Key for Final exam for Math 203

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Abstract

Here are brief solutions to the problems.

Recall that a poset (partially ordered set) is a pair $(P, \geq)$, where P is a nonempty set and $\geq$ is a relation on P satisfying the two properties P1: $x \geq y$ and $y \geq x$ if and only if $x = y$ and P2: $x \geq y, y \geq z$ implies $x \geq z$.

Exercise 1 *Show that in a poset, any zero-object is unique.*

Proof. This is a variant of the argument that identities are unique. Let $0, 0'$ be two zero-objects in a poset. Then each is greater than or equal to the other so by property P1, they are equal. $\square$

A *boolean algebra* is a lattice with 0 and 1 which is distributive and complemented. For example, the family of all subsets of a given set S, with inclusion as the order relation, defines a lattice with S as the 1-object (everything is a subset of S) and the empty set $\emptyset$ as the 0-object. Complements are just set-theoretic complement and $\cup$ (join or lub) means union, while $\cap$ (meet or glb) means intersection. The distributive law holds for the lattice of subsets of S since $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Indeed, any element in the LHS is in A and in either B or C. In the first case, it is in $A \cap B$, etc. (The opposite inclusion is trivial since $A \cap C \subseteq A \cap (B \cup C)$, etc.)

Exercise 2 *Show that, in a boolean algebra, the complement of any element is unique. After this, we will denote the complement of a by a'.*

Proof. If B is a boolean algebra and if x and y are both complements to a; i.e., $a \cup x = 1 = a \cup y$ and $a \cap x = 0 = a \cap y$, then (using the distributive law for the third equality)

$$y = y \cap 1 = y \cap (a \cup x) = (y \cap a) \cup (y \cap x) = 0 \cup (y \cap x) = x \cap y.$$

By symmetry, $x \cap y = y \cap x = x$ so $y = x$. Hence, the complement of an element is unique. □

A ring R is called a *boolean ring* provided that $x^2 = x$ for every element x in R (i.e., every element is multiplicatively idempotent). We dealt with boolean rings on the Midterm, where it was shown that a boolean ring is automatically commutative and of characteristic 2 so $x + x = 0$ for every $x \in R$.

Given a boolean algebra, we can form a boolean ring from it in the following way: Let $R = B$, define multiplication in R to be $\cap$ in B. By L3 (in the written notes) $a \cap a = a$ for any lattice - indeed, the glb of a with itself is clearly again a. The addition for R is a bit less obvious. Define, for all $x, y \in R$, $x + y = (x \cap y') \cup (x' \cap y)$. It is straightforward, though not trivial, to check that with $\cdot = \cap$ and $+$ as just given, R is a ring. By L3, R is a boolean ring and moreover R has 1 for its multiplicative identity (since $1 \cap x = x$ for all x). In the notes we showed that one can also write $x + y = (x \cup y) \cap (x \cap y)'$. For the boolean algebra formed from the subsets of S, the corresponding ring-sum is called *symmetric difference* of sets.

One can show that the converse holds. That is, if R is any boolean ring with identity, there is a boolean algebra B such that R is obtained from B by the process described in the preceding paragraph. Of course, we take $B = R$ as underlying sets and define $x \cap y$ in B to be xy in R - meet and multiplication are the same. Put $x \cup y = x + y - xy$. We actually verified in class that this operation gives an associative operation in any ring with 0 as neutral element.

There are other things to check (i.e., L1 to L4), but I'm only asking

Exercise 3 *Show that B is distributive: Prove that for all a, b, c in B*

$$(a \cup b) \cap c = (a \cap c) \cup (b \cap c).$$

Proof. By definition, $(a \cup b) \cap c = (a + b - ab)c$ and by the distributive law for rings, this equals $ac+bc-abc$, while $(a \cap c) \cup (b \cap c) = ac+bc-(ac)(bc) = ac + bc - abcc = ac + bc - abc$, where the last two equalities use the fact that B is commutative and multiplicatively idempotent. □

Here is a connection with logic. Let B be a set of logical propositions (either true or false) which is closed under formation of "and" and "or" - e.g., the statement "P or Q" is true if and only if either P or Q is true. Calling these operations meet and join, respectively, makes B into a lattice.

Exercise 4 *Determine the corresponding order relation?*

The order associated with lattices defined by meet and join (i.e., sets and binary operations which satisfy L1, ... , L4) is given by $a \geq b$ if and only if $a \cup b = a$. Now it is easy to check (using the definition of "or") that the equality $a \cup b = a$ is false if and only if a is false but b is true, and this is also the only way in which the implication $b \Rightarrow a$ is false. Thus, $b \leq a$ is the same as saying that b logically implies a. Note that transitivity (P2) clearly holds for logical implication, but we should actually replace statements by equivalence classes of statements to guarantee that P1 holds as well. In particular, a statement which is false logically implies any other statement so False corresponds to 0 and similarly True corresponds to 1.