Isolated squares in hypercubes and robustness of commutativity *

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Abstract

It is shown that for any nonempty collection of $d-2$ or fewer squares of a hypercube Q_d there exists a 3-cube subgraph of Q_d which contains exactly one of these squares. As a consequence, a diagram of isomorphisms on the scheme of the d-dimensional hypercube, which has strictly fewer than $d-1$ noncommutative squares, in fact, has no noncommutative faces. Statistical commutativity is considered.
Keywords: Hypercube, commutative diagram of isomorphisms, cube lemma, approximate commutativity, coherence, groupoid, algebraic stability.

1 Introduction

In monoidal categories (or in enriched categories) the problem of testing the commutativity of diagrams of isomorphisms arises (e.g., in the coherence theorems of Mac Lane). Our results in this paper and in [3] show that in some cases, it is possible to reduce the effort needed for such testing and to compensate for the possibility of error in the process.

We prove a combinatorial fact about faces of the hypercube and derive as a corollary an interesting robustness phenomenon for commutativity of diagrams consisting of invertible morphisms. For $d \geq 2$, a d-dimensional hypercube diagram of invertible morphisms which contains $d-2$ or fewer noncommutative faces must actually be commutative.

Our category theory result generalizes a special case of the Cube Lemma (Mitchell [6, p.43]). The Cube Lemma states: If a diagram on the scheme of a 3-dimensional cube has five of the six square faces commutative, all except the back face, and if the morphism from the source of the cube to the source of the back face is an epimorphism, then the back face also commutes. The dual case replaces back by front, source by sink and epimorphism by monomorphism. If all morphisms are invertible, then the cube commutes if any five of the six faces are commutative. Noncommutativity of such a 3-cube diagram means that at least two squares must be noncommutative. The proof of the Cube Lemma is by contradiction, using the cancellation property of epimorphisms (or monomorphisms).

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The result here, Theorem 2, is that in a noncommutative d-cube diagram with all morphisms invertible there must be at least $d-1$ noncommutative squares. Our proof uses the Cube Lemma and inductively proven combinatorial result given in Theorem 1.

Elsewhere [3], we proved a different generalization of the Cube Lemma which shows how commutativity can be *forced*: for $d \geq 2$, there is a particular set of $b_d = 1 + (d-2)2^{d-1}$ square faces of Q_d taken from the set of all $d(d-1)2^{d-3}$ squares, such that if each of these square faces commutes, then Q_d commutes.

In contrast, here we are showing that $d-1$ but no fewer square faces are sufficient to *block* commutativity.

The organization is as follows: In section 2, we give the d-cube graph's description and prove that a nonempty set of square faces which has fewer than $d-1$ members must contain at least one element which is "isolated" by a 3-dimensional subcube. Section 3 gives the notions of diagram and groupoid, and in the next section we prove that when a d-cube diagram fails to commute, it must do so on at least $d-1$ square faces. Section 5 considers the statistical implications. We conclude with some remarks.

2 Hypercubes and isolated square faces

For basic definitions and properties of graphs, see, e.g., Harary [2].

For d a nonnegative integer, the hypercube graph Q_d (or d-cube) has for vertices the binary d-tuples; two such 0/1 strings determine an edge if they differ in exactly one coordinate. Write $\mathbf{O}$ (or $\mathbf{1}$) for the vertex with all coordinates equal to 0 (1), resp. In the usual digraph structure (oriented consistently from 0 to 1 in all coordinates), $\mathbf{O}$ is the source and $\mathbf{1}$ the sink.

Clearly, Q_d contains various subgraphs isomorphic to lower-dimensional hypercubes; for instance, the front and back are copies of Q_{d-1}. Each vertex is a Q_0 subgraph and each edge is a Q_1 subgraph. A Q_2 subgraph is called a *square*. Let $\mathcal{F}_k(Q_d)$ denote the set of Q_k subgraphs of Q_d and put $f_k = |\mathcal{F}_k|$. Clearly, f_k is equal to 2^{d-k} times the number of ways to choose k from d. Also the number b_d of cycles in a cycle basis of Q_d is $b_d = 1 - f_0 + f_1 = 1 + (d-2)2^{d-1}$.

Let Q_d^0 denote the subgraph determined by the vertices of Q_d with last coordinate 0 (called the *back*) and similarly for Q_d^1, the *front* face. The graph $Q_d - (Q_d^0 \cup Q_d^1)$ will be called the *sides* of Q_d with respect to this fixed choice of primary axis, corresponding to the last coordinate. Note that each Q_{k+1} in the sides correspond exactly to one Q_k in the front (and back) face.

Let $k \geq 2$ and suppose $\mathcal{G}$ is any nonempty set of Q_k-subgraphs of Q_d and F is a member of $\mathcal{G}$. We say that F is *isolated* in Q_d with respect to $\mathcal{G}$ if there exists a $k+1$-cube G such that F is the only element of $\mathcal{G}$ which is a subgraph of G; G is said to isolate F for $\mathcal{G}$ and Q_d.

Theorem 1 *Let $d \geq 2$ and $\mathcal{G}$ be any nonempty subset of the squares of Q_d. If $|\mathcal{G}| < d-1$, then $\mathcal{G}$ contains at least one isolated square.*

Proof. We establish the assertion by induction. For $d = 2$, there are no nonempty subsets satisfying the hypotheses. For $d = 3$, $\mathcal{G}$ must consist of a single square, which is isolated by $G = Q_3$.

Now let $d \geq 4$ be any integer and consider a nonempty subset $\mathcal{G}$ of the squares of Q_d which contains fewer than $d-1$ elements. Plainly, since $\mathcal{G}$ is nonempty, it must have a nonempty intersection with the set of squares in some Q_{d-1} subgraph of Q_d; indeed, every square belongs to a $d-1$-cube. By reordering the coordinates, we can assume for convenience that the Q_{d-1} face is Q_d^0.

Let $\mathcal{F} = \mathcal{G} \cap \mathcal{F}_2(Q_d^0)$. If $\mathcal{F} = \mathcal{G}$, then in fact every square s in $\mathcal{G}$ is isolated with respect to $\mathcal{G}$ and Q_d by the unique 3-cube which meets Q_d^0 in s. If $\mathcal{F}$ is a proper subset of $\mathcal{G}$, then it is a nonempty set of fewer than $d-2$ squares in the $d-1$-cube Q_d^0, so, by the inductive hypothesis, there is a square s in $\mathcal{F}$ which is isolated with respect to $\mathcal{F}$ and Q_d^0 by some 3-cube G contained in Q_d^0. Hence, G isolates s for $\mathcal{G}$ and Q_d. $\square$

3 Commutativity of diagrams

In this section, we review the category-theoretic background. See, e.g., Mac Lane [4] for any undefined category theory terms.

An isomorphism is an invertible morphism. A *groupoid* is a category in which every morphism is an isomorphism. A category will be termed *nontrivial* if it contains an object with a nonidentity isomorphism.

Given a category $\mathcal{C}$ and a finite digraph $D = (V, A)$, a diagram δ in $\mathcal{C}$ on the scheme of D is a digraph embedding of D in the underlying digraph of $\mathcal{C}$. That is, a diagram is a labeling of each arc (resp. vertex) of D with a morphism (resp. object) of $\mathcal{C}$ so that morphisms are directed from domain to codomain and morphisms are composable exactly when the corresponding arcs meet head to tail. If, in addition, the diagram has the property that any two directed paths of morphisms joining any ordered pair of objects have identical compositions, it is called a *commutative diagram*. Equivalently, a diagram commutes if and only if it may be extended to a functor from the free category on D to $\mathcal{C}$. For groupoids, we assume that all diagrams are automatically extended to include the inverse morphism for every morphism as well as all the object identity maps. Clearly, a diagram commutes if and only if the corresponding extended diagram commutes.

For commutative diagrams of isomorphisms we can ignore directionality and consider the underlying graph. Commutativity for a diagram amounts to requiring that the composition of all the morphisms in every directed cycle must be an identity morphism. It is easy to check that for a given cycle in the underlying graph, if some directed orientation of the cycle as a directed cycle is equal in composition to the identity, then the same is true for any choice of orientation.

A hypercube commutes if all of its squares do. By symmetry it suffices to check equality for the composition of any two paths p and q from $\mathbf{O}$ to $\mathbf{1}$. Any such paths are determined by a permutation on d (namely, the sequence of coordinates in which $\mathbf{O}$ is changed to $\mathbf{1}$). Let the two paths p, q correspond to permutations σ and τ, respectively. Then each path induces the same morphism since $\sigma \circ \tau^{-1}$ is a product of transpositions, while each transposition leaves the value of the composition unchanged since every square commutes.

4 A minimal blocking set

A *face* of a digraph is a pair of distinct directed paths with the same ordered pair of source and sink vertices. A face commutes with respect to some diagram if the two paths yield identical morphisms. A nonempty set of squares is *blocking* if there is some diagram for which they are the unique noncommutative faces. The following result shows that the minimal size of a blocking set for the d-dimensional hypercube is at least $d-1$.

Let $\beta_{\mathcal{C}}(Q_d)$ denote the smallest number of noncommutative faces in any noncommutative diagram on the scheme of Q_d in the category $\mathcal{C}$.

Theorem 2 *For any nontrivial groupoid category $\mathcal{C}$ and for $d \geq 2$, $\beta_{\mathcal{C}}(Q_d) = d - 1$.*

Proof. To show that fewer than $d-1$ squares can't block commutativity, we use Theorem 1. If a subset $\mathcal{G}$ of fewer than $d-1$ squares blocked commutativity, then the isolated square would belong to a 3-cube in which every other square commutes so by the cube lemma, the isolated square would also commute - a contradiction. Hence, there must be at least $d-1$ noncommutative squares and so $\beta_{\mathcal{C}}(Q_d) \geq d-1$.

To show equality, use the nontriviality of $\mathcal{C}$. Let G be an object with a nonidentity morphism β. Define a diagram on the scheme of Q_d by making all objects equal to G and all arrows are the identity except for one arrow which is the morphism β. Then every square commutes except for the exactly $d-1$ square faces which contain the nonidentity morphism. □

The argument shows that for any diagram in a nontrivial category, any set of faces all of which share exactly one arc is a blocking set.

5 Statistical commutativity

Let $d \geq 3$ be an integer and suppose we are given a diagram δ on the scheme of the d-dimensional hypercube in some nontrivial groupoid. Let C be the event that δ commutes and C' the complementary event that it does not commute. For $0 < k < d-1$ an integer, let $A_{k,d}$ be the event that a randomly chosen subset $\mathcal{F}$ of $n_k = f_2 k/(d-1)$ elements from the set $\mathcal{F}_2(Q_d)$ all commute. Since failure of commutativity ensures at least $d-1$ noncommutative squares, sampling a proportion of $k/(d-1)$ of the squares should yield at least k noncommuting squares on average. Finding none is thus unlikely, as we now show.

Theorem 3 *For d at least 3, $k < d-1$ a positive integer and a diagram δ on the scheme of Q_d in a nontrivial groupoid category, $P(A_{k,d}|C') < e^{-k}$.*

Proof. Suppose that C' holds - that is, that the diagram does not commute. Let m denote the number of noncommutative square faces. Then since the n_k elements of $\mathcal{F}$ are chosen independently, the chance that none of them is noncommutative is bounded above by a product:

$$P(A_k|C') \leq (1 - \frac{m}{f_2})(1 - \frac{m}{f_2 - 1}) \cdots (1 - \frac{m}{f_2 - n_k + 1}).$$

Since for t positive, $1 - t < e^{-t}$, this product is less than $\exp(-mn_k/f_2)$ which suffices by Theorem 1. □

6 Discussion

A possible application of Theorem 2 is in the area of quantum computing. Categories have been used to model computation and system evolution (Manes [5]); groupoids can represent reversible operations such as occur in quantum computations. See [1] for a connection with quantum algebra. Hypercube diagrams could describe the pure states and transitions.

Commutative cubes and other commutative diagrams also arise in various mathematical definitions, as well as in the coherence theory of Mac Lane and Stasheff. Our results show that such algebraic conditions can be checked even when the mechanism for verifying commutativity can give false negatives provided the probability of error is sufficiently small.

Indeed, determination of commutativity might, itself, be subject to incorrect measurement. For instance, through "equipment error" a commutative square might be recorded as noncommutative. Since actual noncommutativity of the hypercube diagram must produce at least $d-1$ noncommutative squares, finding fewer than this number would guarantee that the diagram was, in fact, strictly commutative unless the measurement error also allowed the false conclusion of commutativity for a noncommutative square.

A more subtle form of the commutativity checking effort minimization applies without the isomorphism constraint. Commutativity follows when certain cancellation occurs; e.g., if there is any epimorphism which is not to the hypercube sink or its first neighborhood, then the epimorphism is followed by the two parallel paths of a face and so the comutativity of this face follows from that of a subset of the other faces. The isomorphism constraint ensures that any given square can be made the front or back face of a particular 3-cube.

The preceding can be applied to other diagram schemes. Using a "tetrahedron lemma" and an analogous argument to the hypercube case, one can show that for the complete graph K_n on n vertices, the blocking number for a nontrivial category is $n-2$ (the configuration of $n-2$ triangles with a common edge is a minimum blocking set). In contrast, a particular subset of $(n-1)(n-2)/2$ triangles is sufficient to force commutativity for the entire K_n-diagram and has the minimum cardinality for such subsets.

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