ECON 459 — Game Theory

Lecture Notes — Auctions

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These notes have been used before. If you can still spot any errors or have any suggestions for improvement, please let me know.
1 Loose Comments

• Auctions are pervasive. Trillions of US$ worth of US Treasury debt are sold at auction. Ebay is a multibillion $ corporation. Google sells most of its digital advertising via online auctions. Cell-phone frequencies have been auctioned off in many countries. There are many more examples.

• There are many types of auctions. We will focus on two types.

• A First-Price Auction is one in which bidders submit bids, the highest bidder wins and pays the winning bid.

• A Second-Price Auction is one in which bidders submit bids, the highest bidder wins and pays a price equal to the second highest bid.

• Does the second price format sound crazy? First of all, we will argue that it is not. Second, it may be argued that it is not such a bad (approximate) model for a so called English auction.

• In an English auction the auctioneer increases the price by small increments until there is only one bidder left, who wins. So in some sense the winner pays (just above) the price bid by the last bidder to drop out before him.
2 Ingredients

- A single indivisible object is for sale.
- There is a fixed number $n$ of bidders $i = 1, \ldots, n$.
- The bidders submit sealed bids. So bids are simultaneous and independent. The bid of bidder $i$ is denoted $b_i$.
- Each bidder $i$ has a valuation for the object, denoted by $v_i$. He enjoys $v_i$ if and only if he wins the object.
- The highest bidder wins the object. If there are two or more bids tied at the top, the winner is chosen randomly. All top bidders win with positive probability.
- The price paid depends on whether we are in a second-price or a first-price auction.
- Only the winning bidder pays. The price paid for the object is denoted by $p$.
- Bidders are risk-neutral. If $i$ wins the object and the price is $p$, his utility is $v_i - p$. All non-winners have a utility of 0. If there is uncertainty about who wins, expected payoffs guide the bidders’ behavior.
3 Second-Price, Known Valuations

- Typically, each bidder will know his own $v_i$ and not observe the valuations of other bidders. But to begin with we look at the case in which all bidders know the valuations of all other bidders as well.
- This simplifies things and will be useful later when we look at the case in which $i$ does not observe the other bidders’ valuations.
- Fix $(v_1, \ldots, v_n)$, known to all. The only thing that matters for $i$ in deciding how much to bid is the largest bid of all other bidders. This is because if he wins (because his bid is highest), this bid will be the price he pays.
- For each $i$, let $m_i$ be the largest bid among all bidders but $i$. That is
  \[ m_i = \max_{j \neq i} \{ b_j \} \quad (3.1) \]
- Remember, bids are simultaneous and independent. So, given any possible configuration of the bids of others, how should $i$ set $b_i$? We can proceed on a case-by-case basis as follows.

1. Suppose $m_i > v_i$. 

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(a) Bidding \( b_i \leq v_i \) makes \( i \) lose the auction. He gets a payoff of zero.

(b) Bidding \( v_i < b_i < m_i \) makes \( i \) lose the auction. He gets a payoff of zero.

(c) Bidding \( b_i = m_i \) makes \( i \)'s bid tied as the top bid with one or more other bids. So \( i \) wins with positive probability, and pays a price \( p = m_i \). If he wins he gets \( v_i - p < 0 \). If he does not win he gets zero. So, in this case, his expected payoff is negative.

(d) Bidding \( b_i > m_i \) makes \( i \) win the auction for sure. So, in this case he gets \( v_i - p = v_i - m_i < 0 \).

• From cases (1a) through (1d) we can conclude that if \( m_i > v_i \), then there is no choice of \( b_i \) that is better than setting \( b_i = v_i \). Some other choices are just as good for bidder \( i \), but no choice is better.

2. Suppose \( m_i = v_i \).

(a) Bidding \( b_i < v_i \) makes \( i \) lose the auction. He gets a payoff of zero.

(b) Bidding \( b_i = v_i \) makes \( i \)'s bid tied as the top bid with one or more other bids. So \( i \) wins with positive probability, and pays a price \( p = m_i = v_i \). If he wins he gets \( v_i - p = 0 \). If he does not
win he gets zero. So, in this case, his expected payoff is zero.

(c) Bidding \( b_i > v_i \) makes \( i \) win the auction for sure. His payoff is \( v_i - p \). Since \( p = m_i = v_i \), his payoff is zero.

- From cases (2a) through (2c) we can conclude that if \( m_i = v_i \), then there is no choice of \( b_i \) that is better than setting \( b_i = v_i \). Bidder \( i \) gets a payoff of zero, regardless of his bid \( b_i \).

3. Suppose \( m_i < v_i \).

(a) Bidding \( b_i < m_i \) makes \( i \) lose the auction. He gets a payoff of zero.

(b) Bidding \( b_i = m_i \) makes \( i \)'s bid tied as the top bid with one or more other bids. So \( i \) wins with positive probability, and pays a price \( p = m_i \). If he wins he gets \( v_i - m_i > 0 \). If he does not win he gets zero. So, in this case, his expected payoff positive but below \( v_i - m_i \).

(c) Bidding \( m_i < b_i < v_i \) makes \( i \) win the auction for sure. His payoff is \( v_i - p \). Since \( p = m_i \), his payoff is \( v_i - m_i > 0 \).

(d) Bidding \( b_i = v_i \) makes \( i \) win the auction for sure. His payoff is \( v_i - p \). Since \( p = m_i \), his payoff is \( v_i - m_i > 0 \).
(e) Bidding $b_i > v_i$ makes $i$ win the auction for sure. His payoff is $v_i - p$. Since $p = m_i$, his payoff is $v_i - m_i > 0$.

- From cases (3a) through (3e) we can conclude that if $m_i < v_i$, then there is no choice of $b_i$ that is better than setting $b_i = v_i$. Some other choices are just as good for bidder $i$, but no choice is better.
- Putting together all the cases, (1a) through (3e) we can conclude that, regardless of $m_i$, there is no choice of $b_i$ that is better for $i$ than setting $b_i = v_i$. Depending on $m_i$, some choices may be worse. But no choice is ever better.
- Another way to say this is that bidding $b_i = v_i$ weakly dominates all other strategies.
- Remember that it is okay to rule out weakly dominated strategies (not okay in general to do this iteratively). They are not robust to the possibility of “mistakes.”
- Also, we know that after deleting weakly dominated strategies, at least one Nash equilibrium will still be there.
- Hence we can conclude that the unique Nash equilibrium not involving weakly dominated strategies is for each bidder $i$ to bid $b_i = v_i$. 
• Notice one appealing property of the Nash equilibrium we have found. The bidder with the largest \( v_i \) wins the auction (if there are two or more valuations tied at the top, then one of the top valuers wins for sure).
• An auction with the property that (one of) the highest valuers wins the object is called an efficient auction. More on this later.

4 Second-Price, Unknown Valuations

• We now consider the case where each bidder \( i \) knows \( v_i \) but is uncertain about the valuations of the other bidders.
• The right way to write down the game is to have a tree which starts with a random move by Nature that assigns a valuation to each bidder. This defines a game of incomplete information.
• So, Nature draws a vector \((v_1, \ldots, v_n)\) according to some probability distribution. Let \( P(v_1, \ldots, v_n) \) be the probability of \((v_1, \ldots, v_n)\).
• Also let \( P_i(v_{-i}|v_i) \) be the probability that bidders \(-i\) (all bidders but \(i\)) have valuations \(v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)\), conditional on bidder \(i\) having a valuation of \(v_i\).
To fix ideas about these probability distributions, say there are two bidders $i = 1, 2$. Bidder 1 can have values $v_1 = 2$ or $v_1 = 3$. Bidder 2 can have values $v_2 = 0$ or $v_2 = 4$. Then $P$ must specify the probabilities of the four combinations $(2, 0), (2, 4), (3, 0)$ and $(3, 4)$.

- If, say, all these four probabilities are $1/4$, then we also know that all conditional probabilities are equal to $1/2$ (the draws of $v_1$ and $v_2$ are independent).

- Suppose instead that, for instance, $P(2, 0) = 1/2$ and $P(2, 4) = P(3, 0) = P(3, 4) = 1/6$. Then we have that $P_1(0|2) = P_2(2|0) = 3/4$, while $P_1(4|2) = P_2(3|0) = 1/4$, and $P_1(0|3) = P_1(4|3) = P_2(2|4) = P_2(3|4) = 1/2$.

- After Nature’s draw (four branches in the example above), we write in information sets that ensure that each bidder $i$ can distinguish between any two nodes that give a different $v_i$, but not between nodes that give the same $v_i$ and different $v_{-i}$.

- This is to capture the idea that $i$ knows $v_i$, but does not observe $v_{-i}$.

- The tree looks somewhat messy if you try to draw it. But it can be done without violating the rules for drawing “legitimate” game trees.

- After observing $v_i$, each bidder $i$ now has a probability distribution in mind concerning $v_{-i}$. His beliefs over the
possible $v_{-i}$ are given by $P_i(v_{-i}|v_i)$.

- To “solve” the game we now need to specify how each bidder will chose his bid $b_i$ for each possible value of $v_i$.
- Let $i$’s bid given $v_i$ be denoted by $b_i(v_i)$. These bids have to be optimal given the bidding strategies of all other players ($b_{-i}$), and given the beliefs of $i$ as we specified them above.
- Looks pretty daunting, but luckily we have done most of the work already!
- Fix $v_i$. This also fixes $P_i(v_{-i}|v_i)$. So, given the strategies of the others this fixes a probability distribution over $m_i$ (the maximum bid of the others). So $b_i = b_i(v_i)$ has to be an optimal bid given this probability distribution over $m_i$.
- This looks complicated, but it is not. Clearly, bidding $b_i = v_i$ (in other words setting $b_i(v_i) = v_i$ for all possible $v_i$) must be optimal against any probability distribution over $m_i$.
- The reason $b_i = v_i$ must be optimal is that bidding in this way cannot be worse than any other bid, regardless of $m_i$. This is what we saw in the case of know valuations above.
- So we have reached the following conclusion. **The unique Nash equilibrium not involving weakly...**
dominated strategies is for each bidder \( i \) to set \( b_i(v_i) = v_i \). In other words, the only Nash equilibrium not involving weakly dominated strategies is for each bidder \( i \) to always set his bid equal to his valuation.

- So we have also reached the conclusion that a second-price auction is **efficient** in the general case.

5 First-Price, Unknown Valuations - I

- This case is a lot messier than the second-price case. In particular, analyzing the case of known valuations does not help much. This is why we start directly with the unknown valuations case.

- Let’s try to look at a simple example. (**Warning** the example is so simple that it is deceptive in many ways. Still, it will help us along.)

- Two bidders \( i = 1, 2 \). Each has value either 0 or 1. Each value with probability 1/2.

- Valuations are independent. So all four possible combinations of values, (0, 0) (0, 1), (1, 0) and (1, 1) have probability 1/4.

- Notice that in this case \( P_1(0|0) = P_2(0|0) = P_1(1|0) = \)
\[ P_2(1|0) = P_1(0|1) = P_1(1|1) = P_2(0|1) = P_2(1|1) = 1/2. \]

- Also, for simplicity, we restrict the bids of each bidder to be either 0 or 1. (This is a very strong restriction.)
- Because we are taking everything to be either 0 or 1, the exact way on which ties are broken can matter.
- So, to be specific, we assume that if the two bids are either both 1 or both 0, the winner is bidder 1 with probability 1/2 and bidder 2 with probability 1/2.
- Let’s check if there is an equilibrium like the one in the second-price case in which each bidder sets his bid equal to his valuation.
- We do all the calculations from the point of view of bidder 1. Bidder 2 is symmetric, so we don’t bother.
- Suppose bidder 2 bids 1 if \( v_2 = 1 \) and bids 0 if \( v_2 = 0 \).
- Suppose \( v_1 = 1 \). If bidder 1 sets \( b_1 = 1 \) he gets an expected payoff of
  \[
  \frac{1}{2} P_1(1|1) (1 - 1) + P_1(0|1) (1 - 1) = 0 \quad (5.2)
  \]
  This is because of the following. With probability \( P_1(1|1) \) bidder 2’s valuation is 1, in which case \( b_2 = 1 \), the bids are tied and 1 wins with probability 1/2. If he wins, he pays 1, while his value is 1. With probability \( P_1(0|1) \) bidder 2’s valuation is 0, in which case \( b_2 = 0 \) and 1 wins for sure. If he wins, he pays 1 and his value is 1.
• Still assuming that $v_1 = 1$, suppose that bidder 1 sets $b_1 = 0$. Then he gets and expected payoff of

$$P_1(1|1)0 + \frac{1}{2}P_1(0|1)(1 - 0) = \frac{1}{4} \quad (5.3)$$

This is because of the following. With probability $P_1(1|1)$ bidder 2’s valuation is 1, in which case $b_2 = 1$, 2 wins for sure and 1 gets a payoff of 0. With probability $P_1(0|1)$ bidders 2’s valuation is 0, in which case $b_2 = 0$, the bids are tied and 1 wins with probability $1/2$. If he wins, he pays 0 and his value is 1.

• Now compare (5.2) and (5.3). Clearly, if bidder 2 always sets $b_2 = v_2$, and $v_1 = 1$, then it is better for bidder 1 to bid 0 than to bid 1.

• Hence, in this example there is no Nash equilibrium in which both bidders always set their bids equal to their values.

• Let’s try to work with another candidate equilibrium. This one is that both bidders bid 0, regardless of whether their valuation is 1 or 0.

• As before, we do all the calculations from the point of view of bidder 1. Bidder 2 is symmetric, so we don’t bother.

• Suppose bidder 2 bids 0 if $v_2 = 1$ and bids 0 if $v_2 = 0$. 
• Suppose $v_1 = 1$. If bidder 1 sets $b_1 = 1$ he gets an expected payoff of

$$P_1(1|1) (1 - 1) + P_1(0|1) (1 - 1) = 0 \quad (5.4)$$

This is because of the following. With probability $P_1(1|1)$ bidder 2’s valuation is 1, in which case $b_2 = 0$ and 1 wins for sure. If he wins, he pays 1, while his value is 1. With probability $P_1(0|1)$ bidder 2’s valuation is 0, in which case $b_2 = 0$, and 1 wins for sure. If he wins, he pays 1 and his value is 1.

• Still assuming that $v_1 = 1$, suppose that bidder 1 sets $b_1 = 0$. Then he gets an expected payoff of

$$\frac{1}{2} P_1(1|1) (1 - 0) + \frac{1}{2} P_1(0|1) (1 - 0) = \frac{1}{2} \quad (5.5)$$

This is because of the following. With probability $P_1(1|1)$ bidder 2’s valuation is 1, in which case $b_2 = 0$, the bids are tied and 1 wins with probability 1/2. If he wins, he pays 0 and his value is 1. With probability $P_1(0|1)$ bidder 2’s valuation is 0, in which case $b_2 = 0$, the bids are tied and 1 wins with probability 1/2. If he wins, he pays 0 and his value is 1.

• Now compare (5.4) and (5.5). Clearly, if bidder 2 always sets $b_2 = 0$, and $v_1 = 1$, then it is better for bidder 1 to bid 0 rather than to bid 1.
• So far the candidate equilibrium works. To complete the check we need to consider the case \( v_1 = 0 \).

• Suppose bidder 2 always bids 0. Suppose that \( v_1 = 0 \). If bidder 1 sets \( b_1 = 1 \) he gets an expected payoff of

\[
P_1(1|1) \cdot (0 - 1) + P_1(0|1) \cdot (0 - 1) = -1 \quad (5.6)
\]

This is because of the following. With probability \( P_1(1|1) \) bidder 2’s valuation is 1, in which case \( b_2 = 0 \), and 1 wins for sure. If he wins, he pays 1, while his value is 0. With probability \( P_1(0|1) \) bidder 2’s valuation is 0, in which case \( b_2 = 0 \), and 1 wins for sure. If he wins, he pays 1 and his value is 0.

• Still assuming that \( v_1 = 0 \), suppose that bidder 1 sets \( b_1 = 0 \). Then he gets an expected payoff of

\[
\frac{1}{2}P_1(1|1) \cdot (0 - 0) + \frac{1}{2}P_1(0|1) \cdot (0 - 0) = 0 \quad (5.7)
\]

This is because of the following. With probability \( P_1(1|1) \) bidder 2’s valuation is 1, in which case \( b_2 = 0 \), the bids are tied and 1 wins with probability 1/2. If he wins, he pays 0 and his value is 0. With probability \( P_1(0|1) \) bidder 2’s valuation is 0, in which case \( b_2 = 0 \), the bids are tied and 1 wins with probability 1/2. If he wins, he pays 0 and his value is 0.

• Now compare (5.6) and (5.7). Clearly, if bidder 2 always
sets $b_2 = 0$, and $v_1 = 0$, then it is better for bidder 1 to bid 0 rather than to bid 1.

- Since there are no more cases to consider, we have proved the following. **There is a Nash equilibrium in which both bidders always set** $b_i = 0$, **regardless of their valuations**.

- As we said at the outset, the example is deceptively simple. But it illustrates well an important difference between the second-price auction and the first-price one.

- Consider the binary example we have just looked at in the first-price case, but make the auction a second-price one. Then, we know from before that the only Nash equilibrium not involving weakly dominated strategies is for both bidders to always set $b_i = v_i$.

- Now compare this with the equilibrium we found in the first-price case in which all bids are always 0.

- Clearly, in the first-price case **bids are lower** (at least in some cases) than in the second-price case. Intuitively, this is because in the first-price case the incentives to bid high are reduced, since the winner is charged a higher price.

- The diminished incentives to bid high can be so strong that the **revenue** from selling the object can be lower in
the first-price case than in the second-price case. (Though the simple example is quite special in this sense.)

- In the simple example the (expected) revenue from selling the object using a first-price procedure in the equilibrium we found is zero. This is obvious since all bids are always zero.

- In the simple example the (expected) revenue from selling the object using a second-price auction is $\frac{1}{4}$.

- To see this note that in equilibrium both bidders always set $b_i = v_i$. So, the price paid is zero whenever one or both values are zero. However, if $v_1 = v_2 = 1$ then the price paid is 1. Since the probability that $v_1 = v_2 = 1$ is $\frac{1}{4}$, the expected revenue is $\frac{1}{4}$.

6 First-Price, Unknown Valuations - II

- Let’s look at another example of a first-price auction. This one is still special, but has more of the flavor that one would get in the “general” case.

- Two bidders $i = 1, 2$. Each bidder’s valuation $v_i$ is a continuous random variable distributed uniformly over $[0, 1]$. 
• So, the density of bidder $i$’s valuation is $f_i(v_i) = 1 \forall v_i \in [0, 1]$, while the cumulative distribution of bidder $i$’s valuation is $F_i(v_i) = v_i \forall v_i \in [0, 1]$.

• The bidder’s valuations are independent.

• So, conditional on any $v_1$, bidder 1 believes that $v_2$ is still uniformly distributed over $[0, 1]$, and conditional on any $v_2$ bidder 2 believes that $v_1$ is still uniformly distributed over $[0, 1]$.

• Each bidder is allowed to bid any amount $b_i$ in the entire interval $[0, 1]$.

• In case of tied bids, the winner is selected randomly, say probability $1/2$ each, but the precise randomization turns out to be unimportant in this case.

• To find an equilibrium, we are looking for bidding functions $b_1$ and $b_2$ with the following properties.

• For every $v_1 \in [0, 1]$ the bid $b_1 = b_1(v_1)$ maximizes bidder 1’s expected payoff, given his beliefs about $v_2$, and the bidding function $b_2$.

• For every $v_2 \in [0, 1]$ the bid $b_2 = b_2(v_2)$ maximizes bidder 2’s expected payoff, given his beliefs about $v_1$, and the bidding function $b_1$.

• To get going, we make two simplifying assumptions. These can both be shown to be “without loss of general-
ity.”

• First, we will restrict our search to symmetric equilibria. These are equilibria in which

\[ b_1(v) = b_2(v) \quad \forall v \in [0, 1] \quad (6.8) \]

Note this says that the two bidders will bid the same if their values are the same. It does not say that their bids will necessarily be equal.

• Second, we will restrict our search to the case in which the bidding functions are increasing and differentiable in values. This is stated as the fact that both \( b'_1 \) and \( b'_2 \) exist, and

\[ b'_1(v) = b'_2(v) > 0 \quad \forall v \in [0, 1] \quad (6.9) \]

Note that \( b'_1(v) \) is actually equal to \( b'_2(v) \) because of (6.8). Note also that because of (6.9) both bidding functions are invertible and their inverses are also differentiable.

• Now that we have set up things so that calculus can help, we are ready to look for the optimal bidding functions. We do everything from the point of view of bidder 1. Bidder 2 is completely symmetric.

• Suppose bidder 1 has a value of \( v_1 \). He will then pick a bid \( b_1 \) that solves

\[ \max_{b_1} (v_1 - b_1) \ \text{Prob.} \ [b_2(v_2) \leq b_1] \quad (6.10) \]
• To see this, reason as follows. If bidder 1 wins his payoff is $v_1 - p$. But since we are in a first-price auction $p = b_1$, so 1’s utility if he wins is $v_1 - b_1$.

• What is the probability that 1 wins, given that he is bidding $b_1$? Since the highest bidder wins, this is just the probability that $b_2 \leq b_1$. This is the probability that $b_2(v_2) \leq b_1$. (Notice, we are ignoring ties since (given (6.9)) they happen with probability zero.) So, the objective function of the maximization problem in (6.10) is bidder 1’s expected payoff given $v_1$, $b_1$ and $b_2$

• Since the bidding function $b_2$ can be inverted, (6.10) can be re-written as

$$\max_{b_1} (v_1 - b_1) \text{ Prob. } [v_2 \leq b_2^{-1}(b_1)] \quad (6.11)$$

• Since $F_2$ is the cumulative distribution function of 2’s valuation, (6.11) can be re-written as

$$\max_{b_1} (v_1 - b_1) F_2[b_2^{-1}(b_1)] \quad (6.12)$$

• Since by assumption $F_2(v_2) = v_2$ we can also re-write (6.12) as

$$\max_{b_1} (v_1 - b_1) b_2^{-1}(b_1) \quad (6.13)$$

• Differentiating (6.13) wrt $b_1$, setting equal to zero and re-arranging gives

$$(v_1 - b_1)b_2^{-1}(b_1) = b_2^{-1}(b_1) \quad (6.14)$$
Now remember the rule for finding the derivative of the inverse of a given function. This tells us that
\[ b_2^{-1}'(b_1) = \frac{1}{b_2'[b_2^{-1}(b_1)]} = \frac{1}{b_2'(v_1)} \] (6.15)
where the second equality comes from the fact that \( b_1(v) = b_2(v) \) for any \( v \in [0, 1] \), which we know to be true from (6.8). Given this, it must be that \( b_1 = b_2(v_1) \). Therefore \( b_2^{-1}(b_1) = b_2^{-1}[b_2(v_1)] = v_1 \).

Hence, (6.14) can be re-written as
\[ v_1 - b_1 = b_2'(v_1)b_2^{-1}(b_1) \] (6.16)

Using again the fact that \( b_2^{-1}(b_1) = b_2^{-1}[b_2(v_1)] = v_1 \), we can re-write (6.16) as
\[ v_1 = b_1(v_1) + v_1 b_1'(v_1) \] (6.17)

Now we are almost done. Notice that (6.17) is a differential equation in \( b_1(\cdot) \). We can solve it using standard tricks. In particular, using the rule for differentiating the product of two function, (6.17) tells us that
\[ v_1 = \frac{d}{dv_1}[v_1 b_1(v_1)] \] (6.18)

Integrating each side of (6.18) wrt \( v_1 \) gives
\[ \frac{1}{2} v_1^2 + k = v_1 b_1(v_1) + z \] (6.19)
with \( k \) and \( z \) arbitrary constants.
• So, dividing through by $v_1$ and re-arranging we have

$$b_1(v_1) = \frac{v_1}{2} + \frac{k - z}{v_1} \quad (6.20)$$

• Remember that $v_1$ varies in $[0, 1]$. So, if $k - z \neq 0$ \((6.20)\) says that as $v_1 \to 0$ then $b_1(v_1)$ would tend to $+\infty$ (if $k - z > 0$) or to $-\infty$ (if $k - z < 0$). Neither makes sense. So, we conclude that $k - z = 0$, and we finally have the solution.

• Using symmetry \((6.8)\) once more, the solution \((6.20)\), and $k - z = 0$, we can write

$$b_1(v_1) = \frac{v_1}{2} \quad \text{and} \quad b_2(v_2) = \frac{v_2}{2} \quad (6.21)$$

• Notice again the comparison between the second-price and the first price auction.

• As usual, we know what would happen if we had the same distributions of values as in this example, but we used a second-price auction to sell the object.

• In any Nash equilibrium not involving weakly dominated strategies all bidders would always set their bid equal to their value. Hence with a second-price auction we get

$$b_1(v_1) = v_1 \quad \text{and} \quad b_2(v_2) = v_2 \quad (6.22)$$
Comparing (6.21) and (6.22) we conclude that at least one of our previous conclusions about the comparison of the two auctions still holds. In the first-price auction bids are lower than in the second-price case.

In the simple binary example, the lower bids were enough to reduce the expected revenue. Is this still the case? The answer is quite surprising, and this is what we do next.

7 Revenue Comparison

Consider the case of independently and uniformly distributed values over \([0, 1]\) with two bidders discussed above. We want to compare the expected revenue for the auctioneer in the case of a first-price and of a second-price auction.

The expected revenue for the first-price case can be obtained from the solution for the bidding functions (6.21), the distributions of the values, and the fact that the auction is a first-price one.

Fix a pair values are \(v_1\) and \(v_2\). Then the revenue to the auctioneer is

\[
\max \left\{ \frac{v_1}{2}, \frac{v_2}{2} \right\} = \frac{1}{2} \max \{v_1, v_2\} \quad (7.23)
\]
Hence the expected revenue is given by
\[
\frac{1}{2} E_{v_1,v_2} \{\max\{v_1, v_2\}\} \quad (7.24)
\]

- To evaluate the expectation in (7.24) we need to get the probability distribution of \(\max\{v_1, v_2\}\). This is a standard exercise. It goes as follows.
- Fix any \(r \in [0, 1]\). The probability that \(\max\{v_1, v_2\} \leq r\) is given by

\[
\text{Prob.}[v_1 \leq r \text{ and } v_2 \leq r] = F_1(r) F_2(r) = r^2 \quad (7.25)
\]

where the first equality comes from the fact that the values are independent, and the second comes from the fact that \(F_1(r) = F_2(r) = r\) because the values are uniformly distributed over \([0, 1]\).
- Hence the density of \(r = \max\{v_1, v_2\}\) is given by \(2r\); namely the derivative of its cumulative distribution function given in (7.25).
- Hence, using (7.24) we can now compute the expected revenue for the second-price case as

\[
\frac{1}{2} \int_0^1 r \cdot 2r \, dr = \frac{1}{3} \quad (7.26)
\]

- The expected revenue for the second-price case can be obtained from the solution for the bidding functions (6.22) and the distributions of the values, and the fact that the auction is a second-price one.
• Fix a pair values are \( v_1 \) and \( v_2 \). Then the revenue to the auctioneer is \( \min\{v_1, v_2\} \). Hence the expected revenue is given by

\[
E_{v_1,v_2}\{\min\{v_1, v_2\}\}
\]  
(7.27)

• To evaluate the expectation in (7.27) we need to get the probability distribution of \( \min\{v_1, v_2\} \). This is also a standard exercise. It goes as follows.

• Fix any \( r \in [0, 1] \). The probability that \( \min\{v_1, v_2\} \leq r \) is equal to \( 1 - \text{Prob.}[\min\{v_1, v_2\} > r] \), which can also be written as

\[
1 - \text{Prob.}[v_1 > r \text{ and } v_2 > r]
\]  
(7.28)

• Using the fact that the values are independent, we can write the quantity in (7.28) as

\[
1 - [1 - F_1(r)][1 - F_2(r)] = 1 - [1 - r]^2
\]  
(7.29)

where the equality comes from the fact that \( F_1(r) = F_2(r) = r \) because the values are uniformly distributed over \([0, 1]\).

• Hence the density of \( r = \min\{v_1, v_2\} \) is given by \( 2(1 - r) \); namely the derivative of its cumulative distribution function given in (7.29).

• Hence, using (7.27) we can now compute the expected revenue for the first-price case as

\[
\int_0^1 r \, 2(1 - r) \, dr = \frac{1}{3}
\]  
(7.30)
• Now compare (7.26) and (7.30). The expected revenue is the same in the first-price and second-price auction!

• The fact that a higher price (for given bids) is charged in the first-price case is exactly compensated by the lower equilibrium bids.

• Is this by fluke? It sure looks a bit like it. But it is not. It is an instance of a more general result known as the revenue equivalence theorem.

• This is a result that highlights the role of the efficiency of auctions in terms of the revenue they generate. Clearly both the second-price and the first-price auction we have solved are efficient. The object always goes to the bidder with the highest valuation.

• Very roughly speaking the revenue equivalence theorem says that, under the right assumptions, two auctions that are both efficient will yield the same expected revenue. Among others, the “right assumptions” include the fact that the bidders’ valuations should be independent random variables.