Quadrilateral embedding of $G \times Q_s$ *

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Abstract

It is shown that for any connected graph G and all sufficiently large s, the cartesian product $G \times Q_s$ has a quadrilateral embedding in some surface, where Q_s is the hypercube graph. This answers a question of Pisanski.

Keywords: quadrilateral embedding, Cartesian product, hypercube, stable map, stability number of a graph, closed 2-cell embedding

1 Introduction

Pisanski asked [12]: When are the cartesian products of a given graph with all sufficiently high-dimensional hypercubes embeddable in surfaces so that every region is a quadrilateral? It suffices to find such a quadrilateral embedding in which some subfamily of the regions have boundaries including each vertex once and only once. We call the least dimension of such a hypercube the "stability number" of the graph.

Archdeacon [1] showed that every 4-connected graph has an embedding in some not-necessarily-orientable surface in which all regions have boundaries which are cycles. We use this result to show that every connected graph has finite stability number.

In this paper, graphs are finite and simple (no loops or parallel edges); $G = (V, E)$ denotes a graph with vertex-set V and edge-set E. By a closed surface, we mean a 2-manifold without boundary, which may be orientable or not.

An embedding M of a graph G in a surface S is called a 2-*cell embedding* if the path-connected components (necessarily open) of $S \setminus G$ are homeomorphic to disks (called the *regions* of the embedding); let $\mathcal{R}(M)$ denote the set of regions. For undefined terms, see e.g. Harary [6]; see also [4], [7], [10], [17].

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Two 2-cell embeddings of a graph are *equivalent* if there is a homeomorphism of the surface carrying one embedding onto the other; if the surfaces are orientable, then the homeomorphism must be orientation-preserving. A *map* M is an equivalence class, denoted $M : G < S$; Euler's formula holds: $n - m + r = e(S)$, where $n = |V|$, $m = |E|$, $r = |\mathcal{R}(M)|$, and $e(S)$ is the Euler characteristic of the surface (2 for the sphere, 1 for the projective plane, etc.). When the surface is orientable, $e(S) = 2 - 2\gamma$, where γ denotes the genus of the surface. Let $U(M) = G$.

A map is *closed* provided that the closure of every region is a closed 2-cell; i.e., every region boundary is a cycle. A map is called *quadrilateral* provided that all regions have 4 distinct edges in their boundaries. If $M : G < S$ is a closed map, then G must be 2-connected, and well-known conjectures assert that each 2-connected graph has such maps (even into an orientable surface); see below and Haggard [5]. A quadrilateral map M is closed if $U(M)$ is 2-connected.

A 2-cell embedding of G is *simplest* if the Euler characteristic of the surface is as large as possible. Note that if G is connected and triangle-free, then a quadrilateral map of G on some surface is necessarily a simplest embedding. When the surface is orientable, a simplest embedding is the same as an embedding of minimal genus.

Let $G \times H$ denote cartesian product; $V(G \times H) = VG \times VH$ with $[(v, w), (v', w')] \in E(G \times H)$ if and only if either $v = v'$ and $[w, w'] \in EH$ or $w = w'$ and $[v, v'] \in EG$. For $d \geq 1$, the hypercube Q_d is the d-fold iterated cartesian product of K_2.

Recall that a graph G is k connected if for any two distinct vertices v and w there are at least k paths from v to w in G which are pairwise-disjoint except for their common endpoints. It is easy to see that $G \times Q_1$ is $k+1$-connected if G is k-connected.

A map has a *vertex-disjoint quadrilateral cover* (VDQC) if there exists a set of 4-sided regions such that every vertex is in the boundary of one and only one of these regions. This is a special case of what Pisanski called a "patchwork" (see [12], [4, p. 155]). We call a quadrilateral map with a VDQC *stable*. If G has a stable map, then so do the supergraphs $G \times Q_d$ for all $d \geq 1$. Such maps are used in [2], [8], [9], [14], [15], [16].

It is shown that for any connected graph G there is a positive integer s such that $G \times Q_s$ has a stable map. In section two, the process of extending maps from G to $G \times K_2$ is formalized, and the third section shows that suitable extensions do exist for all connected graphs G. We conclude with some examples.

2 Maps and their extensions

Let $M : G < S$ be any map with region-set $\mathcal{R}(M)$. We call a function $\varphi : V(G) \to \mathcal{R}(M)$ a *region assignment* if $\forall v \in V(G),\ v \in V(\partial\varphi(v))$; i.e., each $v \in V(G)$ belongs to the boundary of the region $\varphi(v)$. We define the *demand* on a region R to be the number $d(R, \varphi)$ of vertices assigned to it by φ,

$$d(R, \varphi) = |\varphi^{-1}(R)|.$$

Given any vertex-disjoint family $\mathcal{F}$ of regions in M which covers every vertex in G, there is a unique region assignment $\varphi = \varphi_{\mathcal{F}}$ such that the demand on each region in $\mathcal{F}$ is the number of vertices in its boundary.

Call a map $N : G \times Q_1 < S'$ the 1-*extension* of M by φ provided that φ is a region assignment for M and (i) S' comes from two copies of S by joining corresponding regions with nonzero demand by a handle, (ii) for each v in $V(G)$, the edge $[(v, 0), (v, 1)]$ of $G \times Q_1$ runs along the handle joining the two copies of region $R = \varphi(v)$, and (iii) all remaining edges are in the two copies of S where they are embedded as in M. The map N is a 1-extension of M if there exists φ such that N is the 1-extension of M by φ, and for $k \geq 2$ a k-extension of M is a 1-extension of a $k-1$-extension of M.

Call a region assignment φ *coherent* for the edge $e = [v, w]$ of G provided that $\varphi(v) = \varphi(w)$, and $[v, w] \in E(\partial\varphi(v))$. The following summarizes our construction.

Lemma 1. *Let $M : G < S$ be a map with region assignment φ which is coherent for some edge $e = [v, w]$ and let N be the 1-extension of M by φ. Then the 4 edges $[(v, 0), (w, 0)]$, $[(v, 1), (w, 1)]$, $[(v, 0), (v, 1)]$, and $[(w, 0), (w, 1)]$ constitute the boundary of a quadrilateral region in N.*

A 1-factor of G is a spanning subgraph which is regular of degree 1. For any map M put $d(M) = \max d(\varphi)$, over all region assignments φ, where $d(\varphi)$ is the minimum of the nonzero $d(R, \varphi)$ for $R \in \mathcal{R}(M)$.

Lemma 2. *Let G be a 2-connected graph with a 1-factor F and let $M : G < S$ be a closed map. Then $d(M) \geq 2$ and M has a closed 1-extension N.*

Proof. For each edge $f \in F$, let R_f be one of the two regions of M with f in their boundary. For $v, w \in V(G)$ with $f = [v, w]$, put $\varphi(v) = R_f = \varphi(w)$. This defines a region assignment φ which is coherent for each edge in F and hence has demand $d(\varphi) \geq 2$. Let $N = N(\varphi)$ be the corresponding 1-extension of M.

Each handle carries at least 2 distinct edges. Since M is a closed embedding, the boundary of a region is a simple cycle and so each added handle is topologically an annulus. The interior of an annulus with two or more radial lines connecting the two components of the boundary is a union of closed disks. Hence, N is also closed. $\square$

Is it true that $d(M) \geq 2$ for every map M?

Lemma 3. *Let $M : G < S$ be closed and let $\mathcal{F}$ be a covering of the vertices of G by pairwise-vertex-disjoint regions of M. Let M' be the 1-extension of M by $\varphi_{\mathcal{F}}$. Then M' is also closed.*

Proof. In the 1-extension, each region in $\mathcal{F}$ is replaced by a handle which is subdivided into quadrilateral regions. The remaining regions of M, and hence of M', have boundaries which are cycles. $\square$

3 Stability numbers

The *stability number $st(M)$ of a map M* is the least nonnegative integer s such that M has an s-extension which is a stable map. The *stability number $st(G)$ of a graph* G is the least s such that $G \times Q_s$ has a stable map in some surface (orientable or not). Note that $st(G) = \min\{r + st(N) \mid N : G \times Q_r < S,\ N \text{ closed}\}$ and $st(G) \leq st(G \times Q_r) + r$.

Theorem 3.1. *Let G be a 2-connected graph with a 1-factor F. Every closed map $M : G < S$ has a 2-extension to a closed map $M_2 : G \times Q_2 < S'$ such that for each $v \in V(G)$, $v \times Q_2$ bounds a cycle in M_2. In particular, M_2 has a VDQC.*

Proof. Use the 1-factor to extend M to N as in Lemmas 1 and 2 above. Let $\mathcal{F}$ be the vertex-disjoint quadrilateral cover $f \times Q_1$ of N determined by the edges f of the matching of G. Then the extension of N with respect to $\varphi_{\mathcal{F}}$, as in Lemma 3, is a closed map M_2 and $v \times Q_2$, $v \in V(G)$, is a VDQC. $\square$

For any map $M : G < S$, let $\mathcal{N}(M)$ denote the family of vertex sets determined by the boundaries of the non-quadrilateral regions of M and let $\mathcal{I}(\mathcal{N}(M))$ denote the *intersection graph* of $\mathcal{N}(M)$ (that is, the vertices are the non-quadrilateral regions and two such regions are adjacent if they share at least one common vertex in their boundaries). Let $c(M)$ denote the chromatic number of $\mathcal{I}(\mathcal{N}(M))$.

Theorem 3.2. *Let G be a 2-connected graph with a 1-factor. Then every closed map $M : G < S$ has an s-extension to a stable map for some $s \leq 2 + c(M_2)$.*

Proof. Let M_2 be an extension of M as in 3.1 above. If $\mathcal{N}(M_2)$ is empty, then M_2 is already stable. Otherwise, the graph $\mathcal{I}(\mathcal{N}(M_2))$ has a finite nonzero chromatic number c and we may partition the family of all non-quadrilateral regions $\mathcal{N}(M_2)$ into c families $\mathcal{F}_1, \ldots, \mathcal{F}_c$, where each $\mathcal{F}_j$ consists of non-quadrilateral regions which are pairwise-vertex-disjoint.

Let $\mathcal{Q}_1$ be the set of regions in M_2 bounded by those $\{v \times Q_2\}$ which are vertex-disjoint with all regions in $\mathcal{F}_1$. The family $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{Q}_1$ covers every vertex of M_2 once and only once. Indeed, the non-quadrilateral regions of M_2 must come from non-quadrilateral regions of N. But each non-quadrilateral region of N is either a copy of some non-quadrilateral region of M which has zero demand with respect to the region assignment φ or else it arises from one of the connected components C of ∂R for some region $R \in \mathcal{R}(M)$ with $d(R, \varphi) > 0$, where C is a path with $r \geq 3$ vertices. The resulting region R' of N has $2r$ sides and both copies of M have the same vertices in the boundary of R.

The 1-extension M_3 of M_2 by $\varphi_{\mathcal{F}}$ is closed and has a VDQC by Lemmas 2 and 3. All regions in $\mathcal{F}_1$ are replaced by quadrilateral regions and the intersection graph of the remaining non-quadrilateral regions (if any) is unchanged except for being replicated, which does not affect the chromatic number of the intersection graph.

Repeating this process, extend the VDQC for M_3 to a VDQC for M_4 where all non-quadrilateral regions in $\mathcal{F}_2$ are replaced by families of quadrilateral regions, and iterate until reaching M_{2+c} which is stable and closed. □

Pisanski [11] proved that for every connected bipartite graph G, $st(G) \leq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. He further showed [12] that if G is a connected r-regular triangle-free graph, then $st(G) \leq 2r + 3$. See also [3] and [13].

Theorem 3.3. *Every connected graph has a finite stability number.*

Proof. If H is any connected graph, then $G = H \times Q_3$ is 4-connected and Archdeacon [1], [10, p. 153], has proved that any 4-connected graph G has a closed 2-cell embedding $M : G < S$. Since G does have a 1-factor, we can apply Theorem 3.2 to M. □

4 Examples

The following is weaker by 1 than the bound of Pisanski but uses an orientable embedding. If $K_{1,n}$ is the star with $n + 1$ vertices, then for $n \geq 3$ $st(K_{1,n}) \leq n + 2$.

Consider $G = K_{1,n} \times Q_1$ which is 2-connected with a 1-factor F, corresponding to the edge of Q_1. There is a unique (closed) map M of G

in the sphere, which has 2 quadrilateral regions and $n-1$ regions with 6 sides. Note that G has $2n+2$ vertices and $3n+1$ edges. We tabulate such information in the sequence

$$\sigma(M) = (2n+2, 3n+1, 4^2, 6^{n-1}, 2),$$

where the last term denotes the Euler characteristic of the sphere. The number of regions is $r = 2+n-1 = n+1$, and Euler's formula holds: $2n+2-(3n+1)+n+1 = 2$.

Define a region assignment $\varphi : V(G) \to \mathcal{R}(M)$, using F, so that, by the construction of Lemma 2, we obtain a map N with sequence

$$\sigma(N) = (4n+4, 8n+4, 4^{n+5}, 12^{n-1}, 4-2n).$$

To see this, note that one of the quadrilateral regions of M has demand 4 and so determines 4 quadrilateral regions in N, while the other quadrilateral region of M is merely replicated. Each of the $n-1$ regions with 6 sides has demand 2 and so determines a quadrilateral region and a region with $12 = 2*6$ sides. Since $2+4+n-1 = n+5$, we obtain the sequence shown, and the reader can easily check that Euler's formula does hold for characteristic $4-2n$. The two copies of M on spheres are joined by n handles so the surface in which N is embedded is orientable with genus $n-1$, and so has Euler characteristic $2-2(n-1) = 4-2n$.

Proceeding as in Theorem 3.1, we obtain M_2 described by

$$\sigma(M_2) = (8n+8, 20n+12, 4^{4n+12}, 12^{2n-2}, 6-6n).$$

Indeed, there are $n+1$ quadrilateral regions in N corresponding to the 1-factor, each of which produces a handle with 4 quadrilateral regions in M_2 and 4 more quadrilateral regions in N which are replicated, so $4n+12 = 4(n+1)+2*4$. Also, $8n+8-(20n+12)+6n+10 = 6-6n = 2-2(3n-2)$ and the genus $3n-2$ of the map M_2 agrees with the calculation $(n-1)+(n-1)+(n+1)-1$.

It is easy to check that $c(M_2) = n-1$ so, by Theorem 3.2, $st(M) \leq n+1$ and hence $st(K_{1,n}) \leq n+2$ for $n \geq 3$.

Our methods work for odd cycles as well. For $r \geq 1$, let $G = C_{2r+1} \times Q_1$ with 1-factor as above and let M be the unique map of G in the sphere. The connected components of the graph $\mathcal{I}(\mathcal{N}(M_2))$ consist of a cycle of length $2r+1$ joined to a pair of non-adjacent vertices, so the corresponding chromatic number is 4. Hence, $st(C_{2r+1}) \leq 7$. This agrees with the bound of Pisanski which holds for $r \geq 2$.

It is clear that many graphs have low stability numbers which result from special, regular embeddings. For example, $st(C_{2r}) = 2$ for $r \geq 3$. Also, the stability number of the complete bipartite graph $st(K_{3,3})$ is 1; indeed,

$K_{3,3}$ has a closed embedding in the projective plane with one 6-sided region and 3 quadrilateral regions. Attaching handles to the hexagonal regions, gives a stable embedding of $K_{3,3} \times Q_1$ in the Klein bottle. One can also show the stability number is at most 6 for the dodecahedron and at most 8 for the Petersen graph.

References

[1] D. Archdeacon, Face colorings of embedded graphs, *J. Graph Theory* **8** (1984) 387–398.

[2] L. Beineke and F. Harary, The genus of the n-cube, *Canad. J. Math.* **17** (1965) 494–496.

[3] C. P. Bonnington and T. Pisanski, On the orientable genus of the Cartesian product of a complete regular tripartite graph with an even cycle, *Ars Combinatoria* **70** (2004), pp.301–307.

[4] J. L. Gross and T. W. Tucker, **Topological Graph Theory**, Dover Publications, New York, 2001.

[5] G. Haggard, Edmonds' characterization of disc embeddings, *Congr. Numer.* **19** (1977), 291–302.

[6] F. Harary, **Graph Theory**, Addison-Wesley, Reading, MA, 1969.

[7] P. C. Kainen, Some recent results in topological graph theory, in **Graphs and Combinatorics**, R. A. Bari and F. Harary, Eds., Springer Lecture Notes in Math. 406, Berlin, 1974, pp. 76–109.

[8] P. C. Kainen, On the stable crossing number of cubes. *Proc. Amer. Math. Soc.* **36** (1972) 55-62. 05C10

[9] P. C. Kainen and A. T. White, On stable crossing numbers, *J. Graph Theory* **2** (1978) 181-187.

[10] B. Mohar and C. Thomassen, **Graphs on Surfaces**, Johns Hopkins U. Press, 2001.

[11] T. Pisanski, Genus of Cartesian products of regular bipartite graphs, *J. Graph Theory* **4** (1980), no. 1, 31–42.

[12] T. Pisanski, Orientable quadrilateral embeddings of products of graphs, Algebraic graph theory (Leibnitz 1989), *Discrete Math.* **109** (1992), no. 1-3, 203–205.

[13] T. Pisanski, Nonorientable genus of Cartesian products of regular graphs, *J. Graph Theory* **6** (1982), no. 4, pp. 391–402.

[14] G. Ringel, Uber drei kombinatorische Probleme am n-dimensionalen Wuerfel und Wuerfelgitter. *Abh. Math. Sem. Univ. Hamburg* **20** (1955) 10–19.

[15] G. Ringel, On the genus of the graph $K_n \times K_2$ or the n-prism, *Discrete Math.* **20** (1977) 287–294.

[16] A. T. White, The genus of repeated cartesian products of bipartite graphs, *Trans. AMS* **151** (1970), no. 2, 393–404.

[17] A. T. White, **Graphs, groups and surfaces**. Second edition. North-Holland, Mathematics Studies Number 8, Amsterdam, 1984.

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