Final exam for Math 203

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Abstract

Please send this in to me by Monday May 15, so I have time to grade it. Show your work and do not discuss the exam with your classmates. Recall that the definitions of poset (partially ordered set), lattice, complemented lattice, and distributive lattice are on our web page - and some of this material was also covered in the written handout on top of the file cabinet outside my office in Reiss 258.

Recall that a poset (partially ordered set) is a pair $(P, \geq)$, where P is a nonempty set and $\geq$ is a relation on P satisfying the two properties P1: $x \geq y$ and $y \geq x$ if and only if $x = y$ and P2: $x \geq y, y \geq z$ implies $x \geq z$.

Exercise 1 *Show that in a poset, any zero-object is unique.*

Define a *boolean algebra* to be a lattice with 0 and 1 which is distributive and complemented. For example, the family of all subsets of a given set S, with inclusion as the order relation, defines a lattice with S as the 1-object (everything is a subset of S) and the empty set $\emptyset$ as the 0-object. Complements are just set-theoretic complement and $\cup$ (join or lub) means union, while $\cap$ (meet or glb) means intersection. The distributive law holds for the lattice of subsets of S since $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Indeed, any element in the LHS is in A and in either B or C. In the first case, it is in $A \cap B$, etc.

Exercise 2 *Show that in a boolean algebra the complement of any element is unique. After this, we will denote the complement of a by a'.*

`You can use the sketch as a hint if you like.`

Sketch of proof. If B is a boolean algebra and if x and y are both complements to a - i.e., $a \cup x = 1 = a \cup y$ and $a \cap x = 0 = a \cap y$, then we must show that $x = y$. But if you begin $y = y \cap 1 = y \cap (a \cup x)$ and now apply the distributive law, you get that $y = y \cap x$. Reversing the roles of x and y and using the symmetry of $y \cap x$, $y = x$. $\square$

A ring R is called a *boolean ring* provided that $x^2 = x$ for every element x in R (i.e., every element is multiplicatively idempotent). We dealt with boolean rings on the Midterm, where it was shown that a boolean ring is automatically commutative and of characteristic 2 so $x + x = 0$ for every $x \in R$.

Given a boolean algebra, we can form a boolean ring from it in the following way: Let $R = B$, define multiplication in R to be $\cap$ in B. By L3 (in the written notes) $a \cap a = a$ for any lattice - indeed, the glb of a with itself is clearly again a. The addition for R is a bit less obvious. Define, for all $x, y \in R$, $x + y = (x \cap y') \cup (x' \cap y)$. It is straightforward, though not trivial, to check that with $\cdot = \cap$ and $+$ as just given, R is a ring. By L3, R is a boolean ring and moreover R has 1 for its multiplicative identity (since $1 \cap x = x$ for all x). In the notes we showed that one can also write $x + y = (x \cup y) \cap (x \cap y)'$. For the boolean algebra formed from the subsets of S, the corresponding ring-sum is called *symmetric difference* of sets.

One can show that the converse holds. That is, if R is any boolean ring with identity, there is a boolean algebra B such that R is obtained from B by the process described in the preceding paragraph. Of course, we take $B = R$ as underlying sets and define $x \cap y$ in B to be xy in R - meet and multiplication are the same. Put $x \cup y = x + y - xy$. We actually verified in class that this operation gives an associative operation in any ring with 0 as neutral element.

There are other things to check (i.e., L1 to L4), but I'm only asking

Exercise 3 *Show that B is distributive: Prove that for all a, b, c in B*

$$(a \cup b) \cap c = (a \cap c) \cup (b \cap c).$$

I mentioned that there is a connection with logic. Let B be a set of logical propositions (either true or false) which is closed under formation of "and" and "or" - e.g., the statement "P or Q" is true if and only if either P or Q is true. Calling these operations meet and join, respectively, makes B into a lattice.

Exercise 4 *Determine the corresponding order relation?*