Book embeddings of graphs and a theorem of Whitney *

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Abstract

It is shown that the number of pages required for a book embedding of a graph is the maximum of the numbers needed for any of the maximal nonseparable subgraphs and that a plane graph in which every triangle bounds a face has a two-page book embedding. The latter extends a theorem of H. Whitney and gives two-page book embeddings for X-trees and square grids.

Key words. Hamiltonian planar graph, book thickness, nicely planar graph, girth, bipartite.

AMS subject classifications: 05C45, 05C10

1 Introduction

In this paper, we consider finite simple graphs (no loops or parallel edges); see, e.g., [11], [19]. Let $\mathrm{bdy}(e) = \{v, w\}$ denote the boundary vertices (endpoints) of an edge e.

By a closed half-plane, we mean a copy of the complex numbers with imaginary part ≥ 0. For k a positive integer, the *book* B_k is the singular surface formed from the union of k closed half-planes (the *pages*) intersecting

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in a line L (the *spine*) which is the boundary of each of the pages. In [15, p. 97], a *k-page book embedding* of a graph $G = (V, E)$ is defined to be an embedding of G into B_k which carries V to L with the property that each edge $e \in E$ is mapped into a single page so that $e \cap L = \text{bdy}(e)$. The *book thickness*, or *pagenumber*, of G, $bt(G)$, is the least number of pages in which G has a book embedding. Equivalently, $bt(G)$ is the least number of colors which suffice to color the edges of G so that no two edges of the same color intersect other than at a common endpoint if the vertices of G are arranged in order around the boundary of a circle and the edges are drawn as straight-line segments, minimized over all possible cyclic orderings of the vertices [3]. It is easy to see that G has book thickness 1 if and only if G is outerplanar and $bt(G) \leq 2$ if and only if G is a subgraph of a planar Hamiltonian graph [3].

Book thickness has turned up in rather diverse applications such as fault-tolerant computing and VLSI (Chung, Leighton, and Rosenberg [5]), computational complexity and graph separators (Galil, Kannan, and Szemeredi [9]), software complexity metrics and vehicle traffic engineering (Kainen [16]), and "bisecondary structures" used for modeling RNA folding energy states (Gleiss and Stadler [10]). In addition, many theoretical questions remain open such as the relationship of book thickness to other invariants [6], [16], and the book thickness of standard and useful families of graphs (e.g., [17], [13], [12], [8]).

Recall that a graph is planar if it can be embedded in the plane; the actual embedding is called a plane graph. We call a graph *nicely planar* if it has a plane embedding in which each triangle bounds a region.

In this paper, we investigate some connections between book thickness and planarity. Yannakakis has shown [21] that the book thickness of a planar graph cannot exceed 4 and examples are known of maximal planar graphs with no Hamiltonian cycle so 3 (and perhaps 4) pages are certainly necessary for some planar graphs.

Our main result is that a nicely planar graph is a subgraph of a Hamiltonian planar graph. We then apply the result to give simple proofs that some useful graphs in computer science have two-page book embeddings, previously shown by direct computations [5]. The proof of our main result requires a lemma of independent interest: $bt(G) = \max bt(B)$, where B is a maximal nonseparable subgraph of G.

In section 2, we study book thickness in terms of the block-cutpoint tree.

Section 3 gives the main result, while section 4 has applications.

2 Book thickness and block decomposition

A graph is nonseparable if it has no cutpoints; a *block* is a maximal nonseparable subgraph [11, p. 26]. A family of subsets of some given set determines an *intersection graph* where vertices correspond to subsets and adjacency holds if and only if the corresponding subsets have nonempty intersection. The intersection graph formed by the cutpoint singletons and the vertex sets of the blocks is a forest, called the *block-cutpoint forest* [11, p. 36]. The points of the intersection graph which have degree 1 must be blocks since every cutpoint belongs to more than one block. A graph is connected if and only if its block-cutpoint forest is a tree.

Theorem 2.1 *The book thickness of a graph is the maximum of the book thicknesses of its blocks.*

Proof. Let G be a graph. Without loss of generality we assume G is connected. We prove the result by induction on the number of blocks. The basis case of one block is trivial.

Consider the block-cutpoint tree of G which by assumption is not trivial. There is a degree one vertex corresponding to a block H with a unique cutpoint v and $G' = G - (H - v)$ has one fewer block than G. By the inductive hypothesis, G' has book thickness equal to the maximum of that of its blocks. So we now need only show that G requires no more than the maximum of the book thicknesses of H and G'. Take minimum book thickness embeddings of H and G' and, by rotating the vertices along the spine if necessary, make v the first vertex of each embedding and place the two embeddings consecutively along the spine of a book, using the maximum number of pages in the two embeddings - WLOG, take G' second. Now pull the copy of v in the second embedding above H and superimpose it on the first copy of v. Hence, $bt(G) = \max\{bt(G'), bt(H)\}$, as required. □

Note that the genus of a graph is the sum of the genera of its blocks (Battle, Harary, Kodama, and Youngs [2]). For book thickness, maximum replaces sum. With respect to block decomposition, genus is to book thickness as the l_1 norm is to the l_∞ norm.

This type of distinction separates invariants, such as the cyclomatic number, which are measures of global complexity (additive over the blocks) from invariants, such as clique size, which are local measures of complexity and have values obtained by maximizing over the blocks.

We wonder whether there are interesting graph theoretic complexity measures corresponding to the l_2 (Hilbert) norm. Note that the global complexity measures given here do not distinguish homeomorphic graphs, while those of local type may have very different values for homeomorphic graphs; see the last section.

3 Extending Whitney's theorem

A *triangulation* is a planar graph with a maximal set of edges. Equivalently, a graph is a triangulation if it is isomorphic to a plane graph in which every face, including the face which contains infinity, has exactly three edges in its boundary. A *triangle* in a graph is a cycle of length 3.

By the Jordan Curve Theorem, in a plane graph, any triangle divides the plane into an interior and an exterior region. A triangle is *bounding* if either its interior or exterior region contains no vertices from the graph. A triangle is *separating* if its deletion increases the number of connected components of the graph. In a triangulation, a triangle is separating if and only if it is not bounding. If there are no separating triangles in a planar graph, then the graph is nicely planar, and the converse holds if the graph is 2-connected.

As an example, consider the plane graph G which consists of two connected components: two disjoint triangles side by side with three additional edges all attached like a bonnet at the top vertex of the first triangle, pointing up into the outer region of the first triangle. Removing the first triangle increases the number of connected components from 2 to 4, the components now being the second triangle and three isolated vertices. The first triangle is separating but also bounding and the second triangle also bounds so G is nicely plane.

A graph is *Hamiltonian* if it has a cycle through all of the vertices. A graph is *subhamiltonian* if it is a subgraph of a planar Hamiltonian graph.

The following theorem is due to H. Whitney [20].

Theorem 3.1 *Every triangulation with no separating triangles is Hamiltonian.*

A plane graph G' is said to be obtained by *stellating* a face F of the plane graph G if G' is obtained from G by adding a new vertex F^* in the interior of F and continuous curves joining it to each vertex in the boundary of the face in such a way that the curves intersect G only at their endpoints and the curves are disjoint except for F^*. See, e.g., [3].

Theorem 3.2 *Every* 3*-connected planar graph with no separating triangles is subhamiltonian.*

Proof. Choose a plane embedding of the given graph G and extend it by stellating any nontriangular faces. The resulting graph G' is clearly a planar triangulation and we claim that it has no separating triangles. Indeed, suppose T were a separating triangle in G'. Then at least one of the vertices is new since G had no separating triangles. But the only new vertices were added as stellation points so $T = \{v, x, y\}$, where v is a stellation point and x,y are vertices in the boundary of the face which was stellated by v. It is straightforward to verify that $\{x, y\}$ would be a separating set for G so no separating triangles exist in G' [18]. By Whitney's Theorem, G' and hence G are subhamiltonian. □

Lemma 3.3 *Let G be a nicely planar block. Then there is a* 3*-connected planar graph G' with no separating triangles such that G is a subgraph of G'.*

Proof. We induct on the number of separating 2-sets among the vertices of G. If there are none, then G is already 3-connected so the basis case of the induction holds with $G' = G$.

Suppose G has $k \geq 1$ separating 2-sets and take $A = \{u, v\}$ to be one of these separating 2-sets. Choose a particular plane embedding of G. Let $G - A = G_1 \cup \ldots \cup G_n$ with G_j denoting the connected components obtained by removing A, listed in the clockwise order of the edges joining them to v. If vu is an edge, we may renumber the components so that the edge from v to u precedes all the edges from v to G_1, and after all those from v to G_n. With respect to the same fixed clockwise order at v there is a last vertex v_j in G_j and a first vertex w_j in G_{j+1}, for $j = 1, \ldots, n-1$. Add $n-1$ vertices y_j together with the edges $y_j v_j$, $y_j w_j$, and $y_j v$, keeping the resulting graph G' plane. Indeed, G' is nicely plane since the added triangles all bound disks.

Hence, it suffices to show that (1) G' is 2-connected and (2) G' has fewer separating 2-sets than G. The assertion (1) follows from the "ear" decomposition characterization, e.g., [19, p. 163]. One can form G' by adding paths (ears) in the order $v_1, y_1, w_1,\ vy_1,\ v_2, y_2, w_2,\ \ldots\ ,vy_n$.

Assertion (2) follows from the claims: (i) No added vertex is a member of a separating 2-set for G', (ii) $A = \{u, v\}$ is not a separating 2-set for G', and (iii) if B is any separating 2-set for G', B is a separating 2-set for G.

These may be established as follows: For (i), first note that for B any separating 2-set of G', exactly one of the vertices must be new. Since the old vertex can't be a cutpoint of the block G, one component of $G' - B$ consists of a single new vertex. But this is impossible since each new vertex is adjacent to 3 old ones. For (ii), let $A = \{u, v\}$ as given above in the definition of G'. By our construction, $G' - A$ is connected so A is not a separating set for G'. Last, for (iii), let B be a separating 2-set for G'. No component of $G' - B$ is a single new vertex by the same degree argument as above. Hence, each component contains at least one vertex of G, so B is a separating 2-set for G. See [18] for a similar and slightly more detailed argument. □

Theorem 3.4 *Every nicely plane graph is subhamiltonian.*

Proof. By Lemma 3.3 and Theorem 3.2, each block is subhamiltonian and this suffices by Theorem 2.1. □

Note that we cannot always add edges alone to a nicely plane graph in a way to satisfy the conditions of Whitney's theorem. For example, a square with one stellated face is nicely plane but extending to a plane graph by adding one more edge forces a separating triangle.

Our method adds both vertices and edges to avoid creation of separating triangles - hence, enabling the use Whitney's Theorem to obtain a Hamiltonian cycle. However, once such a cycle is created, one can delete both the added edges and vertices of the construction, keeping the ordering for the original vertices. By adding edges joining any nonadjacent pairs of consecutive vertices, a Hamiltonian planar graph is obtained which contains the original nicely planar graph as a spanning subgraph. The same argument shows that in the definition of subhamiltonian graph, one can require that the extension only involve the inclusion of new edges.

4 Applications

An X-tree is the plane graph formed by taking a complete plane binary tree oriented with the root at the top (with branching down and to left and right, resp.) and adding to it horizontal paths which join all vertices at the same distance to the root. Let us call an *extended* X-tree the result of using an additional edge for each of the added horizontal paths, joining the endpoints of the path to form a cycle unless these two vertices were already adjacent. The following is an immediate consequence of Theorem 3.4 since an extended X-tree is nicely plane; cf. Chung, Leighton and Rosenberg [5]

Corollary 4.1 *Every extended X-tree (and hence every X-tree) is subhamiltonian.*

In [5] there is also an explicit proof that the product of two paths (called a *square grid*) is subhamiltonian. This follows from the next corollary.

Recall that the *girth* of a graph is the length of its shortest cycle. Graphs with girth > 3 are triangle-free and so if they are planar, then they are nicely planar. In particular, every bipartite planar graph has a 2-page book embedding.

Corollary 4.2 *Every planar graph with girth > 3 is subhamiltonian.*

Theorem 3.4 has implications for two open questions. Barnette asked if every cubic 3-connected planar bipartite graph is Hamiltonian; see, e.g., [14]. By Corollary 4.2 such a graph is at least subhamiltonian. Chartrand, Geller and Hedetniemi [4] conjectured that every planar graph can be written as the edge-disjoint union of two outerplanar graphs. By Theorem 3.4, their conjecture is true for nicely planar graphs.

Recall that two graphs are *homeomorphic* if they have isomorphic subdivisions.

It is well known that every graph is homeomorphic to a graph of book thickness at most three (see Atneosen [1], Bernhart and Kainen [3], and, according to Jozef Przytycki, also a dissertation by G. Hotz, a student of Reidemeister). Applications of three-page embeddings have been made to links by Dynnikov [7].

In contrast, for planar graphs, two pages suffice up to homeomorphism.

Theorem 4.3 *A graph is planar if and only if it is homeomorphic to a graph of book thickness at most two.*

Proof. Let G be any planar graph and let G' be any subdivision of G such that all cycles have even length. For instance, one may take the first barycentric subdivision obtained by subdividing each edge exactly once. By Corollary 4.2, G' is subhamiltonian. □

Enomoto and Miyauchi [8] consider "homeomorphic book embeddings" where edges can cross the spine (i.e., an edge may use more than one page), and they note that a graph is planar if and only if it has a homeormorphic book embedding in two pages. It follows from our results above and some standard topological graph theory arguments that a planar graph with p vertices has a 2-page homeomorphic book embedding with at most $p - 2$ crossings of the spine.

5 Appendix

Although it is well-known, we give a brief sketch of the proof that the genus of a connected graph is the sum of the genuses of its blocks. It suffices to show that the genus of the wedge sum of two graphs is equal to the sum of the genuses of its two parts. Suppose that $G = G_1 \wedge_v G_2$, where v is the only common vertex of the induced subgraphs G_1 and G_2 of G. Take minimum genus embeddings of G_1 and G_2 in surfaces of genus γ_1 and γ_2, resp.

Choose regions R_1 and R_2 in the two embeddings such that both regions contain v in their boundary. Attach a handle to the interior of both these regions to form a surface of genus $\gamma_1 + \gamma_2$, and add an edge e which joins the two distinct copies of v. Contracting e does not increase the genus of the resulting surface in which the wedge-sum G is now embedded. This shows the easier half of the equality.

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