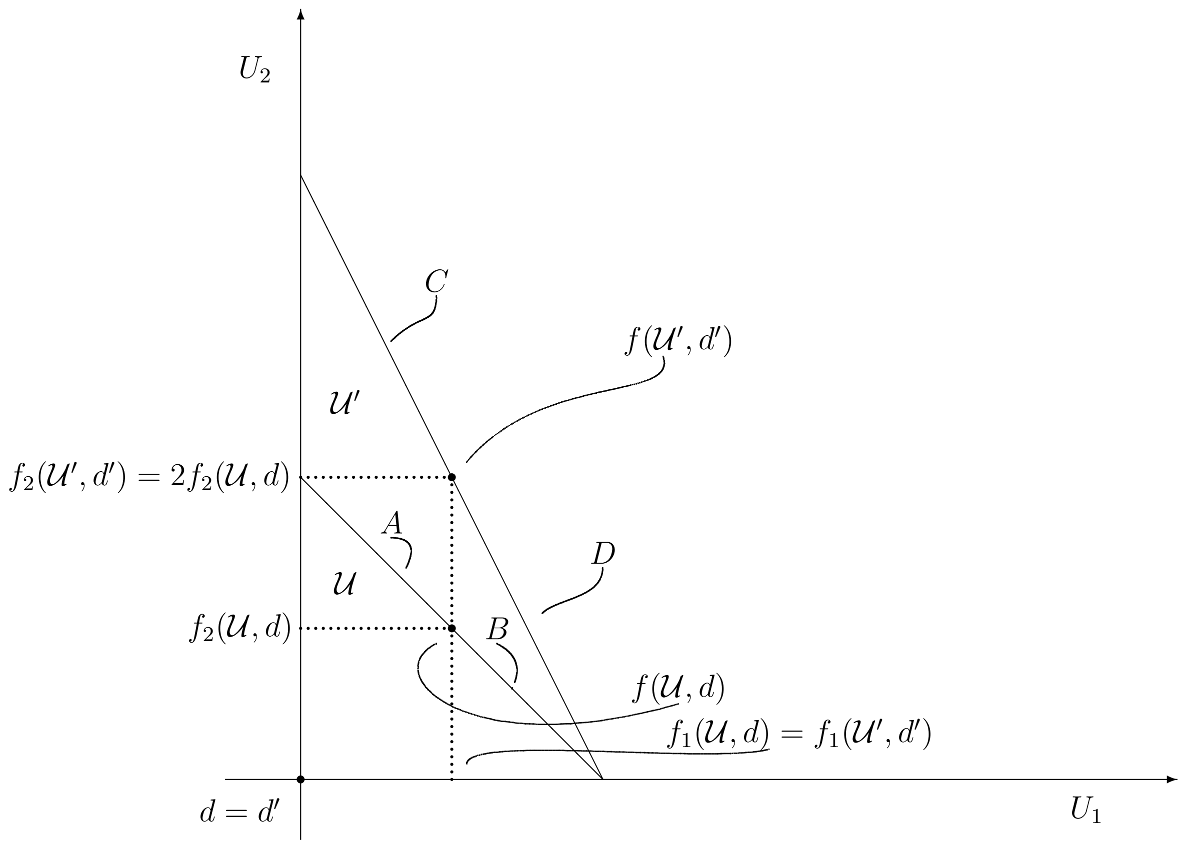



Figure 7: Linear Frontiers

### 2.6.3 Using IIA

• We now know everything there is to know about bargaining problems with a *linear frontier*.

• Using IIA, this will be enough to pin down $f$ in the **general case**.

• Start with *any* bargaining problem $\mathcal{B} = (\mathcal{U}, d)$, not necessarily with a linear frontier and not necessarily symmetric.

• For the time being assume that $d_1 = d_2 = 0$. We will come back to this shortly.

• Now find the **tangent** to the frontier that also has the property that the segment (up) on the left of the tangency point is of the **same length** as the segment (down) on the right of the tangency point. (See figure 8 – the two segments $A$ and $B$ have the same length.)

• Doing this we have constructed a **new** bargaining problem $\mathcal{B}' = (\mathcal{U}', d')$ with $d' = d$, and with $\mathcal{U}'$ the area below the tangent. (See figure 8.)

• Clearly, the bargaining problem $\mathcal{B}' = (\mathcal{U}', d')$ has a linear frontier. We *constructed* it this way! We also have $d' = 0$.

• Hence, we know everything about $f(\mathcal{U}', d')$.

• In particular, the solution $f(\mathcal{U}', d')$ must be as in Figure 8.

• Now we are ready to use IIA.

• Going from $\mathcal{B}' = (\mathcal{U}', d')$ to $\mathcal{B} = (\mathcal{U}, d)$ we shrink the feasible set from $U'$ to $\mathcal{U}$, we do not change the disagreement point, and we do not take out the solution to $\mathcal{B}'$.

• Hence IIA tells us that, as in Figure 8, we must have that

$$f(\mathcal{U}, d) = f(\mathcal{U}', d') \tag{2.15}$$

• To summarize, so far we know the following.

• Consider **any** $\mathcal{B} = (\mathcal{U}, d)$ with $d = (0, 0)$.

• Then to find $f(\mathcal{U}, d)$ we can proceed as follows.

• Find the point on the frontier of $\mathcal{U}$ that has the following property.

• When we draw the **tangent** to $\mathcal{U}$ at this point, the **length** of the **two segments** on the tangent, from the tangency point to the vertical axis, and from the tangency point to the horizontal axis **is the same**. (See figure 8 — the length of $A$ and $B$ is the same.)

### 2.6.4 Using IUO (Again)

• We know how to find $f(\mathcal{U}, d)$, provided that $d = (0, 0)$.

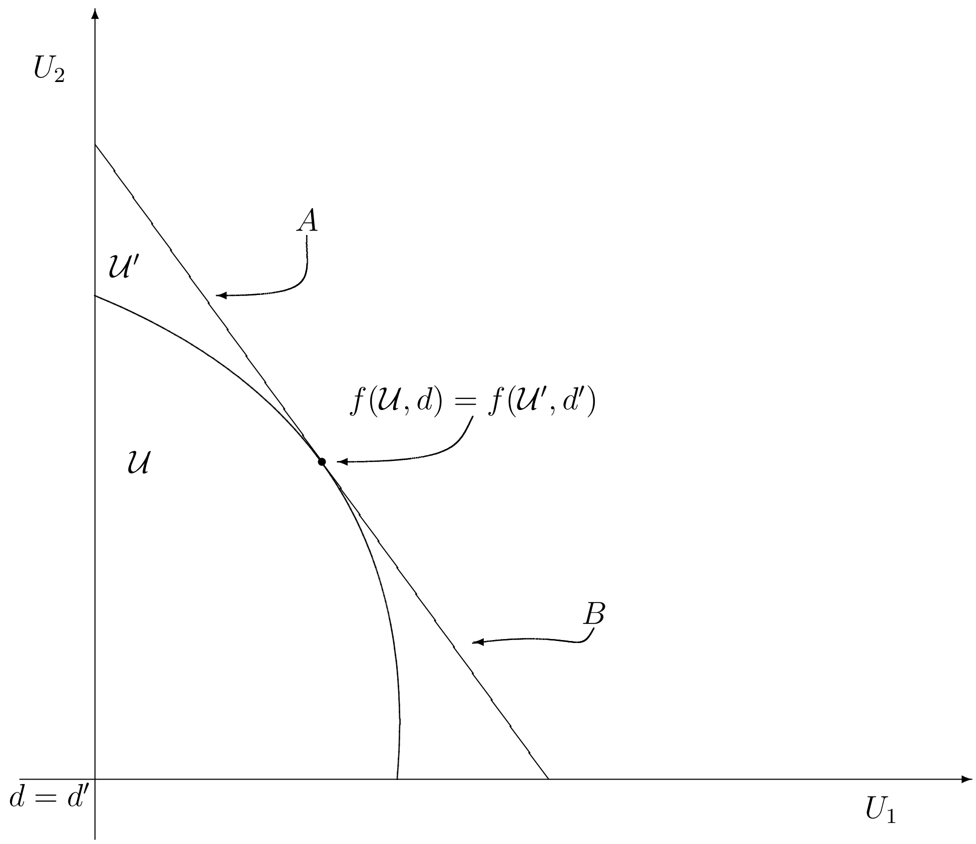



Figure 8: Using IIA

• How do we proceed to find $f(\mathcal{U}, d)$ in the general case in which $d$ may be different from $(0, 0)$?

• Using IUO this is not a hard step to make.

• From IUO, we know that if we subtract $d$ from $\mathcal{U}$ (we move the entire set $\mathcal{U}$ by the vector $-d$), then the solution must also move by the vector $-d$.

• In effect this says that we can consider the vertical line through $d$ as our vertical axis, and the horizontal line through $d$ as our horizontal axis and then apply what we know already about $f(\mathcal{U}, d)$ when $d = 0$ (See Figure 9.)

• So, to find $f(\mathcal{U}, d)$ in the **general case** we can proceed as follows.

• Find the point on the frontier of $\mathcal{U}$ that has the following property.

• When we draw the **tangent** to $\mathcal{U}$ at this point, the **length** of the **two segments** on the tangent, from the tangency point to the vertical line through $d$, and from the tangency point to the horizontal line through $d$ **is the same**. (See figure 9 — the length of $A$ and $B$ is the same.)

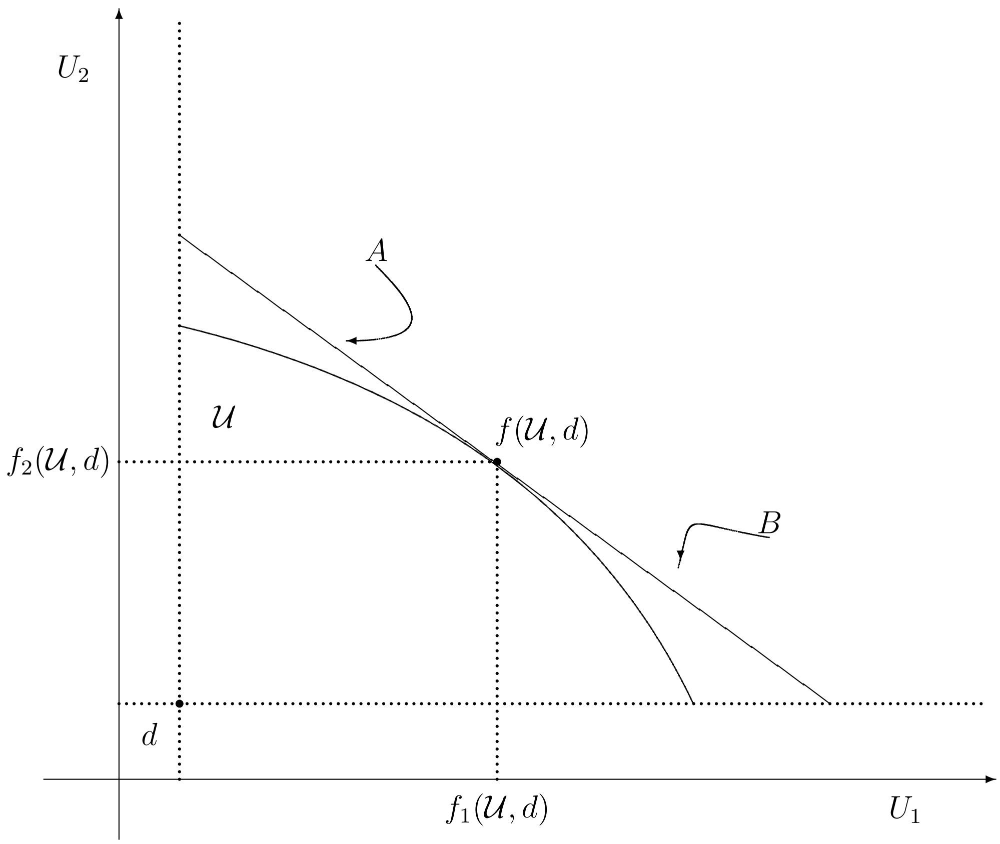



Figure 9: Using IUO Again

## 2.7 Finding $f$ in Practice

• Our analysis so far shows that for any $\mathcal{B} = (\mathcal{U}, d)$, $f(\mathcal{U}, d)$ is pinned down **uniquely** by PAR, SYM, IUO, IUU and IIA.

• Geometrically, we also now know how to find $f(\mathcal{U}, d)$ in the general case.

• This is exemplified in Figure 9.

• We seek a way to *find* the solution using a mathematical method.

• To do this, begin with some facts concerning hyperbolae.

• Recall that the equation of a hyperbola in a "$U_1, U_2$ plane" is given by (in implicit form)

$$u_1 \, u_2 = k \tag{2.16}$$

with $k > 0$ a constant.

• In explicit form (2.16) reads

$$u_2 = \frac{k}{u_1} \tag{2.17}$$

• The hyperbola in (2.16) has asymptotes on the "$U_1$" and "$U_2$" axes in the positive orthant. (It also has a "lower branch" in the negative orthant — but we will ignore lower branches throughout.)

• If we want to write the (implicit) equation of a hyperbola with a vertical asymptote at $v$ and a horizontal asymptote at $h$ we need to subtract these as constants from $u_1$ and $u_2$ respectively.

• So we get

$$(u_1 - v)(u_2 - h) = k \tag{2.18}$$

with $k > 0$ a constant.

• Notice that as we increase $k$ in (2.18) we describe a *family* of hyperbolae which move in the "North-East" direction as $k$ increases. (With given asymptotes if we keep $v$ and $h$ constant.)

• An important **fact** about these hyperbolae is the following.

• If we draw the **tangent** to the hyperbola in (2.18) at **any point**, the **length** of the **two segments** on the tangent, from the tangency point to the vertical asymptote, and from the tangency point to the horizontal asymptote **is the same**. (See figure 10 — the length of $A$ and $B$ is the same.)

• This fact suggests the following method for finding $f(\mathcal{U}, d)$ for a general bargaining problem.

• We should find the furthest hyperbola from the origin (going "North-East") with asymptotes $d_1$ and $d_2$ that touches $\mathcal{U}$.

• This will give us a hyperbola that is **tangent** to $\mathcal{U}$.

• So, if we look at the straight line that is tangent to the hyperbola, it will also be tangent to $\mathcal{U}$, at the same point.

• From what we have just worked out about tangents to hyperbolae, the tangent to $\mathcal{U}$ will have just the right property.

• The tangent to both $\mathcal{U}$ and the hyperbola will have the property that the **length** of the **two segments** on the tangent, from the tangency point to the vertical line through $d$, and from the tangency point to the horizontal line through $d$ **is the same**. (See Figure 11.)

• It follows that $f(\mathcal{U}, d)$ must be the point of tangency between $\mathcal{U}$ and the hyperbola. (See Figure 11.)

• Finding the furthest hyperbola from the origin with asymptotes $d_1$ and $d_2$ that touches $\mathcal{U}$ is the geometric equivalent of a **constrained maximization problem**.

• As $k$ in (2.18) increases the hyperbola moves away from the origin. Therefore we need to solve

$$
\begin{aligned}
&\max_{u_1,u_2} \ (u_1 - d_1)(u_2 - d_2) \\
&\text{s.t.} \ (u_1, u_2) \in \mathcal{U}
\end{aligned} \tag{2.19}
$$

• The objective function in (2.19) is often called the "Nash product."

• To sum up, we have reached the following conclusion.

• Let a bargaining problem $\mathcal{B} = (\mathcal{U}, d)$ be given.

• Assume that the solution function $f$ satisfies PAR, SYM, IUO, IUU and IIA.

• Denote by $(u_1^*, u_2^*)$ the values of $u_1$ and $u_2$ that solve the maximization problem (2.19).

• Then

$$
f_1(\mathcal{U}, d) = u_1^* \text{ and } f_2(\mathcal{U}, d) = u_2^* \tag{2.20}
$$

# 2.8 Canonical Example

• Consider again the canonical interpretation of 2.3 above.

• We pick specific utility functions for the buyer and the seller.

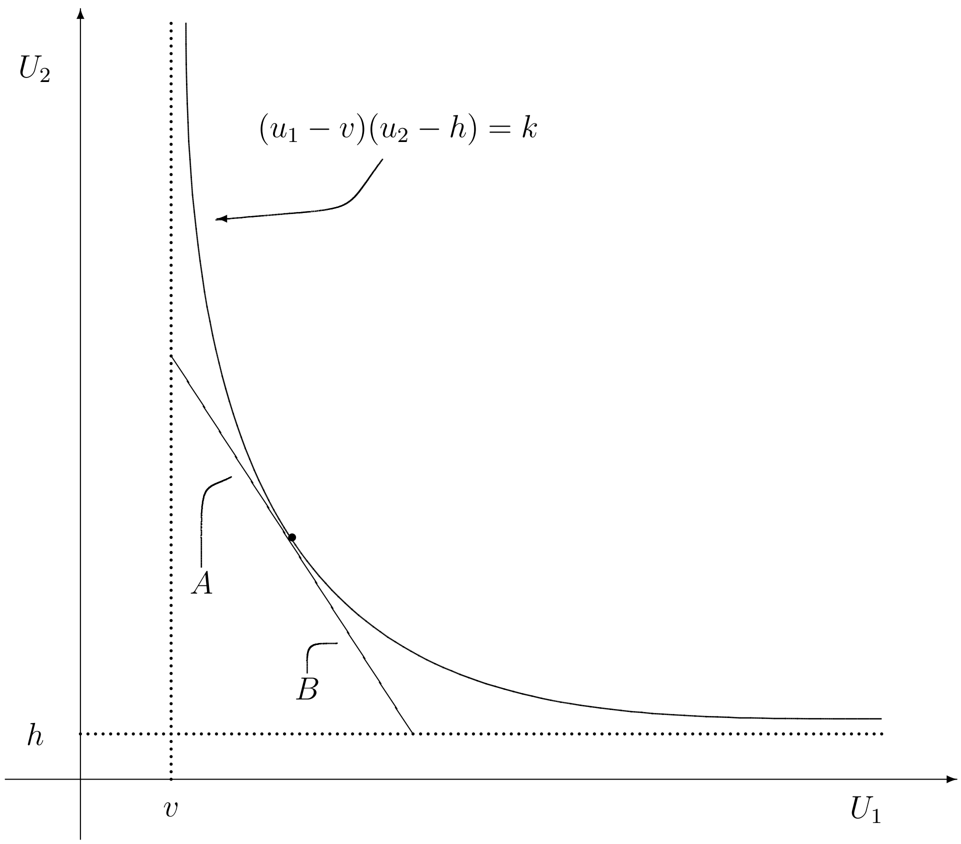



Figure 10: Tangents to Hyperbola

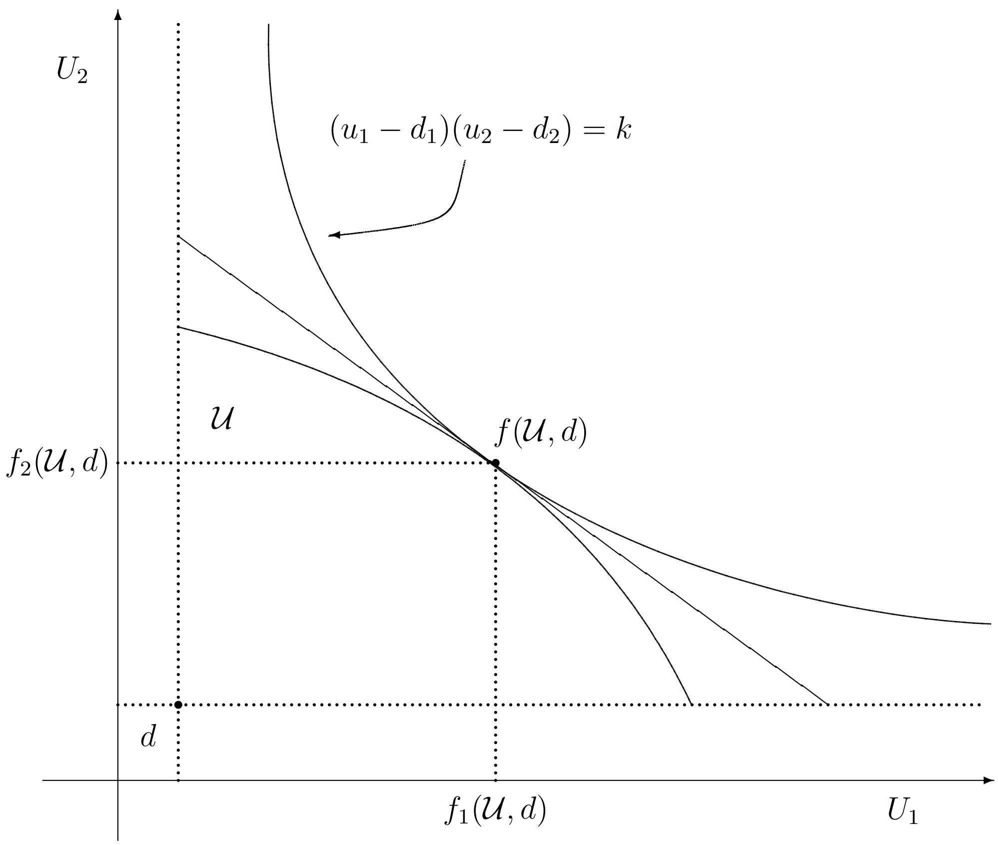



Figure 11: Finding $f(\mathcal{U}, d)$

• If the object is sold at price $p$, then

$$U_S(p - c) = (p - c)^\alpha \tag{2.21}$$

and

$$U_B(v - p) = (v - p)^\beta \tag{2.22}$$

• Remember that we are assuming that $d_1 = d_2 = 0$. If there is no transaction the utility of both is zero.

• Remember that we are assuming that $v > c$. The problem is not interesting otherwise.

• The "Nash product" therefore is

$$(p - c)^\alpha \, (v - p)^\beta \tag{2.23}$$

• So, we are looking for a $p$ that maximizes (2.23), subject to the agreement being feasible.

• So far we have written the constraint the the agreement must be feasible as $(u_1, u_2) \in \mathcal{U}$.

• In this case, there is an easy way to do this in terms of price.

• The feasible utilities are the pairs

$$[(p - c)^\alpha, (v - p)^\beta] \tag{2.24}$$

as $p$ varies in the interval $[c, v]$.

• So, we should be maximizing (2.23) by choice of $p$, subject to the constraint

$$c \, \leq \, p \, \leq \, v \tag{2.25}$$

• We are going to try just maximizing (2.23) without constraints.

• If we find a solution that satisfies (2.25), then this will also be the solution to the *constrained maximization* problem.

• Differentiating (2.23) wrt $p$ and setting equal to 0 gives

$$\alpha(p-c)^{\alpha-1}(v-p)^{\beta} = \beta(p-c)^{\alpha}(v-p)^{\beta-1} \quad (2.26)$$

• Dividing both sides of (2.26) by $(p-c)^{\alpha-1}(v-p)^{\beta-1}$ gives

$$\alpha(v-p) = \beta(p-c) \quad (2.27)$$

• Solving (2.27) for $p$ gives

$$p = v\frac{\alpha}{\alpha+\beta} + c\frac{\beta}{\alpha+\beta} \quad (2.28)$$

• Since both $\alpha$ and $\beta$ are positive, the $p$ in (2.28) clearly satisfies (2.25).

• So, we are done. The price in (2.28) is the one at which the exchange will take place.

• The price at which the exchange will take place is somewhere between $c$ and $v$. Where in this interval depends on the parameters of the buyer's and seller's utility functions as specified in (2.28).

• As $\alpha$ becomes smaller (keeping $\beta$ constant) the price will get closer and closer to $c$.

• As $\beta$ becomes smaller (keeping $\alpha$ constant) the price will get closer and closer to $v$.