Topological constancy in the visual perception of resonance

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Abstract

Topological constancy is a psychophysical phenomenon associated with the visual trace figures produced by two frequencies which are sufficiently close to a low-order rational fraction and it persists even if the oscillators are perturbed. Such effects are described for rotational and vibrational near-harmonies of a dynamical system. Mathematical models are considered for such topological constancy; the optical substrate, neural response, and emergent perception are investigated.

Key words: filling-in, aliasing, resonance, quasi-periodic dynamics, Lissajous figures, topology, reverse Doppler shift.

1 Introduction

The notion of filling in a constant form from sparse information has been extensively studied in visual perception. One of the earliest results in this direction was due to Johansson [17], who showed that basic human activities could be identified merely from the light-trace of independent lights attached to the wrists, elbows, knees, etc., of moving figures in a darkened room.

Something similar happens when periodic forces act upon the direction of a beam of light. Humans are capable of filling in a rather complex mathematical form based only on the transitory information from a moving dot of light. This either reflects well on the intrinsic efficiency of such filling-in mechanisms or, not exclusively, it shows something like robustness in the Platonic forms.

For example, if a dot of light is scanned simultaneously in the x and y directions at the same rate, then one will see an ellipse with eccentricity depending on the phase difference ϕ. When $\phi = 0$ or $\phi = \pi$, a straight line with slope $+1$ or -1 appears; for $\phi = \pi/2$, the figure observed is a circle. On the other hand, if the rate in the y-direction is twice that in the x-direction, then the scanned dot of light traces a figure like the infinity sign ∞.

These are examples of *Lissajous figures*, which are produced by two or more oscillations whose actions are superimposed upon the position of a spot of light, e.g., by causing mirrors to vibrate, and studying the resulting movements of a light beam they are reflecting. Jules Antoine Lissajous (1822–1880) displayed these figures (using tuning forks) and claimed the percept of a transparent surface containing the figure which could be imagined as rotating in space. However, Nathaniel Bowditch (1773–1838) seems to have observed these figures earlier. See, e.g., [8].

A similar phenomenon occurs with respect to circular motion and was described in [18], where the resulting figures are called *mandalas*. These may be described as follows.

Suppose we have a clock with one of its hands going around at a rate of r rotations per second in the counter-clockwise direction. Let $k \geq 1$ and suppose also that there is an analogue-style watch, with its hand going at kr rotations per second in the clockwise direction, and the watch is mounted at the tip of the clock-hand. Put a small but bright light at the tip of the watch hand and record the trace of the light. This can be implemented more effectively by reflecting a laser beam from two rotating mirrors.

If k is a rational number and if the light is projected onto a screen, it repeatedly traverses some closed curve. For instance, if $k = 1$, the resulting trace is an ellipse.

But there are infinitely many different rational numbers near 1 (or any given number): not just 1 but also 11/10, 101/100, 1001/1000, ..., etc. And there are also infinitely many different *irrational* numbers near 1 (such as $1 + \sqrt{2}/10^n$ where n is a positive integer). When k is irrational, the figure traced by the dot is never a closed curve and must eventually fill in an entire 2-dimensional region! How can this be reconciled with the fact that one can't really know that the rate of rotation is *exactly* 1, say, rather than 101/100 or $1 + \pi/10^3$?

Nevertheless, one has a quite clear percept. Beginning with $k = 1$ and gradually increasing k, one sees the ellipse begin to spin (counter-clockwise) faster and faster. Then a discontinuity occurs; the figure actually *changes form* and a new stable figure appears with a reversed direction of spin. Further increase in the value of k causes the spin to slow to zero, and at this point, another rational value for k has been reached. However, the next such "visible rational" after 1 depends on r. If r is low (e.g., say, 20), the next visible rational will be 2 and the form seen will be a triangle (with rounded corners). If r is around 40, the next visible rational k will be $3/2$ corresponding to a five-pointed star; with larger r, one may next see $4/3$ (a seven-pointed star which winds around its center three times before repeating), and so forth.

In the case that the only visible value for k between 1 and 2 is $3/2$, the ellipse spins clockwise, faster and faster, then seems to dissolve into a counterclockwise-spinning five-pointed star which slows, stops, and then reverses spin increasingly fast in the clockwise sense, till another dissolve occurs into a rapidly spinning triangle (going counterclockwise), etc.

The higher the value of r, the higher the resolution of the process. That is, with a very large rotational speed, one can see many rationals near 1. For a fixed value of r, as we continue to increase k, a sequence of rational figures become visible. For example, the sequence might be $1, 3/2, 2, 5/2, 3, 3/2, \ldots$ or, for a larger value of r, one could see $1, 4/3, 3/2, 5/3, 2, 7/3, \ldots$ and so on. In fact, such sequences of rational numbers have been studied in number theory, where they are called *Farey sequences*.

The spins for these figures is analogous to the Doppler shift. Just as the pitch of a siren is increased as an ambulance approaches you while it suddenly falls as the ambulance begins to recede (spatial phase affects frequency), for mandalas a perturbation in frequency is expressed as a corresponding change in spatial phase.

The sequence of these rational "interpolants" may also be influenced by capabilities of the observer. For instance, it is known that the "flicker fusion frequency" (rate at which a blinking light is perceived to be constant - as when watching a movie) will differ from one person to another. How does this influence which mandalas can be seen?

Exactly the same questions arise for Lissajous figures. I've posed them above for mandalas because it is slightly easier to describe the resulting figures and their dynamics. In particular, Lissajous figures often have a 3-dimensional appearance, so additional phenomena may be involved in their perception.

The lower velocity of laser scanning (compared with oscilloscopes) enhances the perception of distinct topological forms [16], [18]). Since the scan rate for an oscilloscope is typically in the kilohertz range while laser scanners are normally in much lower frequencies - a few hundred herz (cycles per second) or less - a larger number of stable figures are able to be distinguished in the case of the oscilloscope so that topological stability is more delicate to observe.

Using digital simulation of the analogue process, Lissajous figures are available as screen savers (and one may also download Java applets which produce these figures). The observer seems to find constancy in the physical form of the perceived shape as though it were an actual entity made out of wire and revolving and twisting in space. The observer has this percept of constancy in spite of change in shape. This is the same as saying that there is constancy in the topology.

2 The phenomena

There are really three separate aspects: the optical display, the perceived image, and the neurobiological dynamics.

Using rotating or vibrating mirrors, e.g., and sunlight or laser light, one can project a bright spot onto a flat surface and cause the dot to trace some dynamical pattern. This can be easily put into some formal model, of Fourier type. As we describe below, the percept is usually of a closed figure which seems to undergo some rotation and other movement even when the actual optical trace is non-repeating. Our main focus will be quantitative. Which forms are seen and how fast do they seem to rotate when a particular optical rhythm is displayed?

There is also the question of how the brain determines the topology of such Lissajous figures and related resonance shapes and of what this tells us about the nature of biological computation. It is generally believed that we see them because of persistence of vision, the phenomenon involved in seeing movies as continuous motion rather than a sequence of still frames. However, in this paper, we argue that perception of these figures requires a more sophisticated type of mental computation.

We consider a dynamical systems model based on a theory due to Kolmogorov, Arnold, and Moser (three leading mathematicians of the 20th century). The theory claims that perturbations of periodic or almost periodic systems maintain a topological coherence due to the presence of the "invariant tori". Indeed, these invariant tori almost play the role of emergent objects.

3 Towards a model

Distinct theories may account for the perception of topological forms by humans. What is the "null hypothesis" for such percept of a closed form? Among all hypotheses, we should, by Occam's razor, prefer that which is simplest.

The dynamic activities of neurons is the superposition of a variety of distinct computations, most unrelated to any specific visual stimulus, so there is no reason to expect that the neural dynamics will be as simple as the dynamics of a spot of light moving on a flat surface. Yet we obtain a stable mental percept which seems to correspond to the changing visual patterns that result when the component oscillation frequencies are changed.

It is possible that apparent closure of a winding strand occurs when and only when the two strands are very near one another and in a parallel direction. For instance, zero is often drawn this way as a nearly closed curve. It would be interesting to see if such fusion occurs when a nearly closed curve is rapidly rotated about a central point. I conjecture that subjects will reliably report whether or not the cycle was closed in spite of the motion. In particular, I suspect that the hyperacuity of vision will detect lack of closure when the figure is not closed.

Put positively, the perception of closure may agree with a computation in which a complex winding non-closed form is *best approximated* by a simpler closed form which is spinning; see Appendix I for the detailed model.

Hypothesis: topological constancy for Lissajous figures utilizes the Kolmogorov-Arnold-Moser (KAM) theory. The KAM theorem shows that, for a small enough perturbation of an integrable system, most of the invariant tori are preserved [33, p. 326]. This can be interpreted in terms of quasiperiodic systems: if a qp-system is sufficiently close to a periodic one, then it is a perturbation of the periodic system. See also [25].

If the neurodynamics of the brain and visual system are driven by a mathematically defined stimulus, the activity patterns of the resulting neural activity should follow the rules of the KAM theory, providing a computational mechanism for the percept of topological constancy.

For an orientation-preserving diffeomorphisms f of the circle S^1, one defines the *rotation number* $\rho(f)$ in a variety of equivalent ways (e.g., [13, p. 296]). The KAM theory shows the existence of "a smooth transformation h ... taking a map with irrational rotation number, α, sufficiently close to rigid rotations, into rigid rotations," [13, p. 303] provided a certain condition holds. A related model studied by Glass and Perez [11] showed how phase locking could be produced by periodic stimulation of a nonlinear oscillator.

If, on the other hand, the discrimination cannot be made, then we can explore whether the same masking effect happens for different closed forms which include self-intersections in projection. As usual with such experiments, one needs to try binocular/monocular, color/black-and-white, and many other experimental parameters, not to mention age and culture of the subject and technology of the display.

A simpler variant of this problem involves the recognition of surfaces on which a variety of curves are most simply embedded - where psychophysical phenomena are used in the aid of mathematical exposition (see, e.g., [29]). More generally, we believe that psychophysical research can contribute to the depiction and understanding of mathematical phenomena.

4 Universality?

It is known that pigeons can respond to a Lissajous figure [7] but this may only depend on movement. Could pigeons learn to discriminate distinct topologies, say as a result of suitable rewards? Could humans from some isolated pre-scientific tribe make such identifications? Presumably they couldn't do it with language. Are some people more sensitive to visual patterns?

Another aspect of universality is the influence of neural and visual deficits. For example, experiments by Land (inventor of the polaroid process) and colleagues [31] shows that when the corpus callosum is cut, there is a decrease in the ability to "discount the illuminant" [24], indicating a role for coordinated cortical activity in the perception of color. Land has shown that one can perceive full color in the presence of only the information from two of the three primaries. But the split brain patient studied here had limited capability for such robust perception of color. Would split-brain patients have a similar decrease in ability to perceive topological constancy in Lissajous figure perception?

It is very difficult to judge the value of such experiments since they usually involve only a small test population due to the cost (in terms of resources) of instructing the subjects in the visual task and the difficulty and expense of collecting information on their performance. All psychophysical research involves a huge number of possibly contributing factors, and in the current era there are additional difficulties due to protocols involving human subjects. Thus, a formidable obstacle to progress is the extreme complexity and expense of experiments involving human capabilities.

5 Color

If one changes the color of a Lissajous figure, whether exactly rational or not, if the color changes are also periodic, with a harmonic relationship to the displayed form, one has the percept of a color flow around the figure.

If all ratios are exactly rational, then the figure seems to be fixed with a stable color; e.g., if both component oscillations are at the same rate, and the color oscillates at that rate being green half the time and red the other half, then one sees a circle, colored green on one half and red on the other. However, if the color rate is slightly off the vibrational rate, then the color will seem to flow around the figure - especially when the color is changing in a gradual fashion. That is, color is somewhat separated from the object, as has been utilized by artists who add an oil wash of color over a line drawing.

Since color is not accurately reproduced by television monitors, studying color effects can best be done with lasers. So this area might be investigated with, say, an argon ion laser which can have an oscillating color output, between green and gold. Mixing this with ordinary red helium neon laser light can produce a wide range of colors, even flesh tones, not normally associated with lasers.

6 Comparison with auditory phenomena

Perturbation of musical chords tends to sound either flat or sharp and if it is more than minimal, such distortion tends to be unpleasant to the ear. However the eyes are much more tolerant, and the perturbation is merely viewed as interesting. Perhaps this can provide some interesting methods for comparing the neural processing of visual and auditory information.

7 History of psychophysics

Origins of psychophysics can be traced back to Weber and Fechner about 1860 [38]. Several leading physicists in the latter half of the 19th century also worked in psychology and physiology: Helmholtz, Maxwell, and Mach, for example; e.g., [12].

However, in the literature I have not found any references to topology in vision (e.g., [34], [35], [36], [37]) with the exception of Zeeman's article on the topology of visual perception [39]. Lissajous figures do not appear to have been studied either.

Lewin introduced a topological model of the "life space" in 1936 [27], [26] but this doesn't seem to have been connected with perception. One might, however, include Gibson's notion of *affordance* or Rene Thom's topological ideas with regard to language [32].

8 Mathematical psychophysics

The study of topological constancy here is a special case of what might be called mathematical psychophysics, in distinction to the ordinary sort of psychophysics. Rather than merely examining the correlation between input data and neural and overt response by a subject, we introduce specific mathematical structure in the stimulus and also look for explicit mathematical structure in response.

Some work in this area has already begun. See, e.g., [9], [10], [1], which deal with the discovery by Glass that random dot patterns, when superimposed with a slight shift in angle or scale, produce a strong percept of mathematical forms such as circles or radial lines. This point of view is also being applied to other forms of moire pattern.

By combining work with neural networks (e.g., [21], [22], [23], [20]), it may be possible to apply the results of mathematical psychophysics to the automated recognition of patterns.

9 Why topological constancy matters

Although Lissajous figures appear to be rather special aspects of vision, it can be argued that they are important for a variety of reasons.

There is an eye movement, called *ocular microtremor* or OMT in the literature, which causes the eyes to vibrate at a rate between 50 and 200 Hz - see, e.g., [5]. OMT is known to be related to basic brain rhythms and, according to some recent research [4], the normal rate is 86 Hz with standard deviation of 6 Hz. Thus, one could expect (assuming a normal distribution) that 95 % of normal subjects will have OMT in the range of 74 to 98 Hz. But it may be that other activities and experimental conditions can affect this. Indeed, since research has only involved small subject populations, the above information is by no means definitive.

The main use made of OMT currently is to assess the depth of anaesthesia. It has also been explored as an indicator of various medical conditions such as multiple sclerosis and Parkinson's disease [4], [3]. Nevertheless, the technology for measuring OMT remains primitive - for instance, using a "strain gauge" which is physically attached to the eyeball. This seems little advanced over the method of using contact-mirrors. However, the natural reflectivity of the ocular surface can be used in conjunction with a very-low-intensity laser or infrared diode to measure eye movements noninvasively; see, e.g., [14] for a discussion of several alternatives.

However, we believe that OMT may also be important in ordinary visual tasks. It is easy to demonstrate that viewing through a screen, rather than through no screen, can emphasize regular patterns in the scene being studied. For example, looking at one screen (or even a brick wall) through another screen can make moiré patterns. Hence, an eye movement like OMT which can produce a screen-like pattern superimposed on the visual field might facilitate the detection of patterns. Since there are three pairs of muscles that control the movement of each eyeball, the resulting vibration should cause a spot of light to scan a Lissajous figure on the retina. The amplitude of OMT is just sufficient to sweep a point of light across the immediate neighbors of a retinal cell. This could also play a role in hyperacuity allowing computation to improve visual accuracy to "subsensor" resolution (as we are able to detect some visual features which subtend arcs below the size that can be distinguished by retinal cells of given diameter subject to diffraction-limited resolution). See, e.g., [2], [28], [15].

In addition, *artificial visual environments*, such as CRTs, may cause various vibrations in a displayed spot of light and, if these are at rates harmonic with the rates of eye movements, again Lissajous figures can result. This can actually be demonstrated by plucking a rubber band, held vertically, in front of a television monitor. Holding the rubber band horizontally, the flickering nature of ordinary CRTs causes one to see a stroboscopic effect so that the vibrating band appears to be in several "frozen" parts. But holding the rubber band vertically, the vibration interacts with the horizontal scan (which is at a much higher frequency) and one can sometimes see Lissajous figures as a result.

Thus, perception of Lissajous figures may be instrumental in the normal functioning of the eyes, both for texture and hyperacuity, as well as for perceiving artificially generated images and in the design of visual displays for education, communication, and entertainment.

In addition, the fact that we perceive non-closed Lissajous figures as though they were closed but spinning or tumbling may be a clue to how other nearly periodic phenomena arise within biology [19]. From an object-oriented point-of-view, closed figures produce smaller and simpler objects.

Further, the basic rhythms of the body are only approximately periodic (such as the circadian cycles of waking and sleeping), so the conversion of quasiperiodicity into perturbed periodicity in vision, as illustrated by topological constancy of Lissajous figures, could shed light on basic mechanism of biological computation.

10 Experiments

The number of distinct rational forms, whether or not hysteresis occurs, binocular disparity, direction of spin, and depth effects are just a few of the possible design parameters for experiments. One must also address issues of how to elicit the information from observers (verbally, by pushing buttons, by pointing to like images or movies on a screen, etc.). Can animals respond and is there actually measurable neural activity? Do distinct theories of dynamical system behavior make different predictions for topological constancy of Lissajous figure perception and related phenomena?

We propose to study first the selection by subjects of a static form which is "most like" a dynamic display. There are various means for displaying the required images, and our laboratory is currently engaged in exploring four of them:

(1) Java applets can provide a widely accessible window into the subject, where one could have sliders (or other virtual knobs) which control the component frequencies and buttons to click on to indicate a best match.

(2) Mathematica is capable of producing animated images and so it may also be able to produce suitable images. However, controlling the program is somewhat arcane (by comparison with Java) so a greater effort will be required for the interface. In contrast, Mathematica is more capable of carrying out related computations and interfacing with statistical packages in order to capture the subjects' behavior.

(3) Hardware-controlled displays offer the most natural means for studying perceptual phenomena since no additional artifacts are introduced. However, the issues of collecting the statistics and recording and controlling the corresponding parameter values must be addressed.

(4) Existing psychophysical software suites can possibly be adapted to our purposes. There are a variety of tools available. For example, many lab-crafted programs, mostly under Linux, are freely available, but these often have a sharp "learning curve" and require substantial set-up time and expertise. There are also commercially available softwares but these require a financial investment.

Since validation of the phenomenon of topological constancy in the perception of these figures of approximate resonance depends on a statistically significant sample of human subjects, the design for such experiments ought to incorporate the statistical requirement in the same way as polling or market research needs a sufficient sample.

For example, suppose we wish to determine a statistic, based on data, such that with probability 95% the actual fraction of people who can recognize two distinct Lissajous figures in a particular frequency range is within .03 of the statistically calculated fraction. It would take a sample size of about $n = 1100$ subjects to achieve this level of accuracy [6, p. 382]. Hence, attempting to make such a determination would require very extensive experimental data.

This is analogous to attempting to find the flicker-fusion frequency at which there is at most a five percent chance of error that the fraction of people who see a moving image is in the range between 47 and 53%.

On the other hand, if we only want to detect the existence of the phenomenon, then many fewer subjects might be needed. That is, we hope to find a way to demonstrate statistical significance for existence without the overhead required to quantify it.

Thus, the goal of our initial experiments will be to explore the phenomenon and to formulate definitive experiments which can be performed given suitable resources.

11 Appendix I

Here we formalize the mandala model. A counterclockwise circular rotation, of unit amplitude and rate r rotations per second is given by the function $\exp(-2\pi rt)$, t in seconds. To give the equation for the result of adding a counterclockwise circular rotation (of unit amplitude and rate r) to a clockwise circular rotation (of positive amplitude $a \leq 1$ and rate kr for $k \geq 1$), we write

$$M_{a,k,r}(t) := \exp(-2\pi irt) + a\exp(2\pi ikrt).$$

First let us suppose that $k = 1$. When $a = 1$, the resulting trace is a line; just as two linear oscillations (with the proper phase relationship) make a circle, two rotations can produce a linear oscillation. When $a < 1$, the figure is an ellipse with horizontal major axis of length $2 + 2a$ and minor axis of length $2 - 2a$, and the spot is rotating around the ellipse counterclockwise. If $a > 1$, the ellipse is oriented vertically instead and rotation of the spot is clockwise. In both cases, the ellipse itself remains fixed.

We may also consider the Lissajous case for comparison. Suppose that $L(t) := L(t; a, r, k, \phi)$ is the parameterized curve

$$x = \cos(\delta_{1,k}\phi + 2\pi rt), \quad y = a\ \sin(2\pi krt),$$

where $\delta_{u,v}$ is the Kronecker-δ (i.e., $\delta_{u,v} = 1$ for $u = v$ and 0 otherwise) so the phase term is zero for $k \neq 1$. If $a = 1$ and $k = 1$, then the resulting parameterized curve $L(t)$ traces a straight-line segment with slope 1 when ϕ is zero and a circle when $\phi = \pi/2$.

For a time-varying phase $\phi = (\pi/2)\ \sin(2\pi st)$, where $0 < s << r$, the resulting Lissajous figure $L(t)$ is seen as a rotating circle in 3-dimensional space, rotating about the slope-1 line in the image plane with s the number of rotations per unit time.

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